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INGARDEN SPACES WITH A SPECIAL NONLINEAR CONNECTION

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Abstract. In this paper we consider a new nonlinear connection constructed from N, a given Lorentz nonlinear connection, and we obtain a condition for the Ingarden space to be a space of scalar curvature.

1. Introduction

Let M be an n-dimensional, real C^{∞} manifold. Denote by (TM, τ, M) the tangent bundle of M and let $F^n = (M, F(x,y))$ be a Finsler space, where $F : TM \to R_+$ is it fundamental function, i.e., F verifies the following axioms:

i) F is a differentiable function on $\tilde{TM} = TM - \{0\}$ and it is continuous on the null section of the projection $\tau : TM \to M$;
ii) F is positively 1- homogeneous with respect to the variables y^i;
iii) $(\forall)(x,y) \in \tilde{TM}$ the Hessian of F^2 with respect y^i is positive defined. Consequently, the d-tensor field $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive defined. It is called the fundamental tensor or metric tensor of F^n .

In 1941 G.Randers first introduced a special fundamental function $F(x,y) = \alpha(x,y) + \beta(x,y)$ where $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i y^j}$ is a Riemannian or pseudo-Riemannian metric and $\beta(x,y) = b_i(x)\, y^i$ is a 1-form. This metric was called a Randers metric by R.S.Ingarden (1957) who used it to study a problem of electron microscope.

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Actually, the Randers spaces are considered as the Finsler space $F^n = (M, F(x,y)) = (M, \alpha(x,y) + \beta(x,y))$ equipped with the Cartan nonlinear connection $\overset{C}{N}$. So, the pair $RF^n = \left(F^n, \overset{C}{N}\right)$ is called the Randers space. Instead of the Cartan nonlinear connection, R.Miron introduced in [5] the Lorentz nonlinear connection N determined by the Lorentz equations of the space F^n. The local coefficients of N are $N^i_j = \gamma^i_{jk} y^k - F^i_j$, where γ^i_{jk} are the Christoffel symbols of the Riemannian structure $a = a_{ij}(x)\, dx^i \otimes dx^j$ and $F^i_j(x) = a^{is} F_{sj}$, $F_{sj} = \frac{\partial b_s}{\partial x^j} - \frac{\partial b_j}{\partial x^s}$.

The Finsler space $F^n = (M, F(x,y)) = (M, \alpha(x,y) + \beta(x,y))$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted by $IF^n = (F^n, N)$.

In this paper we consider a new nonlinear connection constructed from N a given Lorentz nonlinear connection and we obtain conditions for Ingarden space to be of scalar curvature based on this new connection.

2. Lorentz nonlinear connection. Ingarden spaces

Let $F^n = (M, F(x,y))$ be a Finsler space with the fundamental function $F(x,y) = \alpha(x,y) + \beta(x,y)$ where $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i y^j}$ and $\beta(x,y) = b_i(x)\, y^i$. $a = a_{ij}(x)\, dx^i dx^j$ is a pseudo-Riemannian metric on M and $b_i(x)$ is a covector field on the manifold M . We consider the integral of action of the energy $F^2(x,y)$ along a curve $c : t \in [0,1] \to c(t) \in M$:

(2.1) $I(c) = \int_0^1 F^2\left(x, \frac{dx}{dt}\right) dt = \int_0^1 \left[\alpha\left(x, \frac{dx}{dt}\right) + \beta\left(x, \frac{dx}{dt}\right)\right]^2 dt$

The variational problem for $I(c)$ leads to the Euler-Lagrange equations:

(2.2) $E_i(F^2) := \frac{\partial(\alpha+\beta)^2}{\partial x^i} - \frac{d}{dt}\frac{\partial(\alpha+\beta)^2}{\partial y^i} = 0, y^i = \frac{dx^i}{dt}$.

The energy of F^2 is

(2.3) $\varepsilon_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2$

The covector field $E_i(F^2)$ is expressed by

(2.4) $E_i(F^2) = E_i(\alpha^2) + 2\alpha E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i}$.

Let us fix a parametrization of the curve c, by natural parameter s with respect to Riemannian metric $\alpha(x,y)$. It is given by

(2.5) $ds^2 = \alpha^2\left(x, \frac{dx}{dt}\right) dt^2$.

It follows $F^2\left(x, \frac{dx}{ds}\right) = 1$ and $\frac{d\alpha}{ds} = 0$.

Along to an extremal curve c, canonical parametrized by (2.5), $E_i(\beta)$ is expressed by

(2.6) $E_i(\beta) = \left(\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}\right)\frac{dx^j}{ds} = F_{ij}(x)\frac{dx^j}{ds}$.

One obtains [4]:

Theorem 2.1.(Miron-Hassan) *In the canonical parametrization, the Euler-Lagrange equations of the Lagrangian* $(\alpha+\beta)^2$ *are given by*

(2.7) $E_i(\alpha^2) + 2F_{ij}(x)y^j = 0, y^i = \frac{dx^i}{ds}$.

Theorem 2.2. *The Euler-Lagrange equations (2.7) are equivalent to the Lorentz equations:*

(2.8) $\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = \overset{\circ}{F}{}^i_j(x)\frac{dx^j}{ds}$,

where $\overset{\circ}{F}{}^i_j(x) = a^{is}F_{sj}(x)$ *and* γ^i_{jk} *are the Christoffel symbols of the Riemannian metric tensor* $a_{ij}(x)$.

The Euler-Lagrange equations $E_i(F^2) = 0$ determines a canonical semispray or a Dynamical System S on the total space of the tangent bundle :

(2.9) $S = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$

where the coefficients $G^i(x,y)$ are:

(2.10) $2G^i(x,y) = \gamma^i_{jk}(x)y^jy^k - F^i_j(x)y^j$.

Now we can consider the nonlinear connection N with the coefficients $N^i_j = \frac{\partial G^i}{\partial y^j}$. Of course, we have

(2.11) $N^i_j = \gamma^i_{jk}(x)y^k - F^i_j(x)$,

where $F^i_j(x) = \frac{1}{2}\overset{\circ}{F}{}^i_j(x)$.

Since the autoparallel curves of N are given by the Lorentz equations (2.8), we call it the Lorentz nonlinear connection of the Randers metric $\alpha+\beta$.

The nonlinear connection N determines the horizontal distribution, denoted by N too, with the property $T_uTM = N_u \oplus V_u$, $\forall u \in TM$, where V_u being the natural vertical distribution on the tangent manifold TM.

The local adapted basis to the horizontal and vertical vector spaces N_u and V_u is given by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right), i = 1,...,n$, where

(2.12) $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i\frac{\partial}{\partial y^k} = \frac{\partial}{\partial x^i} - \gamma^k_{is}(x)y^s\frac{\partial}{\partial y^k} + F^k_i\frac{\partial}{\partial y^k} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F^k_i\frac{\partial}{\partial y^k}$,

where

(2.13) $\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \gamma^k_{is}(x)y^s\frac{\partial}{\partial y^k}$.

The adapted cobasis to N is $(dx^i, \delta y^i), i = 1,...,n$ with

(2.14) $\delta y^i = dy^i + N^i_jdx^j = dy^i + \gamma^i_{jk}(x)y^kdx^j - F^i_jdx^j = \overset{\circ}{\delta}y^i - F^i_jdx^j$,

where

(2.15) $\overset{\circ}{\delta} y^i = dy^i + \gamma^i_{jk}(x)\, y^k dx^j.$

The weakly torsion of N is

(2.16) $T^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j} = 0.$

The integrability tensor of N is

(2.17) $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}.$

Definition 2.1. *The Finsler space $F^n = (M, F = \alpha + \beta)$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted IF^n.*

The fundamental tensor g_{ij} of IF^n is given by

(2.18) $g_{ij} = \frac{F}{\alpha}(a_{ij} - \tilde{l}_i \tilde{l}_j) + l_i l_j$

where $\tilde{l}_i = \frac{\partial \alpha}{\partial y^i}$, $l_i = \frac{\partial F}{\partial y^i}$, $l_i = \tilde{l}_i + b_i$.

The following results holds [4].

Theorem 2.3. *There exists an unique N-metrical connection $I\Gamma(N) = \left(F^i_{jk}, C^i_{jk}\right)$ of the Ingarden space IF^n which verifies the following axioms:*

i) $\nabla^H_k g_{ij} = 0$; $\nabla^V_k g_{ij} = 0$;

ii) $T^i_{jk} = 0$; $S^i_{jk} = 0$.

The connection $I\Gamma(N)$ has the coefficients expressed by the generalized Christoffel symbols:

(2.19) $$\begin{cases} F^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right) \end{cases}$$

where $\frac{\delta}{\delta x^i}$ are given by (2.12).

3. A special nonlinear connection N^*

Let IF^n be an Ingarden space and N the Lorenz nonlinear connection with the coefficients given by (2.11). Instead of N we now consider a new nonlinear connection $\overset{*}{N}$ [8] with the coefficients

(3.1) $\overset{*}{N}{}^i_j = N^i_j + \frac{F_{|j} y^i}{F},$

where "$|$" denote the covariant differentiation with respect to $I\Gamma(N)$.

The nonlinear connection $\overset{*}{N}$ determines the horizontal distribution, denoted by $\overset{*}{N}$ too, with the property $T_u TM = \overset{*}{N}_u \oplus V_u$, $\forall u \in TM$, V_u being the natural vertical distribution on the tangent manifold TM.

The local adapted basis to the horizontal and vertical vector spaces $\overset{*}{N}_u$ and V_u is given by $\left(\frac{\overset{*}{\delta}}{\delta x^k}, \frac{\partial}{\partial y^k} \right)$, $k = 1, ..., n$, where

(3.2)

$$\begin{aligned}\frac{\overset{*}{\delta}}{\delta x^k} &= \frac{\partial}{\partial x^k} - \overset{*}{N}{}^r_k \frac{\partial}{\partial y^r} = \frac{\partial}{\partial x^k} - N^r_k \frac{\partial}{\partial y^r} - \frac{F_{|k} y^r}{F}\frac{\partial}{\partial y^r} \\ &= \frac{\delta}{\delta x^k} - \frac{F_{|k} y^r}{F}\frac{\partial}{\partial y^r} = \frac{\overset{\circ}{\delta}}{\delta x^k} + F^r_k \frac{\partial}{\partial y^r} - \frac{F_{|k} y^r}{F}\frac{\partial}{\partial y^r} \\ &= \frac{\overset{\circ}{\delta}}{\delta x^k} + \left(F^r_k - \frac{F_{|k} y^r}{F}\right)\frac{\partial}{\partial y^r}\end{aligned}$$

and $\frac{\overset{\circ}{\delta}}{\delta x^k}$ are given by (2.13).

The adapted cobasis to N is $\left(dx^i, \overset{*}{\delta} y^i\right)$, $i = 1, ..., n$ with

(3.3)

$$\begin{aligned}\delta y^i &= dy^i + \overset{*}{N}{}^i_j dx^j = dy^i + N^i_j dx^j + \frac{F_{|j} y^i}{F} dx^j \\ &= dy^i + \gamma^i_{jk}(x)\, y^k dx^j - F^i_j dx^j + \frac{F_{|j} y^i}{F} dx^j \\ &= \overset{\circ}{\delta} y^i - \left(F^i_j - \frac{F_{|j} y^i}{F}\right) dx^j\end{aligned}$$

where $\overset{\circ}{\delta} y^i$are given by (2.15).

Theorem 3.1. *There exists an unique $\overset{*}{N}$- metrical connection $I\overset{*}{\Gamma}\left(\overset{*}{N}\right) = \left(\overset{*}{F}{}^i_{jk}, \overset{*}{C}{}^i_{jk}\right)$ of the Ingarden space IF^n which satisfies the following axioms:*

i) $\overset{*}{\nabla}{}^H_k g_{ij} = 0$; $\overset{*}{\nabla}{}^V_k g_{ij} = 0$;

ii) $\overset{*}{T}{}^i_{jk} = 0$; $\overset{*}{S}{}^i_{jk} = 0$.

The connection $I\overset{*}{\Gamma}\left(\overset{*}{N}\right)$ has the coefficients expressed by the generalized Christoffel symbols

$$(3.4)\quad \begin{cases} \overset{*}{F}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\overset{*}{\delta} g_{sj}}{\delta x^k} + \frac{\overset{*}{\delta} g_{sk}}{\delta x^j} - \frac{\overset{*}{\delta} g_{jk}}{\delta x^s}\right) \\ \overset{*}{C}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{cases}$$

From a direct calculus, using (3.2) and $\frac{\partial g_{ik}}{\partial y^j} y^j = 0$, we get

(3.5) $\overset{*}{F}{}^i_{jk} = F^i_{jk}$.

The hv-torsion $\overset{*}{P}{}^i_{jk}$ is
(3.6)

$$\begin{aligned}\overset{*}{P}{}^i_{jk} &= \frac{\partial \overset{*}{N}{}^i_j}{\partial y^k} - \overset{*}{F}{}^i_{jk} = \frac{\partial N^i_j}{\partial y^k} + \frac{\partial}{\partial y^k}\left(\frac{F_{|j} y^i}{F}\right) - F^i_{jk} \\ &= P^i_{jk} + \frac{\partial}{\partial y^k}\left(F_{|j}\right) l^i + \frac{F_{|j}}{F} - \frac{F_{|j}}{F}\frac{F_{|k}}{F} y^i\end{aligned}$$

The vh-torsion $\overset{*}{R}{}^i_{jk}$ of $I\overset{*}{\Gamma}\left(\overset{*}{N}\right)$ is
(3.7) $\overset{*}{R}{}^{\;i}_{jk} = \frac{\overset{*}{\delta}\overset{*}{N}{}^{\;i}_j}{\delta x^k} - \frac{\overset{*}{\delta}\overset{*}{N}{}^{\;i}_k}{\delta x^j} = R^i_{jk} + \left(B^i_{j|k} - B^r_k \frac{\partial B^i_j}{\partial y^r}\right) - \left(B^i_{k|j} - B^r_j \frac{\partial B^i_k}{\partial y^r}\right)$
where, $B^i_j = \frac{F_{|j} y^i}{F}$.
Equivalently,
(3.8)

$$\begin{aligned}\overset{*}{R}{}^{\;i}_{jk} &= R^i_{jk} + \left(F_{|j|k} l^i - F_{|j} F_{|k} l^i\right) - \left(F_{|k|j} l^i - F_{|k} F_{|j} l^i\right) \\ &= R^i_{jk} + \left(F_{|j|k} - F_{|k|j}\right) l^i\end{aligned}$$

Using the Ricci Identity we get
(3.9) $\overset{*}{R}{}^{\;i}_{jk} = R^i_{jk} - l_s R^s_{jk} l^i$.
Transvecting (3.9) by y^j it results
(3.10) $\overset{*}{R}{}^{\;i}_{jk} y^j = R^i_{jk} y^j - l_s R^s_{jk} l^i y^j$,
or, equivalently
(3.11) $\overset{*}{R}{}^{\;i}_{0k} = R^i_{0k} - l_s R^i_{0k} l^i$.
If the space IF^n is of scalar curvature K, then,
(3.12) $R^i_{0k} = KF^2 h^i_k$
and
(3.13) $\overset{*}{R}{}^{\;i}_{0k} = KF^2 h^i_k - l_s F^2 h^s_k l^i$.
Since $l_s h^s_k = 0$, it results
(3.14) $\overset{*}{R}{}^{\;i}_{0k} = KF^2 h^i_k$.
Now we can state

Theorem 3.2. *If an Ingarden space IF^n is of scalar curvature K, then the space IF^n equipped with $\overset{*}{N}$ nonlinear connection is also of scalar curvature.*

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