"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 22 (2012), No. 1, 125 - 146

CBMO ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR IN HERZ AND MORREY-HERZ SPACES

SHEN YING, HUANG CHUANGXIA AND LIU LANZHE

Abstract. In this paper, we establish CBMO estimates for the multilinear commutator related to the Littlewood-Paley operator in Herz and Morrey-Herz spaces.

1. Introduction

Let $b \in BMO((\mathbb{R}^n)$ and T be the Calderón-Zygmand operator. Consider the commutator defined by

$$[b, T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberg and Weiss [2] state that commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see also [1]). Lu and Yang (see [10]) introduced the central BMO space, that is, CBMO space. Since it is obvious that $BMO(\mathbb{R}^n) \subsetneqq CBMO_q(\mathbb{R}^n)$ for all $1 \leq q < \infty$. However, we know that the (L^p, L^q) boundedness fails with only the assumption $b \in CBMO_q(\mathbb{R}^n)$. Instead, certain boundedness properties on Herz spaces and Morrey-Herz spaces can be proved. In this paper, we will establish CBMO estimates for the multilinear commutator related to the Littlewood-Paley operator in Herz and Morrey-Herz spaces.

Keywords and phrases: multilinear commutator, Littlewood-Paley operator, CBMO, Herz space, Morrey-Herz space.
(2010)Mathematics Subject Classification: 42B20, 42B35.

2. Preliminaries and results

First, let us introduce some notations.

Definition 1. Let $1 \leq q < \infty$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the space $CBMO_q(\mathbb{R}^n)$ if

$$||f||_{CBMO_q} = \sup_{R>0} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x) - f_{B(0,R)}|^q dx \right)^{1/q} < \infty,$$

where, $B = B(0,R) = \{x \in \mathbb{R}^n : |x| < R\}$ and $f_{B(0,R)}$ is the mean value of f on $B(0,R)$.

Let $\vec{q} = (q'_1, \cdots q'_j)$ and $\vec{b} = (b_1, \cdots b_m)$, for $b_j \in CBMO_{q'_j}(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{CBMO_{\vec{q}}} = \prod_{j=1}^{m} ||b_j||_{CBMO_{q'_j}}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C^m_j$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C^m_j$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{CBMO_{\vec{q}}} = ||b_{\sigma(1)}||_{CBMO_{q'_1}} \cdots ||b_{\sigma(j)}||_{CBMO_{q'_j}}$.

Definition 2. Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q < \infty$. For $k \in \mathbb{Z}$, set $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of A_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,\,p}_q(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,\,p}_q} < \infty\},$$

where

$$||f||_{\dot{K}^{\alpha,\,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} \right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,\,p}_q(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n) : ||f||_{K^{\alpha,\,p}_q} < \infty\},$$

where

$$||f||_{K^{\alpha,\,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q} \right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 3. Let $\alpha \in \mathbb{R}$, $0 \leq \lambda < \infty$, $0 < p \leq \infty$ and $0 < q < \infty$. The homogeneous Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) : ||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}} < \infty\},$$

where

$$||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} ||f\chi_k||^p_{L^q} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

Remark 1. Compare the homogeneous Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ with the homogeneous Herz space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and the Morrey space $M^\lambda_q(\mathbb{R}^n)$(see [14]), obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $M^\lambda_q(\mathbb{R}^n) \subset M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n)$. We can see that when $\lambda = 0$, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n)$ is just the homogeneous Herz space.

Definition 4. Fix $\delta > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$;

(2)$|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$;

(3)$|\psi(x + y) - \psi(x)| \leq C|y|^\epsilon(1 + |x|)^{-(n+\epsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear commutator is defined by

$$S^{\vec{b}}_\delta(f)(x) = \left[\int\int_{\Gamma(x)} |F^{\vec{b}}_t(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F^{\vec{b}}_t(f)(x, y) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y - z)f(z)dz,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. We also consider

$$S_\delta(f)(x) = \left(\int\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [16]).

Let H be the space $H = h : ||h|| = (\int\int_{\mathbb{R}^{n+1}_+} |h(y, t)|^2 \frac{dydt}{t^(n+1)})^{1/2} < \infty$, then, for each fixed $x \in \mathbb{R}^n$, $F^{\vec{b}}_t(f)(x, y)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$S_\delta(f)(x) = ||\chi_{\Gamma(x)}F_t(f)(x)||$$

and

$$S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x,y)||.$$

Note that when $b_1 = \cdots = b_m$, $S_\delta^{\vec{b}}$ is just the commutator of order m. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors [3-9],[10], [13], [15]. Our main purpose is to study the boundedness properties for the multilinear commutator on central Morrey spaces.

Now we state our theorems as following.

Theorem 1. *Let $1 < q < \infty$, $b \in CBMO_q(\mathbb{R}^n)$, and $S_\delta^{\vec{b}}$ be defined as in Definition 4 with $0 < \delta < n$, $1 < q_1 < \frac{n}{\delta}$, $1 < q_2 < \infty$. If $0 < p \le \infty$, $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q} - \frac{\delta}{n}$, where $\frac{1}{q} = \frac{1}{q_1'} + \cdots \frac{1}{q_j'}$, and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q} < 1$, $\frac{1}{u} = \frac{1}{q_1} - \frac{\delta}{n}$, $\delta - \frac{n}{q_1} < \alpha_1 < n - \frac{n}{q_1}$ and $\alpha_2 = \alpha_1 - \frac{n}{q}$, then*

$$||S_\delta^{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha_2,p}} \le C||\vec{b}||_{CBMO_{\vec{q}}} ||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.$$

Theorem 2. *Let $\lambda \ge 0$, $1 < q < \infty$, $b \in CBMO_q(\mathbb{R}^n)$, and $S_\delta^{\vec{b}}$ be defined as in Definition 4 with $0 < \delta < n$, $1 < q_1 < \frac{n}{\delta}$, $1 < q_2 < \infty$. If $0 < p \le \infty$, $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q} - \frac{\delta}{n}$, where $\frac{1}{q} = \frac{1}{q_1'} + \cdots \frac{1}{q_j'}$, and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q} < 1$, $\frac{1}{u} = \frac{1}{q_1} - \frac{\delta}{n}$, $\lambda + \delta - \frac{n}{q_1} < \alpha_1 < n - \frac{n}{q_1} + \lambda$ and $\alpha_2 = \alpha_1 - \frac{n}{q}$, then*

$$||S_\delta^{\vec{b}}(f)||_{M\dot{K}_{p,q_2}^{\alpha_2,\lambda}} \le C||\vec{b}||_{CBMO_{\vec{q}}} ||f||_{M\dot{K}_{p,q_1}^{\alpha_1,\lambda}}.$$

Remark 2. *Theorem 1 follows from Theorem 2 when $\lambda = 0$, but it is more convenient to carry out the proof in the particular case of homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, then to generalize it to the homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.*

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.([12]) *Suppose that $f \in CBMO_q(\mathbb{R}^n)$, $1 \le q < \infty$ and $R_1, R_2 > 0$. Then*

$$\left(\frac{1}{|B(0,R_1)|}\int_{B(0,R_1)} |f(x) - f_{B(0,R_2)}|^q dx\right)^{1/q} \le \left(1 + \left|\ln\left(\frac{R_1}{R_2}\right)\right|\right) ||f||_{CBMO_q}.$$

Lemma 2.([16]) *Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof of Theorem 1. We only consider the case $0 < p < \infty$. Let $f \in \dot{K}_{q_1}^{\alpha_1,p}(\mathbb{R}^n)$ and decompose f into

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) \equiv \sum_{l=-\infty}^{\infty} f_l(x).$$

When $m = 1$, we consider

$$\begin{aligned} ||S_\delta^{b_1}(f)||_{\dot{K}_{q_2}^{\alpha_2,p}} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}^p\right)^{1/p} \\ &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3} ||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\ &+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} ||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\ &+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty} ||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\ &= E_1 + E_2 + E_3. \end{aligned}$$

Let us first estimate E_2, note that

$$S_\delta^{b_1}(f_l)\chi_k = (b - b_{B_k})S_\delta(f_l)\chi_k + S_\delta((b - b_{B_k})f_l)\chi_k.$$

We have

$$||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}} \leq ||(b-b_{B_k})S_\delta(f_l)\chi_k||_{L^{q_2}} + ||S_\delta((b-b_{B_k})f_l)\chi_k||_{L^{q_2}} = J_1 + J_2.$$

For J_1, by Hölder's inequality, Lemma 1 and the boundedness of S_δ from $L^{q_1}(\mathbb{R}^n)$ to $L^u(\mathbb{R}^n)$, we have

$$\begin{aligned} J_1 &= \left(\int_{B_k} |b_1(x) - b_{B_k}|^{q_2}|S_\delta(f_l)|^{q_2}dx\right)^{1/q_2} \\ &\leq C\left(\int_{B_k} |b_1(x) - b_{B_k}|^q dx\right)^{1/q}\left(\int_{B_k} |S_\delta(f_l)|^u dx\right)^{1/u} \\ &\leq C|B_k|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}. \end{aligned}$$

For J_2, by Hölder's inequality, Lemma 1 and the boundedness of S_δ from $L^t(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
J_2 &= \left(\int_{B_k} |S_\delta((b-b_{B_k})f_l)|^{q_2}dx\right)^{1/q_2} \\
&\leq C\left(\int_{B_k} |(b-b_{B_k})f_l|^t dx\right)^{1/t} \\
&\leq C\left(\int_{B_k} |b_1(x)-b_{B_k}|^q dx\right)^{1/q}\left(\int_{B_k} |f_l|^{q_1}dx\right)^{1/q_1} \\
&\leq C\left(\int_{B_l} |b_1(x)-b_{B_k}|^q dx\right)^{1/q}||f_l||_{L^{q_1}} \\
&\leq C|B_l|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E_2 &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} J_1\right)^p\right\}^{1/p} + \\
&+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} J_2\right)^p\right\}^{1/p} \\
&= I_1 + I_2.
\end{aligned}
$$

For I_1, if $1<p<\infty$, by Minkowski's inequality and if $0<p\leq 1$, by the inequality $(\sum|a_i|)^p \leq \sum|a_i|^p$, we have

$$
\begin{aligned}
I_1 &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} 2^{kn/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
&\leq C||b_1||_{CBMO_q} \\
&\times \begin{cases} \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p}\right]^{1/p}, \quad 0<p\leq 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p}\right)\left(\sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\le C||b_1||_{CBMO_q}\\
&\times \begin{cases} \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p}\right]^{1/p}, \quad 0<p\le 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \left(\sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p/2}\right)\left(\sum_{k=l-2}^{l+2} 2^{(k-l)\alpha_1 p'/2}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty. \end{cases}\\
&\le C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\right)^{1/p}\\
&\le C||b_1||_{CBMO_q}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.
\end{aligned}
$$

For I_2, similarly to the method for estimating I_1, we have

$$
\begin{aligned}
I_2 &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} 2^{ln/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\le C||b_1||_{CBMO_q}\begin{cases} \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p}\right]^{1/p}, \\ 0<p\le 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p/2}\right)\right. \\ \left.\times\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p'/2}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty. \end{cases}\\
&\le C||b_1||_{CBMO_q}\begin{cases} \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p}\right]^{1/p}, \\ 0<p\le 1 \\ \left[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p/2}\right)\right. \\ \left.\times\left(\sum_{k=l-2}^{l+2} 2^{(k-l)(\alpha_1-\frac{n}{q})p'/2}\right)^{p/p'}\right]^{1/p}, \\ 1<p<\infty. \end{cases}\\
&\le C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\right)^{1/p}\\
&\le C||b_1||_{CBMO_q}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.
\end{aligned}
$$

Thus, we deduce

$$E_2 \leq C||b_1||_{CBMO_q}||f||_{\dot{K}^{\alpha_1,p}_{q_1}}.$$

Now, let us turn to estimate E_1, choosing $(b_1)_B = |B|^{-1}\int_B b_1(x)dx$, by Minkowski's inequality, we have

$$\begin{aligned}
S_\delta^{\vec{b}}(f_l)(x) &= \left(\int\int_{\Gamma(x)} |\int_{A_l}(b_1(x)-b_1(z))\psi_t(y-z)f(z)dz|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\leq \int_{A_l}|b_1(x)-b_1(z)||f(z)|\left[\int\int_{\Gamma(x)}|\psi_t(y-z)|^2\frac{dydt}{t^{n+1}}\right]^{1/2}dz\\
&\leq \int_{A_l}|b_1(x)-b_1(z)||f(z)|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|y-z|)^{2(n+1-\delta)}}dydt\right)^{1/2}dz\\
&\leq \int_{A_l}|b_1(x)-b_1(z)||f(z)|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}2^{2(n+1)}}{(2t+|y-z|)^{2(n+1-\delta)}}dydt\right)^{1/2}dz
\end{aligned}$$

for $|x-y|<t$, $2t+|y-z|>2t+|x-z|-|x-y|>t+|x-z|$,

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2(n+1-\delta)}} = C|x-z|^{-(n-\delta)},$$

thus

$$\begin{aligned}
S_\delta^{\vec{b}}(f_l)(x) &\leq \int_{A_l}|(b_1(x)-b_1(z))f(z)|\\
&\times \left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x-z|)^{2(n+1-\delta)}}dydt\right)^{1/2}dz\\
&\leq \int_{A_l}|(b_1(x)-b_1(z))f(z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2(n+1-\delta)}}\right)^{1/2}dz\\
&\leq \int_{A_l}|(b_1(x)-b_1(z))f(z)||x-z|^{-n+\delta}dz,
\end{aligned}$$

then, using Hölder's inequality, we can get

$$\begin{aligned}
||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}} &\leq \left\{\int_{A_k}\left(\int_{A_l}|(b_1(x)-b_1(z))f(z)||x-z|^{-n+\delta}dz\right)^{q_2}dx\right\}^{1/q_2}\\
&\leq C|B_k|^{\delta/n-1}\left\{\int_{A_k}\left(\int_{A_l}|b_1(x)-b_1(z)||f(z)|dz\right)^{q_2}dx\right\}^{1/q_2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C|B_k|^{\delta/n-1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}\left(\int_{A_l}|f(z)|dz\right)^{q_2}dx\right)^{1/q_2}\\
&\quad +C|B_k|^{\delta/n-1}\left\{\int_{A_k}\left(\int_{A_l}|b_1(z)-(b_1)_B||f(z)|dz\right)^{q_2}dx\right\}^{1/q_2}\\
&\leq C|B_k|^{\delta/n-1}||f_l||_{L^1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}dx\right)^{1/q_2}\\
&+ C|B_k|^{\delta/n-1+1/q_2}\int_{A_l}|b_1(z)-(b_1)_B||f(z)|dz\\
&\leq C|B_k|^{\delta/n-1}||f_l||_{L^{q_1}}|B_l|^{1-1/q_1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q}dx\right)^{1/q}|B_k|^{1/q_2-1/q}\\
&+ C|B_k|^{\delta/n-1+1/q_2}\left(\int_{A_l}|b_1(z)-(b_1)_B|^{q}dz\right)^{1/q}\\
&\times \left(\int_{A_l}|f(z)|^{q_1}dz\right)^{1/q_1}|B_l|^{1-1/q-1/q_1}\\
&\leq C|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}||b_1||_{CBMO_q}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&E_1 \leq C||b_1||_{CBMO_q}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}2^{nk(\delta/n-1+1/q_2)}2^{nl(1-1/q_1)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}\\
&\leq C||b_1||_{CBMO_q}\\
&\times \begin{cases}
\Big[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\\
\quad\times\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^p\Big]^{1/p}, \quad 0<p\leq 1\\
\Big[\sum_{l=-\infty}^{\infty}|B_l|^{\alpha_1 p}||f_l||_{L^{q_1}}^p\left(\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p/2}\right)\\
\quad\times\left(\sum_{k=l+3}^{\infty}\left(2^{nk(\delta/n-1+1/q_2-1/q)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\right)^{p'/2}\right)^{p/p'}\Big]^{1/p}, \ 1<p<\infty.
\end{cases}\\
&\leq C||b_1||_{CBMO_q}
\end{aligned}
$$

$$
\times \begin{cases} \Big[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \\ \times \sum_{k=l+3}^{\infty}\Big(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\Big)^p\Big]^{1/p}, \quad 0<p\leq 1 \\ \Big[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\Big(\sum_{k=l+3}^{\infty}\Big(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\Big)^{p/2}\Big) \\ \times\Big(\sum_{k=l+3}^{\infty}\Big(2^{nk(1/q_1-1)}2^{nl(1-1/q_1)}2^{(k-l)\alpha_1}\Big)^{p'/2}\Big)^{p/p'}\Big]^{1/p}, \ 1<p<\infty. \end{cases}
$$

$$
\begin{aligned}
&\leq C||b_1||_{CBMO_q}\begin{cases} \Big[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p \\ \times\sum_{k=l+3}^{\infty} 2^{(k-l)(n/q_1-n+\alpha_1)p}\Big]^{1/p}, \quad 0<p\leq 1 \\ \Big[\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\Big(\sum_{k=l+3}^{\infty} 2^{(k-l)(\alpha_1-n+\frac{n}{q_1})p/2}\Big) \\ \times\Big(\sum_{k=l+3}^{\infty} 2^{(k-l)(\alpha_1-n+\frac{n}{q_1})p'/2}\Big)^{p/p'}\Big]^{1/p}, \ 1<p<\infty. \end{cases} \\
&\leq C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p}||f_l||_{L^{q_1}}^p\right)^{1/p} \\
&\leq C||b_1||_{CBMO_q}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}.
\end{aligned}
$$

Now, let us turn to estimate E_3, by Hölder's inequality, we have

$$
\begin{aligned}
&||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}} \leq \left\{\int_{A_k}\left(\int_{A_l}(b_1(x)-b_1(z))f(z)|x-z|^{-n+\delta}dz\right)^{q_2}dx\right\}^{1/q_2} \\
\leq\ & C|B_l|^{\delta/n-1}\left\{\int_{A_k}\left(\int_{A_l}|b_1(x)-b_1(z)||f(z)|dz\right)^{q_2}dx\right\}^{1/q_2} \\
\leq\ & C|B_l|^{\delta/n-1}||f_l||_{L^1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^{q_2}dx\right)^{1/q_2} \\
+\ & C|B_l|^{\delta/n-1}|B_k|^{1/q_2}\int_{A_l}|b_1(z)-(b_1)_B||f(z)|dz
\end{aligned}
$$

$$
\begin{aligned}
&\le C|B_l|^{\delta/n-1}||f_l||_{L^{q_1}}|B_l|^{1-1/q_1}\left(\int_{A_k}|b_1(x)-(b_1)_B|^q dx\right)^{1/q}|B_k|^{1/q_2-1/q}\\
&+ C|B_l|^{\delta/n-1}|B_k|^{1/q_2}\left(\int_{A_l}|b_1(z)-(b_1)_B|^q dz\right)^{1/q}\left(\int_{A_l}|f(z)|^{q_1}dz\right)^{1/q_1}\\
&\quad\times|B_l|^{1-1/q-1/q_1}\\
&\le C|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}||b_1||_{CBMO_q}.
\end{aligned}
$$

Thus, in this case, we obtain

$$
E_3 \le C||b_1||_{CBMO_q}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}
$$

$$
\begin{aligned}
&\le C||b_1||_{CBMO_q}\\
&\times\begin{cases}\Big[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\\ \quad\times\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^p\Big]^{1/p}, \quad 0<p\le 1\\ \Big[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^{p/2}\right)\\ \quad\times\left(\sum_{k=-\infty}^{l-3}\left(2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2-1/q)}2^{(k-l)\alpha_1}\right)^{p'/2}\right)^{p/p'}\Big]^{1/p}, \ 1<p<\infty.\end{cases}\\
&\le C||b_1||_{CBMO_q}\begin{cases}\Big[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\\ \quad\times\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p}\Big]^{1/p}, \quad 0<p\le 1\\ \Big[\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\left(\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p/2}\right)\\ \quad\times\left(\sum_{k=-\infty}^{l-3}2^{(l-k)(\delta-\alpha_1-\frac{n}{q_1})p'/2}\right)^{p/p'}\Big]^{1/p}, \ 1<p<\infty.\end{cases}\\
&\le C||b_1||_{CBMO_q}\left(\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}||f_l||^p_{L^{q_1}}\right)^{1/p}\\
&\le C||b_1||_{CBMO_q}||f||_{\dot K^{\alpha_1,p}_{q_1}}.
\end{aligned}
$$

This completes the proof of the case $m=1$.

When $m>1$, we consider

$$\begin{aligned}
||S_\delta^{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha_2,p}} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}^p\right)^{1/p} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&+ C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&= G_1+G_2+G_3.
\end{aligned}$$

Let us first estimate G_2, set $\vec{b}_B=((b_1)_B,\cdots,(b_m)_B)$, where $(b_j)_B=|B|^{-1}\int_B|b_j(x)|dx,\quad 1\le j\le m$, we have

$$\begin{aligned}
&F_t^{\vec{b}}(f_l)(x,y)=\int_{\mathbb{R}^n}\prod_{j=1}^m\left[(b_j(x)-(b_j)_B)-(b_j(z)-(b_j)_B)\right]\psi_t(y-z)f_l(z)dz \\
=\ &\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_B)_\sigma\int_{\mathbb{R}^n}(b(z)-(b)_B)_{\sigma^c}\psi_t(y-z)f_l(z)dz \\
=\ &\prod_{j=1}^m(b_j(x)-(b_j)_B)F_t(f_l)(y)+(-1)^m F_t(\prod_{j=1}^m(b_j-(b_j)_B))f_l)(y) \\
+\ &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_B)_\sigma\int_{\mathbb{R}^n}(b(z)-b(x))_{\sigma^c}\psi_t(y-z)f_l(z)dz \\
=\ &\prod_{j=1}^m(b_j(x)-(b_j)_B)F_t(f_l)(y)+(-1)^m F_t(\prod_{j=1}^m(b_j-(b_j)_B))f_l)(y) \\
+\ &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-(b)_B)_\sigma F_t^{\vec{b}_{\sigma^c}}(f_l)(x,y),
\end{aligned}$$

thus

$$\begin{aligned}
&S_\delta^{\vec{b}}(f_l)(x) = ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f_l)(x)|| \le ||\chi_{\Gamma(x)}\prod_{j=1}^m(b_j(x)-(b_j)_B)F_t(f_l)(x)|| \\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(b(x)-(b)_B)_\sigma F_t^{\vec{b}_{\sigma^c}}(f_l)(x)|| \\
&+||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_B)f_l)(x)|| \le \prod_{j=1}^m(b_j(x)-(b_j)_B))S_\delta(f_l)(x) \\
+ \quad & (-1)^m S_\delta(\prod_{j=1}^m(b_j-(b_j)_B))_B)f_l)(x) \\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma S_\delta((b-\ b_B)_{\sigma^c}f_l)(x) \\
= \quad & H_1+H_2+H_3.
\end{aligned}$$

For H_1, taking $1<q_1<n/\delta$ and u such that $1/u=1/q_1-\delta/n$, choosing $1/q=1/q_1'+\cdots 1/q_j'$, by Hölder's inequality and the boundedness of S_δ from $L^{q_1}(\mathbb{R}^n)$ to $L^u(\mathbb{R}^n)$, we have

$$\begin{aligned}
&||\prod_{j=1}^m(b_j(x)-(b_j)_B))S_\delta(f_l)(x)\chi_k||_{L^{q_2}} \\
\le\ & C\Big(\int_{A_k}\Big|\prod_{j=1}^m(b_j(x)-(b_j)_B))\Big|^q dx\Big)^{1/q}\Big(\int_{A_k}|S_\delta f(x)|^u dx\Big)^{1/u} \\
\le\ & C\prod_{j=1}^m\Big(\int_{A_k}|b_j(x)-(b_j)_B|^{q_j'}dx\Big)^{1/q_j'}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1} \\
\le\ & C|B_k|^{1/q_1'+\cdots 1/q_j'}\prod_{j=1}^m\Big(\frac{1}{|B_k|}\int_{B_k}|b_j(x)-(b_j)_B|^{q_j'}dx\Big)^{1/q_j'}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1} \\
\le\ & C|B_k|^{1/q}\prod_{j=1}^m||b_j||_{CBMO_{q_j}}||f_l||_{L^{q_1}} \\
\le\ & C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}$$

For H_2, taking $1<t<n/\delta$ and u such that $1/q_2=1/t-\delta/n$, choosing $1/t=1/q+1/q_1$, by Hölder's inequality and the boundedness of S_δ

from $L^t(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&||(-1)^m S_\delta(\prod_{j=1}^m(b_j-(b_j)_B)))(x)\chi_k||_{L^{q_2}}\\
\leq\ & C||\prod_{j=1}^m(b_j-(b_j)_B))f_l\chi_k||_{L^t}\\
\leq\ & C\Big(\int_{A_k}\Big|\prod_{j=1}^m(b_j(x)-(b_j)_B))\Big|^q dx\Big)^{1/q}\Big(\int_{A_k}|f_l(x)|^{q_1}dx\Big)^{1/q_1}\\
\leq\ & C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}$$

For H_3, choosing $1/q_2 = 1/q_1' + 1/\omega$ and $1/\omega = 1/q_2' + 1/q_1 - \delta/n$, using Hölder's inequality and the boundedness of S_δ, we have

$$\begin{aligned}
&||\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma S_\delta((b-\ b_B)_{\sigma^c}f_l)(x)\chi_k||_{L^{q_2}}\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big(\int_{A_k}|(b(x)-\ b_B)_\sigma|^{q_1'}dx\Big)^{1/q_1'}\\
\times\ & \Big(\int_{A_k}|S_\delta((b-\ b_B)_{\sigma^c}f_l)(x)|^\omega dx\Big)^{1/\omega}\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big(\int_{A_k}|(b(x)-\ b_B)_\sigma|^{q_1'}dx\Big)^{1/q_1'}\\
\times\ & \Big(^{1/q_2'}\Big(\int_{A_k}|f_l(x)|^{p_1}dx\Big)^{1/q_1}\\
\leq\ & C|B_k|^{1/q_1'}||\vec{b}_\sigma||_{CBMO_{\vec{q}}}|B_k|^{1/q_2'}||\vec{b}_{\sigma^c}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}\\
\leq\ & C|B_k|^{1/q}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}$$

Then, similarly to the method estimating for I_1, we get $G_2 \leq C||\vec{b}||_{CBMO_q}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}$.

Next, let us estimate G_1, let $\tau, \tau' \in \mathbb{N}$ such that $\tau + \tau' = m$, we have

$$
\begin{aligned}
||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}} &\le \left\{ \int_{A_k} \left(\int_{A_l} \prod_{j=1}^m |b_j(x) - b_j(z)||f(z)||x-z|^{-n+\delta} dz \right)^{q_2} dx \right\}^{1/q_2} \\
&\le C|B_k|^{\delta/n-1} \sum_{j=0}^m \sum_{\sigma\in C_j^m} \left(\int_{A_k} |(b(x) - \ b_B)_\sigma|^{q_2} dx \right)^{1/q_2} \int_{A_l} |(b(z) - b_B)_{\sigma^c}||f(z)|dz \\
&\le C|B_k|^{\delta/n-1} \sum_{j=0}^m \sum_{\sigma\in C_j^m} \sum_{\tau+\tau'=m} \left(\int_{A_k} |(b(x) - \ b_B)_\sigma|^{\tau} dx \right)^{1/\tau} |B_k|^{1/q_2-1/\tau} \\
&\quad \times \left(\int_{A_l} |(b(z) - \ b_B)_{\sigma^c}|^{\tau'} dz \right)^{1/\tau'} \left(\int_{A_l} |f(z)|^{q_1} dz \right)^{1/q_1} |B_l|^{1-1/\tau'-1/q_1} \\
&\le C|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}
$$

Then, similarly to the method estimating for E_1, we can obtain $G_1 \le C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}$.

Finally, let us estimate G_3, since

$$
\begin{aligned}
||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}} &\le \left\{ \int_{A_k} \left(\int_{A_l} \prod_{j=1}^m |b_j(x) - b_j(z)||f(z)||x-z|^{-n+\delta} dz \right)^{q_2} dx \right\}^{1/q_2} \\
&\le C|B_l|^{\delta/n-1} \sum_{j=0}^m \sum_{\sigma\in C_j^m} \left(\int_{A_k} |(b(x) - \ b_B)_\sigma|^{q_2} dx \right)^{1/q_2} \int_{A_l} |(b(z) - b_B)_{\sigma^c}||f(y)|dz \\
&\le C|B_l|^{\delta/n-1} \sum_{j=0}^m \sum_{\sigma\in C_j^m} \sum_{\tau+\tau'=m} \left(\int_{A_k} |(b(x) - \ b_B)_\sigma|^{\tau} dx \right)^{1/\tau} |B_k|^{1/q_2-1/\tau} \\
&\quad \times \left(\int_{A_l} |(b(z) - \ b_B)_{\sigma^c}|^{\tau'} dz \right)^{1/\tau'} \left(\int_{A_l} |f(y)|^{q_1} dz \right)^{1/q_1} |B_l|^{1-1/\tau'-1/q_1} \\
&\le C|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||\vec{b}||_{CBMO_{\vec{q}}}||f_l||_{L^{q_1}}.
\end{aligned}
$$

Thus, similarly to the method estimating for E_3, we can get $G_3 \le C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{\dot{K}_{q_1}^{\alpha_1,p}}$.
This completes the proof of Theorem 1.

Proof of Theorem 2. Let $f \in M\dot{K}_{p,q_1}^{\alpha_1,\lambda}(\mathbb{R}^n)$ and decompose f into

$$
f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) \equiv \sum_{l=-\infty}^{\infty} f_l(x).
$$

When $m=1$, we consider

$$\begin{aligned}
||S_\delta^{b_1}(f)||_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}||S_\delta^{b_1}(f_l)\chi_k||^p_{L^{q_2}}\right)^{1/p} \\
&\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&+ C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&+ C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}||S_\delta^{b_1}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
&= U_1+U_2+U_3.
\end{aligned}$$

Let us first estimate U_2, similarly to the estimate of E_2, we have

$$\begin{aligned}
U_2 &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} J_1\right)^p\right\}^{1/p} \\
&\quad +C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} J_2\right)^p\right\}^{1/p} \\
&= V_1+V_2.
\end{aligned}$$

For V_1, we have

$$\begin{aligned}
V_1 &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} |B_k|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
&\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=k-2}^{k+2} 2^{kn/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
&\qquad \leq C||b_1||_{CBMO_q}\times\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\quad \left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2} 2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l} 2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p} \\
&\leq C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2} 2^{(k-l)(\alpha_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p} \\
&\qquad \leq C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

Similarly, for V_2, we have

$$V_2 \le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}|B_l|^{1/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}$$

$$\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=k-2}^{k+2}2^{ln/q}||b_1||_{CBMO_q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p}$$

$$\le C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} \times 2^{-k_0\lambda}$$

$$\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)\alpha_1}2^{(l-k)n/q}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p}$$

$$\le C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2}2^{(k-l)(\alpha_1-n/q-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p}$$

$$\le C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.$$

Therefore $U_2 \le C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}$.

Then, let us estimate U_1, similarly to the estimate of E_1, we get

$$\begin{aligned}
& U_1 \le C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
\times\ & \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}|B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
\le\ & C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
\times\ & \left\{\sum_{k=-\infty}^{\infty} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=-\infty}^{k-3}2^{kn(\delta/n-1+1/q_2)}2^{ln(1-1/q_1)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
\le\ & C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right. \\
\times\ & \left.\left[\sum_{l=-\infty}^{k-3}2^{(k-l)(n/q_1-n)}2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l}2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p} \\
\le\ & C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=-\infty}^{k-3}2^{(k-l)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p} \\
\le\ & C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

Last, let us estimate U_3, similarly to the estimate of E_3, we get

$$\begin{aligned}
& U_3 \leq C||b_1||_{CBMO_q} \\
\times \quad & \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}|B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
\leq \quad & C||b_1||_{CBMO_q} \\
\times \quad & \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty} 2^{k(\alpha_1-n/q)p}\left(\sum_{l=k+3}^{\infty} 2^{ln(\delta/n-1/q_1)}2^{kn(1/q_2)}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\
\leq \quad & C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(U_3'\right)^p\right\}^{1/p} \\
\leq \quad & C||b_1||_{CBMO_q}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(U_3''\right)^{1/p} \\
\leq \quad & C||b_1||_{CBMO_q}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}.
\end{aligned}$$

We denoted

$$U_3'=\sum_{l=k+3}^{\infty} 2^{(l-k)(\delta-n/q_1)}2^{(k-l)\alpha_1}2^{(l-k)\lambda}2^{-l\lambda}\left(\sum_{i=-\infty}^{l} 2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p},$$

and

$$U_3''=\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k+3}^{\infty} 2^{(l-k)(\delta-n/q_1-\alpha_1+\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p$$

This completes the proof of the case $m=1$.
When $m>1$, we consider

$$\begin{aligned}
||S_\delta^{\vec{b}}(f)||_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}} \leq \; & C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
+ \; & C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
+ \; & C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty}||S_\delta^{\vec{b}}(f_l)\chi_k||_{L^{q_2}}\right)^p\right\}^{1/p} \\
= \; & W_1+W_2+W_3.
\end{aligned}$$

For W_2, similarly to G_2, we have

$$\begin{aligned} W_2 &\le C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=k-2}^{k+2} |B_k|^{1/q}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\ &\le C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k-2}^{k+2} 2^{(k-l)(\alpha_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p} \\ &\le C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}. \end{aligned}$$

For W_1, similarly to the estimate of G_1, we have

$$\begin{aligned} & W_1 \le C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\ \times & \left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-3} |B_k|^{\delta/n-1+1/q_2}|B_l|^{1-1/q_1}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\ \le & C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\ \times & \left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=-\infty}^{k-3} 2^{(k-l)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p} \\ \le & C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}. \end{aligned}$$

For W_3, similarly to the estimate of G_3, we have

$$\begin{aligned} & W_3 \le C||\vec{b}||_{CBMO_{\vec{q}}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+3}^{\infty} |B_l|^{\delta/n-1/q_1}|B_k|^{1/q_2}||f_l||_{L^{q_1}}\right)^p\right\}^{1/p} \\ \le & C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right. \\ \times & \left.\left[\sum_{l=k+3}^{\infty} 2^{(l-k)(\delta-n/q_1-\alpha_1+\lambda)} 2^{-l\lambda}\left(\sum_{i=-\infty}^{l} 2^{i\alpha_1 p}||f_i||^p_{L^{q_1}}\right)^{1/p}\right]^p\right\}^{1/p} \\ \le & C||\vec{b}||_{CBMO_{\vec{q}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left[\sum_{l=k+3}^{\infty} 2^{(l-k)(\alpha_1-n+n/q_1-\lambda)}||f||_{M\dot{K}^{\alpha_1,p}_{p,q_1}}\right]^p\right\}^{1/p} \\ \le & C||\vec{b}||_{CBMO_{\vec{q}}}||f||_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}}. \end{aligned}$$

This completes the proof of Theorem 2.

Acknowledgement

The authors would like to express their gratitude to the referee for his comments and suggestions. This work was jointly supported by the National Natural Science Foundation of China (No.11101053), the Key Project of Chinese Ministry of Education (No.211118), the Excellent Youth Foundation of Educational Committee of Hunan Provincial (No.10B002).

References

[1] W. G. Chen, **Besov estimates for a class of multilinear singular integrals**, Acta Math.Sinica 16(2000), 613-626.
[2] R. R. Coifman, R. Rochberg, G. Weiss, **Factorization theorems for Hardy spaces in several variables,** Ann. of. Math., 103(1976), 611-635.
[3] S. Janson, **Mean oscillation and commutators of singular integral operators**, Ark.Math.16(1978), 263-270.
[4] L. Z. Liu, **Triebel-Lizorkin space estimates for multilinear operators of sublinear operators**, Proc.Indian Acad.Sci.(Math.Sci.) 113(2003), 379-393.
[5] L. Z. Liu, **The continuity of commutators on Triebel-Lizorkin spaces**, Integral Equations and Operator Theory 49(2004), 65-76.
[6] L. Z. Liu, **Boundedness of multilinear operator on Triebel-Lizorkin spaces**, Inter J.of Math.and Math.Sci.5(2004), 259-272.
[7] L. Z. Liu, **Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces**, Extracta Math.19,2(2004), 243-255.
[8] L. Z. Liu, S. Z. Lu and J. S. Xu, **Boundedness for commutators of Littlewood-Paley operators**, Adv.in Math.32(2003), 473-480.
[9] S.Z.Lu,Y.Meng and Q.Wu, **Lipschitz estimates for multilinear singular integrals**, Acta Math.Scientia 24,B(2004), 291-300.
[10] S. Z. Lu and D. C. Yang, **The central BMO spaces and Littlewood-Paley operators**, Approximation Theory and its Applications 11(1995), 72-94.
[11] S. Z. Lu and Q. Wu, **CBMO estimates for commutators and multilinear singular integrals**, Math.Nachr. 276(2004), 75-88.
[12] S. Z. Lu, Q. Wu and D. C. Yang, **Boundedness of commutators on Hardy type spaces**, Sci.in China(ser.A) 32(2002), 232-244.
[13] S. Z. Lu, D. C. Yang and Z. S. Zhou, **Sublinear operators with rough kernel on generalized Morrey spaces**, Hokkaido Math.J.27(1998), 219-232.
[14] M. Paluszynski, **Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss**, Indiana Univ.Math.J.44(1995), 1-17.
[15] C. Pérez and R. Trujillo-Gonzalez, **Sharp weighted estimates for multilinear commutators**, J.London Math.Soc.65(2002), 672-692.

[16] A. Torchinsky, **Real variable methods in harmonic analysis**, Pure and Applied Math. 123, Academic Press, New York, 1986.

Shen Ying
Department of Mathematics, Changsha University of Science and Technology, Changsha, 410077, P.R.of CHINA

Huang Chuangxia
Department of Mathematics, Changsha University of Science and Technology, Changsha, 410077, P.R.of CHINA

Liu Lanzhe
Department of Mathematics, Changsha University of Science and Technology, Changsha, 410077, P.R.of CHINA, e-mail: lanzheliu@163.com