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A COMMON FIXED POINT THEOREM FOR SET-VALUED MAPPINGS USING δ-DISTANCE IN COMPLETE METRIC SPACES

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Abstract. A common fixed point theorem for set-valued mappings on a complete metric space is established using δ-distance function.

1. Preliminaries

We let (X, d) be a complete metric space and let $B(X)$ be the set of all nonempty bounded subsets of X. As in [1, 2, 4, 5], we define the function $\delta(A, B)$ with A and B in $B(X)$ by $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$ and if B also consists of single point b, we write $\delta(A, B) = \delta(a, b) = d(a, b)$. It follows immediately that $\delta(A, B) = \delta(B, A) \geq 0$, and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all A, B and $C \in B(X)$. If $\delta(A, B) = 0$, then $A = B = \{a\}$.

Now if $\{A_n : n = 1, 2, \ldots\}$ is a sequence of sets in $B(X)$, we say that it converges to the set A in $B(X)$ if

(i) each point $a \in A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \ldots\}$,

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

The set A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved in [1].

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Lemma 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of nonempty, bounded subsets of a complete metric space (X,d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Now, let T be a mapping of X into $B(X)$. We say that the mapping T is continuous at a point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx in $B(X)$. We say that T is continuous mapping of X into $B(X)$ if T is continuous at each point x in X. We say that a point z in X is a fixed point of T if z is in Tz. If A is in $B(X)$, we define the set $TA = \bigcup_{a\in A} Ta$. If S is a second mapping of X into $B(X)$, we define the composition $(ST)(x) = \bigcup_{y\in T(x)} S(y)$.

2. Main results

We now prove the following theorem.

Theorem 2.1. *Let S and T be mappings of a complete metric space (X,d) into $B(X)$ satisfying the following inequalities*

$$\delta(TSx,Sx) \leq \varphi(\delta(Sx,x)), \tag{1}$$
$$\delta(STx,Tx) \leq \varphi(\delta(Tx,x)) \tag{2}$$

for all x in X, where $\varphi:[0,\infty)\to[0,\infty)$ is a nondecreasing function with $\sum_{n=1}^{\infty}\varphi^n(t)<\infty$ for all $t>0$. If S or T is continuous, then S and T have a common fixed point z. Further, $Tz=Sz=\{z\}$.

Proof. Let x_0 be an arbitrary point in X. Define sequence $\{x_n\}$ in X as follows. Choose a point x_1 in Tx_0 and then a point x_2 in Sx_1. In general, having chosen x_n in X choose x_{2n+1} in Tx_{2n} and then x_{2n+2} in Sx_{2n+1} for $n=0,1,\dots$.

Then using inequalities (1) and (2), we have

$$\begin{aligned} d(x_{2n+1},x_{2n}) &\leq \delta(Tx_{2n},x_{2n}) \\ &\leq \delta(TSx_{2n-1},Sx_{2n-1}) \\ &\leq \varphi(\delta(Sx_{2n-1},x_{2n-1})) \\ &\leq \varphi(\delta(STx_{2n-2},Tx_{2n-2})) \\ &\leq \varphi^2(\delta(Tx_{2n-2},x_{2n-2})) \\ &\quad\dots \\ &\leq \varphi^{2n}(\delta(Tx_0,x_0)) \end{aligned} \tag{3}$$

and, similarly,

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &\leq \delta(Sx_{2n+1}, x_{2n+1}) \\ &\leq \delta(STx_{2n}, Tx_{2n}) \\ &\leq \varphi(\delta(Tx_{2n}, x_{2n})) \\ &\leq \varphi(\delta(TSx_{2n-1}, Sx_{2n-1})) \\ &\leq \varphi^2(\delta(Sx_{2n-1}, x_{2n-1})) \\ &\leq \varphi^2(\delta(STx_{2n-2}, Tx_{2n-2})) \\ &\leq \varphi^3(\delta(Tx_{2n-2}, x_{2n-2})) \\ &\quad \ldots \\ &\leq \varphi^{2n+1}(\delta(Tx_0, x_0)). \end{aligned} \tag{4}$$

Then from inequalities (3) and (4), we obtain

$$d(x_{n+1}, x_n) \leq \varphi^n(\delta(Tx_0, x_0)) \tag{5}$$

for $n = 0, 1, \ldots$.

Now let $m > n$. Then from inequality (5), we have

$$\begin{aligned} d(x_n, x_m) \;\; &\leq \;\; d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \varphi^n(\delta(Tx_0, x_0)) + \varphi^{n+1}(\delta(Tx_0, x_0)) + \ldots + \varphi^{m-1}(\delta(Tx_0, x_0)) \\ &= \sum_{k=n}^{m-1} \varphi^k(\delta(Tx_0, x_0)) \leq \sum_{k=n}^{\infty} \varphi^k(\delta(Tx_0, x_0)). \end{aligned}$$

Take any $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, we can choose a sufficiently large natural number N such that

$$d(x_n, x_m) \leq \sum_{k=N}^{\infty} \varphi^k(\delta(Tx_0, x_0)) < \varepsilon,$$

for all $m > n \geq N$. It follows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Now we suppose that the mapping T is continuous. Then the sequence $\{Tx_{2n}\}$ in $B(X)$ converges to Tz in $B(X)$. Using inequality (3), we now have

$$\begin{aligned} \delta(z, Tx_{2n}) &\leq d(z, x_{2n}) + \delta(x_{2n}, Tx_{2n}) \\ &\leq d(z, x_{2n}) + \varphi^{2n}(\delta(Tx_0, x_0)). \end{aligned}$$

Letting n tends to infinity, we have

$$\lim_{n \to \infty} \delta(z, Tx_{2n}) = 0.$$

Further, using Lemma 1.1, we have $Tz = \{z\}$. Then from inequality (2), we have

$$\delta(Sz, z) \leq \delta(STz, Tz) \leq \varphi(\delta(Tz, z)) = \varphi(0) = 0$$

which implies

$$Sz = Tz = \{z\}.$$

The same result of course holds if S is continuous instead of T.

Putting $T = S$ in Theorem 2.1, then we get the following corollary;

Corollary 2.2. *Let T be mapping of a complete metric space (X, d) into $B(X)$ satisfying the following inequality*

$$\delta(T^2x, Tx) \leq \varphi(\delta(Tx, x))$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If T is continuous, then T has a fixed point z. Further, $Tz = \{z\}$.

If we let S and T be single valued self -mappings of X, we obtain the following result given in [3].

Corollary 2.3. *Let (X, d) be a complete metric space and let S and T be self-mappings of X satisfying the following inequalities*

$$\begin{aligned} d(TSx, Sx) &\leq \varphi(d(Sx, x)), \\ d(STx, Tx) &\leq \varphi(d(Tx, x)) \end{aligned}$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If S or T is continuous, then S and T have a common fixed point.

Example 2.4. Let $X = [0, 1]$ with usual metric d and let

$$Tx = [0, x/2], \quad Sx = \begin{cases} [0, x/4] & \text{if } x < 1/2 \\ \{0\} & \text{if } x \geq 1/2 \end{cases}.$$

Then we have,

$$\delta(TSx, Sx) = 0, \delta(STx, Tx) = x/2, \delta(Sx, x) = x \text{ and } \delta(Tx, x) = x$$

for all $x \geq 1/2$.

For all $x < 1/2$, we have also

$$\delta(TSx, Sx) = x/4, \delta(STx, Tx) = x/2, \delta(Sx, x) = x \text{ and } \delta(Tx, x) = x.$$

Then, assuming $\varphi(t) = t/2$ for all $t \geq 0$, all the conditions of Theorem 2.1 are satisfied. Also $T0 = S0 = \{0\}$.

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