

“Vasile Alecsandri” University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 22 (2012), No. 2, 5 - 16

STRONG CONVERGENCE FOR FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

BALWANT SINGH THAKUR

Abstract. The purpose of this paper is to prove strong convergence theorems for finite family of asymptotically nonexpansive mappings in the intermediate sense using implicit iteration process with error term. The results in this paper extends and generalizes corresponding results.

1. INTRODUCTION

In 1993, Bruck, Kuczumow and Reich [1] introduced the concept of asymptotically nonexpansive mappings in intermediate sense. This class of mappings properly contains the class of asymptotically nonexpansive mappings which itself includes a class of nonexpansive mappings as a proper subclass [3].

Let K be a nonempty subset of a real normed space E . Recall that a mapping $T : K \rightarrow K$ is said to be:

(i) *Nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, y in K .

Keywords and phrases: Implicit iteration process with error, asymptotically nonexpansive mapping in intermediate sense.

(2010) Mathematics Subject Classification: 47H09, 47H10.

- (ii) *Asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| ,$$

for all x, y in K and $n \in \mathbb{N}$, where \mathbb{N} denotes the set of positive integers.

- (iii) *Asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and the following inequality holds:

$$(1.1) \quad \limsup_{n \rightarrow \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 .$$

Xu and Ori [8] introduced implicit iteration process involving finite family of nonexpansive mappings and proved weak convergence of this iteration in a Hilbert space and posed the following question for strong convergence:

What assumptions on the mappings and/or the control parameters are sufficient to guarantee the strong convergence of the sequence.

Zhou and Chang [9] studied the strong convergence of the implicit iteration process to a common fixed point of semi-compact nonexpansive mappings.

More recently Chidume and Shahzad [2] proved a strong convergence theorem for finite family of nonexpansive mappings using implicit iteration process, in which just one of the mappings is semi-compact.

In this paper we study strong convergence of modified implicit iteration process with errors to a common fixed point of a finite family of asymptotically nonexpansive mappings in the intermediate sense in uniformly convex Banach space. The iteration scheme is defined as below:

Let K be a nonempty closed subset of a normed space E , and let $\{T_1, T_2, \dots, T_N\} : K \rightarrow K$ be N asymptotically nonexpansive mappings in the intermediate sense, for arbitrary chosen $x_0 \in K$, construct the

sequence $\{x_n\} \subset K$ by

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned}$$

where, $\{u_n\}$ is a bounded sequence in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

We can write down above table in the following compact form:

$$(1.2) \quad x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n,$$

for all $n \geq 1$, where $n = (k(n) - 1)N + i(n)$, $k(n) \geq 1$ is an integer such that $k(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $i(n) \in I = \{1, 2, \dots, N\}$.

2. PRELIMINARIES

From now on we will denote by $F(T)$ the set of fixed points of a mapping $T : K \rightarrow K$, i.e., $F(T) = \{x \in K : Tx = x\}$. For any $x \in E$, define $d(x, K) = \inf_{x \in K} \|x - y\|$.

We now give some results which will be used in the rest of this paper.

Lemma 2.1. [4] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2. [5] *Let E be a uniformly convex Banach space, b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a real sequence*

in $[b, c]$ and $\{x_n\}, \{y_n\}$ are two sequences in E such that:

$$\left. \begin{aligned} \limsup_{n \rightarrow \infty} \|x_n\| &\leq a; \\ \limsup_{n \rightarrow \infty} \|y_n\| &\leq a; \\ \lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| &= a, \end{aligned} \right\}.$$

where $a \geq 0$ is some constant. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

3. MAIN RESULTS

Lemma 3.1. *Let E be a real normed space, K be a nonempty closed convex subset of E . Let $\{T_1, T_2, \dots, T_N\} : K \rightarrow K$ be N asymptotically nonexpansive mappings in the intermediate sense with*

$$d_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x, y \in K} (\|T_j^n x - T_j^n y\| - \|x - y\|), 0 \right\}$$

such that $\sum_{n=1}^{\infty} d_n < \infty$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.2) with

- (i) $0 < \alpha \leq \beta_n \leq \beta < 1$, for all $n \geq 1$, where $\alpha, \beta \in (0, 1)$;
- (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$.

Proof. For any $p \in F$, according to (1.2) we have

$$\|x_n - p\| = \left\| \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n - p \right\|.$$

Since $p \in F(T_{i(n)})$ and $\alpha_n + \beta_n + \gamma_n = 1$, we have

$$\begin{aligned} \|x_n - p\| &= \left\| \alpha_n (x_{n-1} - p) + \beta_n (T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p) + \gamma_n (u_n - p) \right\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left\| T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p \right\| + \gamma_n \|u_n - p\|. \end{aligned}$$

By definition of d_n , we have

$$(3.1) \quad \left\| T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p \right\| \leq \|x_n - p\| + d_{k(n)}.$$

It follows that

$$\begin{aligned} \|x_n - p\| &\leq \frac{\alpha_n}{1 - \beta_n} \|x_{n-1} - p\| + \frac{\beta_n}{1 - \beta_n} d_{k(n)} + \frac{\gamma_n}{1 - \beta_n} \|u_n - p\| \\ &= \left(1 - \frac{\gamma_n}{1 - \beta_n} \right) \|x_{n-1} - p\| + t_n, \end{aligned}$$

where $t_n := \frac{\beta_n}{1 - \beta_n} d_{k(n)} + \frac{\gamma_n}{1 - \beta_n} \|u_n - p\|$.

Hence

$$\|x_n - p\| \leq \|x_{n-1} - p\| + t_n.$$

But $0 < \alpha \leq \beta_n \leq \beta < 1$ implies that $\frac{\alpha}{1-\alpha} \leq \frac{\beta_n}{1-\beta_n} \leq \frac{\beta}{1-\beta}$, hence

$$\sum_{n=1}^{\infty} \frac{\beta_n}{1-\beta_n} d_{k(n)} \leq \frac{\beta}{1-\beta} \sum_{n=1}^{\infty} d_{k(n)} = \frac{\beta}{1-\beta} N \sum_{n=1}^{\infty} d_n < +\infty.$$

Since $\{u_n\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n < \infty$, we can see that $\sum_{n=1}^{\infty} t_n < \infty$. By Lemma 2.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

This completes the proof. \square

Lemma 3.2. *If all the assumptions of the Lemma 3.1 hold, then $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i \in \{1, 2, \dots, N\}$.*

Proof. For any $p \in F$, it follows from Lemma 3.1 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Denote $a := \lim_{n \rightarrow \infty} \|x_n - p\|$ for some $p \in F$.

By the proof of Lemma 3.1,

$$\|x_n - p\| \leq \|x_{n-1} - p\| + t_n,$$

where $\{t_n\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n < \infty$.

We have,

$$(3.2) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \\ \leq \limsup_{n \rightarrow \infty} \|x_{n-1} - p\| + \limsup_{n \rightarrow \infty} \gamma_n \|u_n - x_{n-1}\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \gamma_n = 0$ and the sequence $\{u_n - x_{n-1}\}$ is bounded,

$$(3.3) \quad \lim_{n \rightarrow \infty} \gamma_n \|u_n - x_{n-1}\| = 0.$$

Hence, by (3.2) and (3.3), we get

$$(3.4) \quad \limsup_{n \rightarrow \infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \leq a.$$

Similarly, using (3.1) we have

$$(3.5) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \|T_{i(n)}^{k(n)} x_n - p + \gamma_n(u_n - x_{n-1})\| \\ \leq \limsup_{n \rightarrow \infty} \|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p\| + \limsup_{n \rightarrow \infty} \gamma_n \|u_n - x_{n-1}\| \\ \leq \limsup_{n \rightarrow \infty} (\|x_n - p\| + d_{k(n)}) + \limsup_{n \rightarrow \infty} \gamma_n \|u_n - x_{n-1}\| \\ = a. \end{aligned}$$

Now,

$$\begin{aligned}
 (3.6) \quad & \lim_{n \rightarrow \infty} \left\| (1 - \beta_n) [x_{n-1} - p + \gamma_n(u_n - x_{n-1})] \right. \\
 & \quad \left. + \beta_n \left[T_{i(n)}^{k(n)} x_n - p + \gamma_n(u_n - x_{n-1}) \right] \right\| \\
 &= \lim_{n \rightarrow \infty} \left\| \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n - p \right\| \\
 &= \lim_{n \rightarrow \infty} \|x_n - p\| \\
 &= a.
 \end{aligned}$$

It follows from (3.4)-(3.6) and Lemma 2.2 that

$$(3.7) \quad \limsup_{n \rightarrow \infty} \left\| T_{i(n)}^{k(n)} x_n - x_{n-1} \right\| = 0.$$

Moreover,

$$\begin{aligned}
 \|x_n - x_{n-1}\| &= \left\| \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n - x_{n-1} \right\| \\
 &\leq \beta_n \left\| T_{i(n)}^{k(n)} x_n - x_{n-1} \right\| + \gamma_n \|u_n - x_{n-1}\|.
 \end{aligned}$$

From above inequality and (3.3) and (3.7), we have

$$(3.8) \quad \lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0,$$

and so,

$$(3.9) \quad \lim_{n \rightarrow \infty} \|x_n - x_{n+j}\| = 0, \quad \text{for all } j = 1, 2, \dots, N.$$

From (3.7) and (3.8), we have

$$(3.10) \quad \lim_{n \rightarrow \infty} \left\| T_{i(n)}^{k(n)} x_n - x_n \right\| = 0.$$

We take $T_n = T_{n \bmod N}$ by definition, i.e. $T_n = T_{i(n)}$ for $n \geq 1$. For each $n > N$, we have $n = (k(n) - 1)N + i(n)$, where $k(n) \geq 1$, $k(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $i(n) \in \{1, 2, \dots, N\}$, also $i(n) = i(n + N)$ and $k(n) + 1 = k(n + N)$.

We have the following estimates

$$\begin{aligned}
\|x_n - T_n x_n\| &\leq \|x_n - x_{n+N}\| + \left\| x_{n+N} - T_{i(n)}^{k(n)+1} x_{n+N} \right\| \\
&\quad + \left\| T_{i(n)}^{k(n)+1} x_{n+N} - T_{i(n)}^{k(n)+1} x_n \right\| + \left\| T_{i(n)}^{k(n)+1} x_n - T_n x_n \right\| \\
&= \|x_n - x_{n+N}\| + \left\| x_{n+N} - T_{i(n+N)}^{k(n+N)} x_{n+N} \right\| \\
&\quad + \left\| T_{i(n+N)}^{k(n+N)} x_{n+N} - T_{i(n+N)}^{k(n+N)} x_n \right\| + \left\| T_{i(n)}^{k(n)+1} x_n - T_{i(n)} x_n \right\| \\
&\leq 2 \|x_n - x_{n+N}\| + \left\| x_{n+N} - T_{i(n+N)}^{k(n+N)} x_{n+N} \right\| \\
&\quad + d_{k(n+N)} + \left\| T_{i(n)}^{k(n)+1} x_n - T_{i(n)} x_n \right\|.
\end{aligned}$$

(3.10) implies

$$(3.11) \quad \lim_{n \rightarrow \infty} \left\| T_{i(n)}^{k(n)+1} x_n - T_{i(n)} x_n \right\| = 0,$$

since $\left\| T_{i(n)}^{k(n)+1} x_n - T_{i(n)} x_n \right\| = \left\| T_{i(n)} \left(T_{i(n)}^{k(n)} x_n \right) - T_{i(n)} x_n \right\| \leq \max_{1 \leq i \leq N} \left\| T_i \left(T_{i(n)}^{k(n)} x_n \right) - T_i x_n \right\|$ for each $n \geq 1$ and all T_i , $i = 1, \dots, N$ are uniformly continuous.

It follows from (3.9) (3.10) and (3.11) that

$$(3.12) \quad \lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Since

$$\begin{aligned}
\|x_n - T_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \\
&\quad + \|T_{n+j} x_{n+j} - T_{n+j} x_n\|,
\end{aligned}$$

from (3.9), (3.12) and the uniform continuity of T_i for any $i = 1, 2, \dots, N$, we have $\lim_{n \rightarrow \infty} \|x_n - T_{n+j} x_n\| = 0$, for $j = 1, 2, \dots, n$.

Let $l \in \{1, 2, \dots, N\}$. For any $n \geq 1$ there exists some $j(n) \in \{1, 2, \dots, N\}$ such that $(n + j(n)) \bmod N = l$. Thus

$$\|x_n - T_l x_n\| = \|x_n - T_{n+j(n)} x_n\| \leq \max_{1 \leq j \leq n} \|x_n - T_{n+j} x_n\|,$$

for every $n \geq 1$. Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad \text{for all } l = 1, 2, \dots, N.$$

This completes the proof. \square

Let $f : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function with $f(0) = 0$ and $f(r) > r$ for all $r \in (0, \infty)$.

A mapping $T : K \rightarrow K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [6] on K if, for all $x \in K$

$$\|x - Tx\| \geq f(d(x, F(T))) .$$

A finite family $\{T_1, T_2, \dots, T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition (B) on K if, for all $x \in K$

$$\max_{1 \leq l \leq N} \|x - T_l x\| \geq f(d(x, F)) .$$

Note that condition (B) reduces to condition (A) when $T_l = T$ for all $l = 1, 2, \dots, N$.

It is well known that every continuous and demicompact mapping must satisfy condition (A) (see [6]). Since every completely continuous mapping $T : K \rightarrow K$ is continuous and demicompact, so it satisfies condition (A). Therefore to study strong convergence of $\{x_n\}$ defined by (1.2) we use condition (B) instead the complete continuity of mappings T_1, T_2, \dots, T_N .

Theorem 3.3. *If all the assumptions of the Lemma 3.1 hold, and $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B), then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N\}$.*

Proof. By Lemma 3.1, we see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n \rightarrow \infty} \|x_n - p\| = a$ for some $a \geq 0$. If $a = 0$, there is nothing to prove, so assume that $a > 0$. As proved in Lemma 3.1, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + t_{n+1} ,$$

where $\{t_n\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n < \infty$.

The above inequality implies

$$d(x_{n+1}, F) \leq d(x_n, F) + t_{n+1} .$$

Applying Lemma 2.1, we obtain that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists.

Also by Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ for all $l \in \{1, 2, \dots, N\}$. Since $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B), we conclude that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, for a given $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\varepsilon}{3}$ for all $n \geq n_0$. Since $\sum_{k=1}^{\infty} t_k < \infty$ and $\{t_k\}$ is a sequence of real numbers, we may take n_0 sufficiently large such that $\sum_{k=1}^{n+m} t_k - \sum_{k=1}^{n_0} t_k < \frac{\varepsilon}{4}$ for all $n \geq n_0$ and $m \geq 1$.

Let $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{4}$.

We have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| .$$

As proved in Lemma 3.1, for all $n \geq 2$, we have $\|x_n - p^*\| \leq \|x_{n-1} - p^*\| + t_n$, so we get by induction for all $n \geq n_0$ and $m \geq 1$, that

$$\|x_{n+m} - p^*\| \leq \|x_{n_0} - p^*\| + t_{n_0+1} + \cdots + t_{n+m},$$

and

$$\|x_n - p^*\| \leq \|x_{n_0} - p^*\| + t_{n_0+1} + \cdots + t_n.$$

Hence,

$$\|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| + 2\left(\sum_{k=1}^{n+m} t_k - \sum_{k=1}^{n_0} t_k\right).$$

It follows that

$$\|x_{n+m} - p^*\| + \|x_n - p^*\| < 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon,$$

hence $\|x_{n+m} - x_n\| < \varepsilon$, whenever $n > n_0$ and $m \geq 1$.

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n \rightarrow \infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$.

Since each T_i , $i \in \{1, 2, \dots, N\}$ is uniformly continuous, for every $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $\|T_i x - T_i y\| < \varepsilon$ whenever $x, y \in K$ satisfy $\|x - y\| < \delta(\varepsilon)$.

Let $\varepsilon > 0$ and let $\varepsilon_1 = \min\{2\delta(\frac{\varepsilon}{2}), \varepsilon\}$. Then there exists a natural number n_1 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_1$. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, there exists a natural number $n_2 \geq n_1$ such that for all $n \geq n_2$ we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_2}, F) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$.

Let $i \in \{1, 2, \dots, N\}$. Since

$$\|q^* - w^*\| \leq \|q^* - x_{n_2}\| + \|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} = \frac{\varepsilon_1}{2} \leq \delta\left(\frac{\varepsilon}{2}\right),$$

it follows that

$$\|T_i q^* - T_i w^*\| < \frac{\varepsilon}{2}.$$

Then,

$$\|T_i q^* - q^*\| \leq \|T_i q^* - T_i w^*\| + \|w^* - q^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon_1}{2},$$

but $\varepsilon_1 \leq \varepsilon$, hence $\|T_i q^* - q^*\| < \varepsilon$. Letting $\varepsilon \rightarrow 0$, we get $q^* \in F(T_i)$. Since $i \in \{1, 2, \dots, N\}$ is arbitrary, we get that $q^* \in F$.

This completes the proof. \square

A finite family $\{T_1, T_2, \dots, T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition (C) on K if, for all $x \in K$

$$\frac{1}{N} \sum_{l=1}^N \|x - T_l x\| \geq f(d(x, F)).$$

Remark 3.4. Theorem 3.3 holds true if we replace condition (B) with condition (C).

We recall that a mapping $T : K \rightarrow K$ is called *semi-compact* (or *hemcompact*) if any sequence $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence.

Theorem 3.5. *If all the assumptions of the Lemma 3.1 hold, and one of the mappings in $\{T_1, T_2, \dots, T_N\}$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \dots, N\}$. By Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|x_n - T_{i_0} x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j \rightarrow \infty} x_{n_j} \rightarrow p \in K$. Now Lemma 3.2 guarantees that $\lim_{n_j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l \in \{l = 1, 2, \dots, N\}$ and so $\|p - T_l p\| = 0$ for all $l \in \{l = 1, 2, \dots, N\}$. This implies that $p \in F$. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 3.3, that $\{x_n\}$ converges strongly to some common fixed point in F .

This completes the proof. \square

Remark 3.6. Theorem 3.5 generalizes and improves Theorem 1 of Zhou and Chang [9] for the family of asymptotically nonexpansive mappings in the intermediate sense also the key assumption (v) There exists a constant $L > 0$ such that for any $i, j \in \{1, 2, \dots, N\}$ $i \neq j$

$$\|T_i^n x - T_j^n y\| \leq L \|x - y\| \quad \text{for all } n \geq 1, x, y \in K,$$

is not required.

Remark 3.7. Theorem 3.5 generalizes and improves Theorem 3.3 and Theorem 3.4 of Sun [7] for the family of asymptotically nonexpansive mappings in the intermediate sense and modified implicit iteration process with errors considered here.

Acknowledgements. The author is greatly indebted to the anonymous referee for several helpful comments, pointing out important flaws and for many stimulating hints. This work is supported by the UGC India, Project 41-1390/2012(SR).

REFERENCES

- [1] R.E.Bruck, Y.Kuczumow and S.Reich, **Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property**, Colloq. Math. 65(1993),169-179.
- [2] C.E.Chidume and N.Shahzad, **Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings**, Nonlinear Anal. 62(2005),1149-1156.
- [3] K.Goebel and W.A.Kirk, **A fixed point theorem for asymptotically nonexpansive mappings**, Proc. Amer. Math. Soc. 35(1972),171-174.
- [4] M.O.Osilike, S.C.Aniagbosor and B.G.Akuchu, **Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces**, Panamer. Math. J. 12(2002),77-88.
- [5] J.Schu, **Weak and strong convergence to fixed points of asymptotically nonexpansive mappings**, Bull. Austral. Math. Soc. 43(1991),153-159.
- [6] H.F.Senter and W.G.Dotson,Jr., **Approximating fixed points of nonexpansive mappings**, Proc. Amer. Math. Soc. 44(1974),375-380.
- [7] Z.H.Sun, **Strong convergence of an implicit iteration process for a finite family of asymptotically quasi- nonexpansive mappings**, J. Math. Anal. Appl. 286(2003),351-358.
- [8] H.K.Xu and R.G.Ori, **An implicit iteration process for nonexpansive mappings**, Numer. Funct. Anal. Optim. 22(2001),767-773.
- [9] Y.Zhou and S.S.Chang, **Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces**, Numer. Funct. Anal. Optim. 23(2002),911-921.

Balwant Singh Thakur

Pt.Ravishankar Shukla University, School of Studies in Mathematics,
Raipur - 492010 (C.G.), India, e-mail: balwantst@gmail.com

