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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 22 (2012), No. 2, 17 - 30

ON TEMPERED DISTRIBUTIONS WHOSE GABOR TRANSFORM IN LORENTZ-KARAMATA SPACES

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Abstract. In this paper, some fundamental properties of Lorentz-Karamata (LK) spaces are examined by using the properties of Lorentz spaces. Also, we define and show some properties of the spaces of tempered distributions where Gabor transform of these tempered distributions are in Lorentz-Karamata spaces in analog to modulation spaces.

1. Introduction and Preliminaries

Gabor transform is a special case of the short-time Fourier transform (STFT) that is used to determine the sinusoidal frequency and phase content of local sections of a signal as it changes over time. The function to be transformed is first multiplied by a Gaussian function, which can be regarded as a window function, and the resulting function is then transformed with a Fourier transform to derive the time-frequency analysis. [6]. Since the Fourier transform of a function in $L^1\left(\mathbb{R}^d\right)$ is not so useful to get information about spectrum of the transform, it is effective to use Gabor transform or STFT. In Gabor transform, a (window) function g is taken and fixed. Then the Gabor transform of any function f according to this window g is found by

$$V_g f\left(x,w\right)=\int_{\mathbb{R}^d} f\left(u\right)\overline{g\left(u-x\right)}e^{-2\pi i\langle t,w\rangle}dt$$

for $x,w\in\mathbb{R}^d$ where $\langle t,w\rangle$ is the usual product on $\mathbb{R}^d$.

Keywords and phrases: Lorentz-Karamata spaces, Modulation, Tempered Distribution, Gabor Transform.
(2000) Mathematics Subject Classification: 46E30, 94A14, 26A12, 43A22.

For any measurable, complex valued function on $\mathbb{R}^d$, the translation and modulation operators are defined by $L_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i \langle t,w \rangle} f(t)$ for any $x, w \in \mathbb{R}^d$, respectively. The Fourier transform $\widehat{f}$ of any $f \in L^1(\mathbb{R}^d)$ is given by

$$\widehat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,t \rangle} dx.$$

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by D.E.Edmunds, R.Kerman and L.Pick in [8]. By using Karamata theory, they introduced Lorentz-Karamata (LK) spaces and compared quasinorms on these spaces. Also J.S.Neves studied LK spaces $L_{p,q;b}(R,\mu)$ in [15] where $p, q \in (0,\infty]$, b is a slowly varying function on $[1,\infty)$ and (R,μ) is a measure space. These spaces cover the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$ (introduced in [7]), Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$ (introduced in [1]), Zygmund spaces $L^p(\log L)^{\alpha}(R)$ (introduced in [2, 18]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^p(R)$ under convenient choices of slowly varying functions and parameters p, q. In [9,15], it is proved that $L_{p,q;b}(R,\mu)$ space endowed with a convenient norm, is a rearrangment-invariant Banach function space and has an associate space $L_{p',q';b^{-1}}(R,\mu)$ if (R,μ) is a resonant measure space, $p \in (1,\infty)$ and $q \in [1,\infty]$. Also it is showed that when $p \in (1,\infty)$ and $q \in [1,\infty)$, LK spaces have absolutely continuous norm.

Throughout this paper, $G, \mathbb{R}^d$ and dx will stand for locally compact abelian group, Euclidean d-dimensional space and Lebesgue measure, respectively. Besides these, $C_0(\mathbb{R}^d)$ will denote the space of all continuous functions that vanish at infinity. For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that A and B are equivalent.

Definition 1. *Let f be a measurable function defined on a measure space (X,μ) and finite valued almost everywhere. The distribution function λ_f of f is defined by*

$$\lambda_f(y) = \mu\{x \in X : |f(x)| > y\}. \tag{1.1}$$

The nonnegative rearrangement of f is given by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, \ t \geq 0 \tag{1.2}$$

where we assume that $\inf \phi = \infty$ *and* $\sup \phi = 0$. *Also the average(maximal) function of* f *on* $(0, \infty)$ *is given by*

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds. \tag{1.3}$$

Note that $\lambda_f(\cdot), f^*(\cdot)$ *and* $f^{**}(\cdot)$ *are nonincreasing and right continuous functions.*

Definition 2. *A positive and Lebesgue measurable function* b *is said to be slowly varying (s.v.) on* $[1, \infty)$ *in the sense of Karamata if, for each* $\varepsilon > 0$, $t^{\varepsilon} b(t)$ *is equivalent to a non-decreasing function and* $t^{-\varepsilon} b(t)$ *is equivalent to a non-increasing function on* $[1, \infty)$.

Given a s.v. function b on $[1, \infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right) \quad \text{for all } t > 0.$$

It is known that any s.v. function b on $(0, \infty)$ is equivalent to a s.v. continuous function $\widetilde{b}$ on $(0, \infty)$. Consequently, without loss of generality, we assume that all s.v. functions in question are continuous functions on $(0, \infty)$ [10]. The detailed study of Karamata Theory, properties and examples of s.v. functions can be found in [3], [8], [14], [15] and [18, Chapter V, pp.186].

Definition 3. *Let* $p, q \in (0, \infty]$ *and* b *be a s.v. function on* $[1, \infty)$. *The Lorentz-Karamata (LK) space* $L_{p,q;b}(G)$ *is defined to be the set of all measurable functions* f *such that*

$$\|f\|_{(p,q;b)} := \left\| t^{\frac{1}{p} - \frac{1}{q}} \gamma_b(t)\, f^*(t) \right\|_{q;(0,\infty)} \tag{1.4}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ *stands for the usual* L_q *(quasi-) norm over the interval* $(0, \infty)$.

Let us introduce the functional $\|f\|_{p,q;b}$ defined by

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p} - \frac{1}{q}} \gamma_b(t)\, f^{**}(t) \right\|_{q;(0,\infty)}; \tag{1.5}$$

this is identical with that defined in (1.4) except that f^* is replaced by f^{**}. It is easy to see that LK spaces $L_{p,q;b}(G)$ endowed with a convenient norm (1.5), are rearrangment-invariant Banach function spaces and have absolutely continuous norm when $p \in (1, \infty)$ and $q \in [1, \infty)$. It is clear that, for $0 < p < \infty$, LK spaces contain the characteristic function of every measurable subset of G with finite measure and hence, by linearity, every $\mu-$simple function. From the definition of $\|\cdot\|_{(p,q;b)}$, it follows that if $f \in L_{p,q;b}(G)$ and $p,q \in (0,\infty)$, then the function $\lambda_f(y)$ is finite valued. In this case, with a little thought, it is easy to see that it is possible to construct a sequence of (simple) functions which satisfy Lemma 1.1 in [4]. Therefore, if we use the same method as employed in the proof of proposition 2.4 in [13], we can show that Lebesgue dominated convergence theorem holds and so the set of simple functions is dense in LK space. Also, we can see the density of $C_c(G)$, the set of all continuous and complex-valued functions with compact support.

Lemma 1. [15, *Lemma* 3.1] *Let b be a s.v. function on $[1,\infty)$. Then*

(i) b^r is also a s.v. function on $[1,\infty)$ for any $r \in \mathbb{R}$ and $\gamma_{b^r}(t) = \gamma_b^r(t)$ for all $t > 0$.
(ii) Let $\alpha > 0$. Then

$$\int_0^\tau t^{\alpha-1}\gamma_b(t)\,dt \approx \sup_{0<t<\tau} t^\alpha \gamma_b(t) \approx \tau^\alpha \gamma_b(\tau) \quad \textit{for all } \tau > 0; \tag{1.6}$$

$$\int_\tau^\infty t^{-\alpha-1}\gamma_b(t)\,dt \approx \sup_{\tau<t<\infty} \tau^{-\alpha}\gamma_b(t) \approx \tau^{-\alpha}\gamma_b(\tau) \quad \textit{for all } \tau > 0. \tag{1.7}$$

Lemma 2. [18, *Lemma* 5.1(i)] *Let $1 < p < \infty$, $1 \le q < \infty$ and b be a s.v. function. Then $C_0^\infty(\mathbb{R}^d)$, the space of all smooth functions with compact support, is dense in $L_{p,q;b}(\mathbb{R}^d)$.*

In anolog to modulation spaces, Gürkanlı defined $M(p,q)(\mathbb{R}^d)$ spaces of tempered distributions whose Gabor transform in Lorentz spaces and showed some properties of these spaces. Also, a new Segal algebra $S(p,q)(\mathbb{R}^d)$ is established and examined which is obtained by intersecting $M(p,q)(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ in [11].

In the next section, we will show some new properties of Lorentz-Karamata (LK) spaces that are examined by using the properties of Lorentz spaces. Later, we will define and examine $M(p,q;b)(\mathbb{R}^d)$ spaces of tempered distributions whose Gabor transform in LK spaces. For this part, we mostly benefit from [11] and [12, Chapter 11].

2. Main Results

2.1. Some Properties of LK spaces.

Lemma 3. *The Schwartz space $S(\mathbb{R}^d)$ is contained by $L_{p,q;b}(\mathbb{R}^d)$.*

Remark 1. *If we consider* [18, *Lemma* 5.1(*iii*)] *with* $\sigma = 0$, *we see the density of* $S\left(\mathbb{R}^d\right)$ *in* $L_{p,q;b}\left(\mathbb{R}^d\right)$ *for* $p \in (1,\infty)$ *and* $q \in [1,\infty)$. *However, we may give an alternative proof to* [18, *Lemma* 5.1(*iii*)].

Proof. Let us take any $f \in S\left(\mathbb{R}^d\right)$. Then for every $m \in \mathbb{N}$, there is a positive constant c_m such that

$$|f(u)| \leq c_m \left(1+|u|^2\right)^{-m/2} \tag{2.1}$$

for all $u \in \mathbb{R}^d$. If we take $m = 0$, then it shows that $f^*(t) \leq c_0$ for all $t > 0$. Therefore

$$\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t) f^*(t)\right\|_{q;(0,1)} \leq c_0^q \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\right\|_{q;(0,1)} < \infty \tag{2.2}$$

by Lemma 1. Also, if we choose $m > \frac{d}{p}$, then we get

$$f^*(t) \leq c_m w_d^{m/d}\left(w_d^{2/d} + t^{2/d}\right)^{-m/2}$$

where w_d is the volume of the unit ball in $\mathbb{R}^d$ and

$$\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t) f^*(t)\right\|_{q;(1,\infty)} \leq \int_1^\infty t^{-qm/d+q/p-1}\gamma_{b^q}(t)\,dt < \infty \tag{2.3}$$

by Lemma 1. Hence $\|f\|_{p,q;b;G} < \infty$ by (2.2) and (2.3). $\square$

Proposition 1. *Let* f *be a scalar valued, measurable functions on* (G,μ). *Then for any* $s \in G$, *we have the following:*

(i) $\lambda_{L_s f}(y) = \lambda_f(y)$ *for all* $y \geq 0$,
(ii) $(L_s f)^*(t) = f^*(t)$ *for all* $t \geq 0$ *and* $(L_s f)^{**}(t) = f^{**}(t)$ *for all* $t > 0$,
(iii) *If* $p,q \in (0,\infty)$, *then* $\|L_s f\|_{(p,q;b)} = \|f\|_{(p,q;b)}$, $\|L_s f\|_{p,q;b} = \|f\|_{p,q;b}$.

Proof. It is straightforward by taking into consideration [5, Lemma 3.1]. $\square$

Proposition 2. *For any* $f \in L_{p,q;b}(G)$, $1 < p < \infty$ *and* $1 \leq q < \infty$, *the function* $s \to L_s f$ *is continuous from* G *into* $L_{p,q;b}(G)$.

Proof. Since the set of simple functions is dense in $L_{p,q;b}(G)$, it is sufficient to show the mapping $s \to L_s f$ is continuous for any simple function f. Let $f = \sum_{i=1}^n k_i \chi_{E_i}$ where χ_E is the characteristic function of E. Then we have $L_s f = \sum_{i=1}^n k_i \chi_{E_i+s}$,

$$\left|\chi_{E_i+s} - \chi_{E_i}\right|(t) = \begin{cases} 1, & t \in (E_i+s) \triangle E_i \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

and

$$\lambda_{\chi_{E_i+s}-\chi_{E_i}}(y) = \left\{ \begin{array}{cc} \mu\left(\left(E_i+s\right) \vartriangle E_i\right), & t<1 \\ 0, & t \geq 1 \end{array} \right. \tag{2.5}$$

where $\vartriangle$ denotes the symmetric difference of sets. Therefore, we write

$$\left(\chi_{E_i+s}-\chi_{E_i}\right)^*(t) = \left\{ \begin{array}{cc} 1, & t<\mu\left(\left(E_i+s\right) \vartriangle E_i\right) \\ 0, & t \geq \mu\left(\left(E_i+s\right) \vartriangle E_i\right) \end{array} \right. \tag{2.6}$$

and

$$\begin{aligned} \left\|\chi_{E_i+s}-\chi_{E_i}\right\|_{(p,q;b)}^q &= \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\left(\chi_{E_i+s}-\chi_{E_i}\right)^*(t)\right\|_{q;(0,\infty)}^q \\ &= \int_0^\infty \left|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\left(\chi_{E_i+s}-\chi_{E_i}\right)^*(t)\right|^q dt \qquad (2.7) \\ &= \int_0^{\mu((E_i+s)\vartriangle E_i)} t^{\frac{q}{p}-1}\gamma_b^q(t)\,dt \\ &= \int_0^{\mu((E_i+s)\vartriangle E_i)} t^{\frac{q}{p}-1}\gamma_{b^q}(t)\,dt \\ &\approx \sup_{0<t<\mu((E_i+s)\vartriangle E_i)} t^{\frac{q}{p}}\gamma_{b^q}(t) \\ &\approx \mu\left(\left(E_i+s\right) \vartriangle E_i\right)^{\frac{q}{p}}\gamma_{b^q}\left(\mu\left(\left(E_i+s\right) \vartriangle E_i\right)\right) \end{aligned}$$

by (1.6). By using the absolutely continuity of the norm, we get $\|L_sf-f\|_{(p,q;b)} \to 0$ as $s \to 0$ and similarly $\|L_sf-f\|_{p,q;b} \to 0$ as $s \to 0$. □

In [4], a convolution operator T is defined and the necessary conditions for convolution of two simple functions are found. By the help of O'Neil Theorem (see [17], pp.133), convolution theorems for Lorentz spaces are established in [4]. Now we will give a proposition for LK spaces by the same method used in there.

Proposition 3. *Let T be a convolution operator defined as in [4, Definition 2.1] and $h = T(f,g)$. T can be uniquely extended so that if $f \in L_{p,q;b}(G)$, $1<p,q<\infty$ and $g \in L^1(G)$, then $h \in L_{p,s;b}(G)$, where $q \leq s$. Moreover $\|h\|_{p,q;b} \precsim \|f\|_{p,q;b}\|g\|_1$.*

Proof. Since $q \leq s$ implies $\|h\|_{p,s;b} \precsim \|h\|_{p,q;b}$, it is sufficient to assume $q = s$. Let f and g be simple functions. By Lemma 2.2 and Theorem

2.4 in [4], we have

$$\begin{aligned} h^{**}(t) &\precsim tf^{**}(t)\,g^{**}(t)+\int_t^{\infty} f^{*}(u)\,g^{*}(u)\,du \\ &\precsim tf^{**}(t)\,g^{**}(t)+f^{**}(t)\int_t^{\infty} g^{*}(u)\,du \\ &= tf^{**}(t)\,\frac{1}{t}\int_0^{t} g^{*}(u)\,du+f^{**}(t)\int_t^{\infty} g^{*}(u)\,du \\ &= f^{**}(t)\int_0^{\infty} g^{*}(u)\,du \end{aligned}$$

and

$$h^{**}(t)\precsim f^{**}(t)\,\|g\|_1 .$$

Thus, we get the result by the definition of $\|\cdot\|_{p,q;b}$ and the density of the simple functions in $L^1(G)$ and LK spaces. □

2.2. **The Space $\mathbf{M}(p,q;b)(\mathbb{R}^d)$.** Let us choose and fix a non zero window $g\in S(\mathbb{R}^d)$, $p\in(1,\infty)$ and $q\in[1,\infty)$. Now, we will define a space $M(p,q;b)(\mathbb{R}^d)$ of tempered distributions by using the Gabor transform with respect to rapidly decreasing functions.

Lemma 4. *Let $g\in S(\mathbb{R}^d)$. Then for any $f\in S'(\mathbb{R}^d)$, the following are equivalent:*
(i) $f\in S(\mathbb{R}^d)$
(ii) $V_g f\in S(\mathbb{R}^{2d})$
(iii) For every $m\in\mathbb{N}$, there is a positive constant c_m such that

$$|V_g f(u,w)|\le c_m(1+|u|+|w|)^{-m} \tag{2.8}$$

for all $u,w\in\mathbb{R}^d$ [12, *Theorem* 11.2.5].

Definition 4. *For fixed window $g\in S(\mathbb{R}^d)$, $M(p,q;b)(\mathbb{R}^d)$ will denote the subpace of tempered distributions $S'(\mathbb{R}^d)$ consisting of $f\in S'(\mathbb{R}^d)$ such that the Gabor transform $V_g(f)$ of f is in $L_{p,q;b}(\mathbb{R}^{2d})$ space. We endow the vector space $M(p,q;b)(\mathbb{R}^d)$ with the norm*

$$\|f\|_{M(p,q;b)}:=\|V_g(f)\|_{p,q;b}=\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\,(V_g(f))^{**}(t)\right\|_{q;(0,\infty)} . \tag{2.9}$$

Although it seems that this definition depends on the choice of the window function, we know from [12] that choosing different windows just give equivalent norms. Therefore, we will use a fixed window function and measure all norms with respect to this window. Before starting to study properties of $M(p,q;b)(\mathbb{R}^d)$ spaces, recall the adjoint operator of V_g. For a given non zero window h and a function F on $\mathbb{R}^{2d}$, it is known that

$$\left\langle V_h^* F, f\right\rangle = \left\langle F, V_h f\right\rangle . \tag{2.10}$$

Lemma 5. *Let $1 < p < \infty$, $1 \leq q < \infty$ and $\gamma_b^q(1) = \gamma_{b^q}(1) = b^q(1) < \infty$. If $f \in L^1\left(\mathbb{R}^d\right)$ and bounded, then $f \in L_{p,q;b}\left(\mathbb{R}^d\right)$.*

Remark 2. *By* [2, *Theorem II*.6.6], *we know that $L^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$ is the smallest rearrangement invariant Banach function space. Therefore we can easily get the result. Nevertheless, we may give an alternative proof similar to* [11].

Proof. Let $1 < p < \infty$, $1 \leq q < \infty$ and $\gamma_b^q(1) = \gamma_{b^q}(1) < \infty$. Then, we have

$$\begin{aligned} \|f\|_{(p,q;b)}^q &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,\infty)}^q \\ &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,1)}^q + \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(1,\infty)}^q . \end{aligned}$$

By using the (right) continuity of f^*, we get

$$\begin{aligned} \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,1)}^q &= \int_0^1 t^{\frac{q}{p}-1} \gamma_b^q(t) \left(f^*(t)\right)^q dt \\ &\leq \sup_{x \in [0,1]} \left(\left(f^*(t)\right)^q\right) \int_0^1 t^{\frac{q}{p}-1} \gamma_{b^q}(t)\, dt. \end{aligned} \tag{2.11}$$

Since $\frac{q}{p} > 0$, using (2.11) and (1.6), we have

$$\begin{aligned} \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,1)}^q &\leq \sup_{x \in [0,1]} \left(\left(f^*(t)\right)^q\right) \int_0^1 t^{\frac{q}{p}-1} \gamma_{b^q}(t)\, dt \\ &\precsim \sup_{x \in [0,1]} \left(\left(f^*(t)\right)^q\right) \sup_{0<t<1} t^{\frac{q}{p}} \gamma_{b^q}(t) \precsim \gamma_{b^q}(1) < \infty. \end{aligned} \tag{2.12}$$

We know that $\left(f^*(t)\right)^q \precsim t^{-q} \|f\|_1^q$ for all $t > 0$ by [15, Lemma 3.6]. Therefore $f \in L^1\left(\mathbb{R}^d\right)$ implies that

$$\begin{aligned} \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(1,\infty)}^q &= \int_1^\infty t^{\frac{q}{p}-1} \gamma_b^q(t) \left(f^*(t)\right)^q dt \\ &\precsim \int_1^\infty t^{\frac{q}{p}-1} \gamma_b^q(t)\, t^{-q} \|f\|_1^q\, dt \\ &= \|f\|_1^q \int_1^\infty t^{\frac{q}{p}-q-1} \gamma_{b^q}(t)\, dt. \end{aligned} \tag{2.13}$$

Since $p > 1$ causes $\frac{q}{p} - q < 0$, using (2.13) and (1.7), we get

$$\begin{aligned} \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(1,\infty)}^q &\precsim \|f\|_1^q \int_1^\infty t^{\frac{q}{p}-q-1} \gamma_{b^q}(t)\, dt \\ &\approx \|f\|_1^q \sup_{1<t<\infty} t^{\frac{q}{p}-q} \gamma_{b^q}(t) \precsim \gamma_{b^q}(1) < \infty. \end{aligned} \tag{2.14}$$

So, using (2.12) and (2.14) we see that $\|f\|_{(p,q;b)} < \infty$ and so $\|f\|_{p,q;b} < \infty$. $\square$

Proposition 4. *Let $1 < p < \infty$, $1 \le q < \infty$ and $g \in S(\mathbb{R}^d)$. Then $S(\mathbb{R}^d)$ is dense in $M(p,q;b)(\mathbb{R}^d)$.*

Proof. Let us take any $f \in S(\mathbb{R}^d)$. Since $f \in S(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$, we have $V_g(f) \in S(\mathbb{R}^{2d})$ and

$$\sup_{z\in\mathbb{R}^{2d}} \left\{ (1+|z|)^m V_g f(z) \right\} < \infty \tag{2.15}$$

by Lemma 4. Also, by (2.8)

$$\begin{aligned} \|f\|_{M(p,q;b)} &= \|V_g(f)\|_{p,q;b} = \left\| (1+|z|)^m (1+|z|)^{-m} V_g(f) \right\|_{p,q;b} \\ &\le \sup_{z\in\mathbb{R}^{2d}} \left\{ (1+|z|)^m V_g f(z) \right\} \left\| (1+|z|)^{-m} \right\|_{p,q;b} \end{aligned} \tag{2.16}$$

can be found. Since the right hand side of (2.16) is finite for sufficiently large m by (2.1), (2.2) and Lemma 5, it is obvious that $\|f\|_{M(p,q;b)} < \infty$ and $S(\mathbb{R}^d) \subset M(p,q;b)(\mathbb{R}^d)$.

For the density part of the proof, one can use the techniques mutatis mutandis applied in [12, Proposition 11.3.4]. $\square$

Theorem 1. *Let $g_1,\ g_2 \in S(\mathbb{R}^d)$ be two non zero windows and $1 < p < \infty$, $1 \le q < \infty$. Then*

(i) $V_{g_1}^*$ is a map from $L_{p,q;b}(\mathbb{R}^{2d})$ into $M(p,q;b)(\mathbb{R}^d)$ with

$$\left\| V_{g_1}^* F \right\|_{M(p,q;b)} \le \|V_{g_2} g_1\|_1 \|F\|_{p,q;b}.$$

(ii) The inversion formula

$$f = \frac{1}{\langle g_1, g_2 \rangle} \iint_{\mathbb{R}^{2d}} V_{g_2}(f)(x,w)\, M_w L_x g_1\, dx dw$$

holds in $M(p,q;b)(\mathbb{R}^d)$.

Proof. (i) We must first show that $V_{g_1}^* F$ is a tempered distribution. We can easily see that $V_{g_1}(f) \in L_{p',q';b^{-1}}\left(\mathbb{R}^{2d}\right)$ for all $f \in S\left(\mathbb{R}^d\right)$ by definition of $L_{p',q';b^{-1}}\left(\mathbb{R}^{2d}\right)$ where $\frac{1}{p}+\frac{1}{p'}=1=\frac{1}{q}+\frac{1}{q'}$. We also have

$$\begin{aligned}\left|\left\langle V_{g_1}^* F, f\right\rangle\right| &= \left|\left\langle F, V_{g_1} f\right\rangle\right| = \left|\iint_{\mathbb{R}^{2d}} F(x,w)\,\overline{V_{g_1}(f)(x,w)}\,dxdw\right| \\ &\le \|F\|_{p,q;b}\,\|V_{g_1} f\|_{p',q';b^{-1}} \qquad (2.17)\end{aligned}$$

for all $f \in S\left(\mathbb{R}^d\right)$ with Hölder's inequality for LK spaces. By (2.17), we get

$$\begin{aligned}\left|\left\langle V_{g_1}^* F, f\right\rangle\right| &\le \|F\|_{p,q;b}\,\|V_{g_1} f\|_{p',q';b^{-1}} \\ &\le \|F\|_{p,q;b} \sup_{z\in\mathbb{R}^{2d}} \left\{(1+|z|)^m V_{g_1} f(z)\right\} \left\|(1+|z|)^{-m}\right\|_{p',q';b^{-1}}. \qquad (2.18)\end{aligned}$$

Since the right hand side of (2.18) is finite for sufficiently large m by Lemma 5 and [12, Corollary 11.2.6], we see that $V_{g_1}^* F \in S'\left(\mathbb{R}^d\right)$. So it possesses Gabor transform such that

$$\begin{aligned}V_{g_2} V_{g_1}^* F(u,v) &= \left\langle V_{g_1}^* F, M_v L_u g_2\right\rangle = \left\langle F, V_{g_1}(M_v L_u g_2)\right\rangle \\ &= \iint_{\mathbb{R}^{2d}} F(x,w)\, V_{g_2}(g_1)(u-x, v-w)\, e^{-2\pi i x(v-w)}\,dxdw. \qquad (2.19)\end{aligned}$$

By (2.19), we obtain

$$\left|V_{g_2} V_{g_1}^* F(u,v)\right| \le \left(|F| * |V_{g_2}(g_1)|\right)(u,v)$$

and

$$\begin{aligned}\left\|V_{g_1}^* F\right\|_{M(p,q;b)} &= \left\|V_{g_2} V_{g_1}^* F\right\|_{p,q;b} \le \left\||F| * |V_{g_2}(g_1)|\right\|_{p,q;b} \\ &\le \|F\|_{p,q;b}\,\|V_{g_2}(g_1)\|_1\end{aligned}$$

since $V_{g_2}(g_1) \in S\left(\mathbb{R}^{2d}\right) \subset L^1\left(\mathbb{R}^{2d}\right)$ and $L_{p,q;b}\left(\mathbb{R}^{2d}\right)$ is $L^1\left(\mathbb{R}^{2d}\right)$ −module.

(ii) From the definition of $M(p,q;b)\left(\mathbb{R}^d\right)$, we see the result by Theorem 11.2.3 and Corollary 11.2.7 in [12]. □

We know that $L_{p,q;b}\left(\mathbb{R}^{2d}\right)$ is a rearrangement invariant Banach space for $1<p,q<\infty$ and it is easy to show that it is a solid translation invariant function space. Therefore $M(p,q;b)\left(\mathbb{R}^d\right)$ is a coorbit space and a Banach space where b is a s.v. function and $1<p,q<\infty$. Also $M(p,q;b)\left(\mathbb{R}^d\right)$ spaces are strongly invariant under time-frequency shifts, i.e. $\|L_x M_w f\|_{M(p,q;b)} = \|f\|_{M(p,q;b)}$.

Proposition 5. *The mapping $s \to L_s f$ is continuous from $\mathbb{R}^d$ into $M(p,q;b)\left(\mathbb{R}^d\right)$ for all $f \in M(p,q;b)\left(\mathbb{R}^d\right)$, $1 < p < \infty$ and $1 \leq q < \infty$.*

Proof. Let $1 < p < \infty$, $1 \leq q < \infty$ and $f \in M(p,q;b)\left(\mathbb{R}^d\right)$. Then, we have
(2.20)

$$\begin{aligned}
\|L_s f - f\|_{M(p,q;b)} &= \|V_g(L_s f - f)\|_{p,q;b} \\
&= \|V_g(L_s f) - V_g f\|_{p,q;b} = \left\|e^{-2\pi wsi} L_{(s,0)}(V_g f) - V_g f\right\|_{p,q;b} \\
&\leq \left\|e^{-2\pi wsi} L_{(s,0)}(V_g f) - e^{-2\pi wsi} V_g f\right\|_{p,q;b} + \left\|e^{-2\pi wsi} V_g f - V_g f\right\|_{p,q;b} \\
&= \left\|e^{-2\pi wsi}\left(L_{(s,0)} V_g f - V_g f\right)\right\|_{p,q;b} + \left\|\left(e^{-2\pi wsi} - 1\right) V_g f\right\|_{p,q;b} \\
&= \left\|L_{(s,0)}(V_g f) - V_g f\right\|_{p,q;b} + \left\|\left(e^{-2\pi wsi} - 1\right) V_g f\right\|_{p,q;b}
\end{aligned}$$

for all $g \in S\left(\mathbb{R}^d\right)$. Now, if we let $h_s(x,w) = \left|e^{-2\pi wsi} - 1\right| \left|V_g f(x,w)\right|$, then it is easily seen that $h_s(x,w) \to 0$ as $s \to 0$ for all $(x,w) \in \mathbb{R}^{2d}$. Therefore, we can see that the rearrangement of $\left(e^{-2\pi wsi} - 1\right) V_g f(x,w)$ also converges to zero as $s \to 0$. Since $h_s(x,w) \leq 2\left|V_g f(x,w)\right|$ and $V_g f \in L_{p,q;b}\left(\mathbb{R}^{2d}\right)$, we have $h_s^* \leq \left(2\left|V_g f\right|\right)^*$ and $\|h_s\|_{p,q;b} = \left\|\left(e^{-2\pi wsi} - 1\right) V_g f\right\|_{p,q;b} \to 0$ as $s \to 0$ by Lebesgue dominated convergence theorem. On the other hand, by proposition 2, we can deduce that the mapping $s \to L_s(V_g f)$ is continuous for all $f \in L_{p,q;b}\left(\mathbb{R}^{2d}\right)$ and $s \in \mathbb{R}^{2d}$. Hence, we get the result with the last two assertions. □

Theorem 2. *$M(p,q;b)\left(\mathbb{R}^d\right)$ is an essential Banach $L^1\left(\mathbb{R}^d\right)$ −module.*

Proof. Let $h \in M(p,q;b)\left(\mathbb{R}^d\right)$ and $f \in L^1\left(\mathbb{R}^d\right)$. It is known by [12, Lemma 3.1.1] that

$$V_g(f * h)(x,w) = e^{-2\pi ixw}(f * h) * M_w \widetilde{g} \tag{2.21}$$

where $M_w\widetilde{g}(x)=e^{2\pi ixw}\overline{g}(x)$. Therefore, if we use the strongly translation invariance property of LK spaces and (2.21) together, we have

$$\begin{aligned}
\|f*h\|_{M(p,q;b)} &= \|V_g(f*h)\|_{p,q;b}=\|f*h*M_w\widetilde{g}\|_{p,q;b}\\
&= \left\|\int_{\mathbb{R}^d} f(t)\left(h*M_w\widetilde{g}\right)(x-t)\,dt\right\|_{p,q;b}\\
&\leq \int_{\mathbb{R}^d}|f(t)|\,\|L_t\left(h*M_w\widetilde{g}\right)(x)\|_{p,q;b}\,dt\\
&= \int_{\mathbb{R}^d}|f(t)|\,\|\left(h*M_w\widetilde{g}\right)(x)\|_{p,q;b}\,dt\\
&= \|\left(h*M_w\widetilde{g}\right)(x)\|_{p,q;b}\int_{\mathbb{R}^d}|f(t)|\,dt\\
&= \|h\|_{p,q;b}\,\|f\|_1\,.
\end{aligned}$$

For the essentiality part, take any $h\in M(p,q;b)\left(\mathbb{R}^d\right)$. We know by Proposition 5 that the mapping $s\to L_sh$ is continuous of $\mathbb{R}^d$ into $M(p,q;b)\left(\mathbb{R}^d\right)$ for all $h\in M(p,q;b)\left(\mathbb{R}^d\right)$. Therefore for any $\varepsilon>0$, there exits a compact neighborhood U of $0\in\mathbb{R}^d$ such that $\|L_sh-h\|_{M(p,q;b)}<\varepsilon$ for all $s\in U$. Now let $g\in L^1\left(\mathbb{R}^d\right)$ be a positive and continuous function with compact support such that supp$g\subset U$ and $\int_{\mathbb{R}^d}g(x)\,dx=1$. Then, we get

$$\begin{aligned}
\|g*h-h\|_{M(p,q;b)} &= \left\|\int_{\mathbb{R}^d} g(z)\,h(u-z)\,dz-\right.\\
&\qquad\left.-\int_{\mathbb{R}^d} h(u)\,g(z)\,dz\right\|_{M(p,q;b)}\\
&\leq \int_{\mathbb{R}^d} g(z)\,\|L_zh-h\|_{M(p,q;b)}\,dz\\
&\leq \varepsilon\int_{\mathbb{R}^d} g(z)\,dz=\varepsilon.
\end{aligned}$$

Since $M(p,q;b)\left(\mathbb{R}^d\right)$ is a Banach $L^1\left(\mathbb{R}^d\right)-$module and $L^1\left(\mathbb{R}^d\right)*M(p,q;b)\left(\mathbb{R}^d\right)\subset M(p,q;b)\left(\mathbb{R}^d\right)$, we say that $L^1\left(\mathbb{R}^d\right)*M(p,q;b)\left(\mathbb{R}^d\right)$ is dense in $M(p,q;b)\left(\mathbb{R}^d\right)$. Hence, $M(p,q;b)\left(\mathbb{R}^d\right)$ is an essential Banach $L^1\left(\mathbb{R}^d\right)-$module by Module Factorization Theorem. □

Theorem 3. *Let $p,q\in(1,\infty)$ and b be s.v. function. The associate space of $M(p,q;b)\left(\mathbb{R}^d\right)$ is $M\left(p',q';b^{-1}\right)\left(\mathbb{R}^d\right)$ with the norm*

$\|\cdot\|_{M(p',q';b^{-1})}$ *where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *In other words, these spaces are reflexive.*

Proof. Let $h \in M(p',q';b^{-1})(\mathbb{R}^d)$. It is known that $L_{p,q;b}(\mathbb{R}^{2d})$ is a Banach space with associate space $L_{p',q';b^{-1}}(\mathbb{R}^{2d})$ and both of them have absolutely continuous norms where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. It is known (generally) that the dual form is

$$\langle u,v\rangle = \int_{\mathbb{R}^{2d}} u(x)\, v(x)\, dx \tag{2.22}$$

where $u \in L_{p,q;b}(\mathbb{R}^{2d})$ and $v \in L_{p',q';b^{-1}}(\mathbb{R}^{2d})$. If we define a linear functional on $M(p,q;b)(\mathbb{R}^d)$ with $h \in M(p',q';b^{-1})(\mathbb{R}^d)$, then

$$H_h(f) = \int_{\mathbb{R}^{2d}} V_g f(x)\, V_g h(x)\, dx \tag{2.23}$$

can be written. Also, if we use Hölder's inequality and [9, Theorem 3.2.10], then we get

$$|H_h(f)| \le \|V_g h\|_{p',q';b^{-1}} \|V_g f\|_{p,q;b}$$

for all $f \in M(p,q;b)(\mathbb{R}^d)$. Hence the linear functional H_h is bounded.

Conversely, let us take any $H \in \left(M(p,q;b)(\mathbb{R}^d)\right)^*$. By proposition 4, it is easy to see that $M(p,q;b)(\mathbb{R}^d)$ is isometrically isomorphic to the closed subspace

$$N = \left\{V_g f \in L_{p,q;b}(\mathbb{R}^{2d}) : f \in M(p,q;b)(\mathbb{R}^d)\right\} \tag{2.24}$$

of $L_{p,q;b}(\mathbb{R}^{2d})$. Hence $\widetilde{H}(V_g f) := H(f)$ defines a continuous linear functional on N and $\widetilde{H}$ can be extended continuously to $L_{p,q;b}(\mathbb{R}^{2d})$. Thus by [9, Theorem 3.4.41], there exists $k \in L_{p',q';b^{-1}}(\mathbb{R}^{2d})$ such that

$$\widetilde{H}(V_g f) = \int_{\mathbb{R}^{2d}} V_g f(x)\, k(x)\, dx = H(f). \tag{2.25}$$

On the other hand, since $k \in L_{p',q';b^{-1}}(\mathbb{R}^{2d})$, we can find $K \in M(p',q';b^{-1})(\mathbb{R}^d)$ by Theorem 1 such that $K = V_g^* k$. As a result, every continuous linear functional on $M(p,q;b)(\mathbb{R}^d)$ is of the form (2.25) and $\left(M(p,q;b)(\mathbb{R}^d)\right)^* = M(p',q';b^{-1})(\mathbb{R}^d)$. □

Lastly, we are going to give a theorem without its proof. It can be proved by the techniques used in [5].

Theorem 4. *Let* $g \in S(\mathbb{R}^d)$ *and* $p,q \in (1,\infty)$. *Then the multipliers space* $M\left(L^1(\mathbb{R}^d), M(p,q;b)(\mathbb{R}^d)\right)$ *is isometrically isomorphic to* $M(p,q;b)(\mathbb{R}^d)$.

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