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COUPLED FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED GENERALIZED METRIC SPACES

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Abstract. In this paper, we prove some coupled fixed point theorems for nonlinear contractive mappings having the mixed monotone property in partially ordered G - metric spaces.

1. Introduction

In recent years, many studies in the area of fixed point theory in partially ordered metric spaces have been performed. Many well-known fixed point theorems in this area can be found in [1], [2], [4], [7 - 15], [21 - 26]. Some of these theorems were given and proved by Bhaskar and Lakshmikantham in [10]. In this paper, the authors introduced the notions of mixed monotone mapping and coupled fixed point and discussed the existence and uniqueness of a solution for periodic boundary value problem. Coupled fixed point theorems and coupled coincidence point results are given in [3 - 5], [9], [13 - 15], [26]. Mustafa and Sims [17] introduced a new structure of generalized metric spaces, namely G-metric space. As a result, many fixed point theorems for various mappings in this space was established [6], [17 - 19], [27]. In this research stream, Choudhury and Maity [5] proved several fixed point theorems for mixed monotone mappings satisfying a contractive condition. In this paper, we prove some coupled fixed point theorems for nonlinear contractive mappings in partially ordered G-metric spaces, which generalize results in [5].

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2. Preliminaries

Definition 2.1. ([17]) Let X be a non-empty set and $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, (symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2.2. ([17]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and one says that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in the G-metric space (X, G) then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m > N$.
In [17], the authors have shown that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to a point.

Definition 2.3. ([17]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Lemma 2.4. *([17]) If (X, G) is a G-metric space, then the following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$,*

(4) *$G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.*

Lemma 2.5. *([17]) If (X, G) be a G-metric space, then the following are equivalent:*

(1) *The sequence $\{x_n\}$ is G-Cauchy,*

(2) *For every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Lemma 2.6. *([17]) If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Lemma 2.7. *If (X, G) is a G-metric space then*

$$G(x, x, y) \leq G(x, x, z) + G(z, z, y)$$

for all $x, y, z \in X$.

Proof. For all $x, y, z \in X$, by Definition 2.1 (iv) and (v), we have

$$\begin{aligned} G(x, x, y) = G(y, x, x) &\leq G(y, z, z) + G(z, x, x) \\ &= G(x, x, z) + G(z, z, y) \end{aligned}$$

This ends the proof. □

Definition 2.8. ([17]) Let $(X, G), (X', G')$ be two G-metric spaces. Then a function $f : X \to X'$ is said to be G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Lemma 2.9. *([17]) Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.10. ([17]) A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.11. ([17]) A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is convergent in X.

Definition 2.12. ([5]) Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

Definition 2.13. ([10]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non - decreasing in x and is monotone non - increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 2.14. ([10]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x)$$

The following Lemma will be useful in the sequel.

Lemma 2.15. *(See e.g. [16]) Let $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real numbers such that*

$$\lim_{n\to\infty}(x_n+y_n)=\alpha>0$$

Then there exists subsequences $\{x_{n_{k_j}}\}$ of $\{x_n\}$ and $\{y_{n_{k_j}}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty} x_{n_{k_j}}=\alpha_1,\ \lim_{j\to\infty} y_{n_{k_j}}=\alpha_2 \ \text{ and } \alpha_1+\alpha_2=\alpha$$

Proof. Since the sequence $\{x_n+y_n\}$ is convergent, it is bounded.
On other hand, due to $0\le x_n, y_n\le x_n+y_n$, $\{x_n\}$ and $\{y_n\}$ are also bounded.
Since $\{x_n\}$ is bounded, by Bolzano - Weierstrass theorem, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$. Assume that $\lim_{k\to\infty} x_{n_k}=\alpha_1$.
Also, due to $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $\lim_{j\to\infty} y_{n_{k_j}}=\alpha_2$. Since $\lim_{k\to\infty} x_{n_k}=\alpha_1$, we have $\lim_{j\to\infty} x_{n_{k_j}}=\alpha_1$.
Finally, we have

$$\alpha=\lim_{j\to\infty}(x_{n_{k_j}}+y_{n_{k_j}})=\alpha_1+\alpha_2.$$

□

3. Main results

Let Θ denote the family of all functions $\theta:[0,\infty)^2\to[0,\infty)$ for which there exists

$$\lim_{\substack{t_1\to r_1\\ t_2\to r_2}}\theta(t_1,t_2)>0 \ \text{ for all } (r_1,r_2)\in[0,\infty)^2 \text{ with } r_1+r_2>0$$

For example,
$\theta(t_1,t_2)=k\max\{t_1,t_2\}, k>0$, $\theta(t_1,t_2)=at_1^p+bt_2^q, a,b,p,q>0$ for all $(t_1,t_2)\in[0,\infty)^2$ are in Θ.

Now, we prove our main results.

Theorem 3.1. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a G-metric G on X such that (X,G) is a complete G-metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\theta\in\Theta$ such that*

$$\begin{aligned}&G(F(x,y),F(u,v),F(w,z))+G(F(y,x),F(v,u),F(z,w))\\ &\le G(x,u,w)+G(y,v,z)-\theta\left(G(x,u,w),G(y,v,z)\right)\end{aligned}\tag{3.1}$$

for all $x \succeq u \succeq w$ and $y \preceq v \preceq z$. Suppose that either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}$ is G-convergent to x, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\}$ is G-convergent to y, then $y \preceq y_n$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We construct the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n), \text{ for all } n \geq 0 \tag{3.2}$$

We shall show that

$$x_n \preceq x_{n+1}, \tag{3.3}$$

and

$$y_n \succeq y_{n+1}, \tag{3.4}$$

for all $n \geq 0$.

Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = f(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (3.3) and (3.4) hold for $n = 0$.

Suppose that (3.3) and (3.4) hold for some $n \geq 0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ and by the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) \tag{3.5}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) \tag{3.6}$$

Now from (3.5) and (3.6), we obtain

$$x_{n+1} \preceq x_{n+2} \text{ and } y_{n+1} \succeq y_{n+2}$$

Thus by mathematical induction we conclude that (3.3) and (3.4) hold for all $n \geq 0$.

Let $n \geq 1$. Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, from (3.1) and (3.2), we have

$$\begin{aligned} G(x_{n+1}, x_{n+1}, x_n) &+ G(y_{n+1}, y_{n+1}, y_n) \\ &= G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\quad +G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq G(x_n, x_n, x_{n-1}) + G(y_n, y_n, y_{n-1}) \\ &\quad -\theta\left(G(x_n, x_n, x_{n-1}), G(y_n, y_n, y_{n-1})\right) \end{aligned} \tag{3.7}$$

As $\theta(t_1, t_2) \geq 0$, for all $(t_1, t_2) \in [0, \infty)^2$, we have

$$G(x_{n+1}, x_{n+1}, x_n)+G(y_{n+1}, y_{n+1}, y_n) \leq G(x_n, x_n, x_{n-1})+G(y_n, y_n, y_{n-1}) \tag{3.8}$$

Set $\delta_n = G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)$, then the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta \tag{3.9}$$

We shall show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. By Lemma 2.15, the sequences $\{G(x_{n+1}, x_{n+1}, x_n)\}$ and $\{G(y_{n+1}, y_{n+1}, y_n)\}$ have convergent sequences that be still denoted $\{G(x_{n+1}, x_{n+1}, x_n)\}$ and $\{G(y_{n+1}, y_{n+1}, y_n)\}$, respectively. Assume that

$\lim_{n\to\infty} G(x_{n+1}, x_{n+1}, x_n) = \delta_1$ and $\lim_{n\to\infty} G(y_{n+1}, y_{n+1}, y_n) = \delta_2$, then $\delta_1 + \delta_2 = \delta > 0$.

Then taking the limit as $n \to \infty$ of both sides of (3.8), we have

$$\begin{aligned} \delta &= \lim_{n\to\infty} \delta_n \\ &\leq \lim_{n\to\infty} \left[G(x_n, x_n, x_{n-1}) + G(y_n, y_n, y_{n-1})\right] \\ &\quad - \lim_{n\to\infty} \theta\left(G(x_n, x_n, x_{n-1}), G(y_n, y_n, y_{n-1})\right) \\ &= \delta - \lim_{\substack{r_1\to\delta_1 \\ r_2\to\delta_2}} \theta\left(r_1, r_2\right) \\ &< \delta, \end{aligned}$$

in which $r_1 = G(x_n, x_n, x_{n-1}), r_2 = G(y_n, y_n, y_{n-1})$. This is a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)] = 0 \tag{3.10}$$

In what follows, we shall show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, on the contrary, that at least one of the sequences $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$G(x_{n(k)}, x_{n(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)}, y_{m(k)}) \geq \varepsilon \tag{3.11}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ such that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (3.11). Then

$$G(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) + G(y_{n(k)-1}, y_{n(k)-1}, y_{m(k)}) < \varepsilon \tag{3.12}$$

By rectangle inequality, Definition 2.1 (v), we have

$$G(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq G(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) \tag{3.13}$$

and

$$G(y_{n(k)}, y_{n(k)}, y_{m(k)}) \leq G(y_{n(k)}, y_{n(k)}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{n(k)-1}, y_{m(k)}) \tag{3.14}$$

From (3.11) - (3.14), we obtain

$$\begin{aligned} \varepsilon &\leq G(x_{n(k)}, x_{n(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)}, y_{m(k)}) \\ &< G(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + G(y_{n(k)}, y_{n(k)}, y_{n(k)-1}) + \varepsilon \end{aligned}$$

Letting $k \to \infty$ and using (3.10), we have

$$\lim_{k\to\infty} [G(x_{n(k)}, x_{n(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)}, y_{m(k)})] = \varepsilon \tag{3.15}$$

By Lemma 2.7 , we have

$$\begin{aligned} G(x_{n(k)}, x_{n(k)}, x_{m(k)}) &\leq G(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \\ &\quad + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)}) \\ &\leq G(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \\ &\quad + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \\ &\quad + G(x_{m(k)+1}, x_{m(k)+1}, x_{m(k)}) \end{aligned}$$

On the other hand, $G(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) \leq 2G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)})$, since by Definition 2.1, $G(x, x, y) \leq G(x, y, y) + G(y, x, y) = 2G(y, y, x)$. Thus,

$$\begin{aligned} G(x_{n(k)}, x_{n(k)}, x_{m(k)}) &\leq 2G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}) \\ &\quad + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \\ &\quad + G(x_{m(k)+1}, x_{m(k)+1}, x_{m(k)}) \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned}
G(y_{n(k)}, y_{n(k)}, y_{m(k)}) &\le 2G(y_{n(k)+1}, y_{n(k)+1}, y_{n(k)}) \\
&\quad +G(y_{n(k)+1}, y_{n(k)+1}, y_{m(k)+1}) \\
&\quad +G(y_{m(k)+1}, y_{m(k)+1}, y_{m(k)})
\end{aligned} \tag{3.17}$$

From (3.16), (3.17), we have

$$\begin{aligned}
&G(x_{n(k)}, x_{n(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)}, y_{m(k)}) \\
\le\ & 2\delta_{n(k)} + \delta_{m(k)} + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) \\
&+G(y_{n(k)+1}, y_{n(k)+1}, y_{m(k)+1})
\end{aligned} \tag{3.18}$$

Since $n(k) > m(k)$, we have $x_{n(k)} \succeq x_{m(k)}$ and $y_{n(k)} \preceq y_{m(k)}$, hence from (3.1) and (3.2),

$$\begin{aligned}
&G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) + G(y_{n(k)+1}, y_{n(k)+1}, y_{m(k)+1}) \\
&= G(F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) \\
&+G(F(y_{n(k)}, x_{n(k)}), F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)})) \\
&\le G(x_{n(k)}, x_{n(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)}, y_{m(k)}) \\
&-\theta\left(G(x_{n(k)}, x_{n(k)}, x_{m(k)}), G(y_{n(k)}, y_{n(k)}, y_{m(k)})\right)
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we have

$$\theta\left(G(x_{n(k)}, x_{n(k)}, x_{m(k)}), G(y_{n(k)}, y_{n(k)}, y_{m(k)})\right) \le 2\delta_{n(k)} + \delta_{m(k)} \tag{3.20}$$

By Lemma 2.15 and (3.15), the sequences $\{G(x_{n(k)}, x_{n(k)}, x_{m(k)})\}$ and $\{G(y_{n(k)}, y_{n(k)}, y_{m(k)})\}$ have subsequences converging to, say, ε_1 and ε_2, respectively, and $\varepsilon_1 + \varepsilon_2 = \varepsilon > 0$. By passing to subsequences, we may assume that $\lim_{k\to\infty} G(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \varepsilon_1$ and $\lim_{k\to\infty} G(y_{n(k)}, y_{n(k)}, y_{m(k)}) = \varepsilon_2$.
Taking $k \to \infty$ in (3.20) and using (3.10), we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} [2\delta_{n(k)} + \delta_{m(k)}] \\
&\ge \lim_{k\to\infty} \theta\left(G(x_{n(k)}, x_{n(k)}, x_{m(k)}), G(y_{n(k)}, y_{n(k)}, y_{m(k)})\right) \\
&= \lim_{\substack{r_1\to\varepsilon_1 \\ r_2\to\varepsilon_2}} \theta(r_1, r_2).
\end{aligned}$$

in which $r_1 = G(x_{n(k)}, x_{n(k)}, x_{m(k)})$ and $r_2 = G(y_{n(k)}, y_{n(k)}, y_{m(k)})$. That is a contradiction. Thus, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since (X, G) is a G-complete space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y \tag{3.21}$$

Thus

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} x_n = x\,;\ \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} y_n = y \tag{3.22}$$

Now, suppose that assumption (a) holds. From (3.2), we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x)$$

Finally, suppose that (b) holds. Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to x$ and as $\{y_n\}$ is a non-increasing sequence and $y_n \to y$, we have $x_n \preceq x$ and $y_n \succeq y$ for all n.
If $x_n = x$ and $y_n = y$ for some n, then, by our construction, $x_{n+1} = x$ and $y_{n+1} = y$ and (x, y) is a coupled fixed point of F. So we can assume either $x_n \neq x$ or $y_n \neq y$.
Then we have

$$\begin{aligned}
& G(F(x,y),x,x) + G(F(y,x),y,y) \\
\leq\ & G(F(x,y),F(x_n,y_n),F(x_n,y_n)) + G(F(x_n,y_n),x,x) \\
& +G(F(y,x),F(y_n,x_n),F(y_n,x_n)) + G(F(y_n,x_n),y,y) \\
=\ & G(F(x_n,y_n),F(x_n,y_n),F(x,y)) + G(F(y_n,x_n),F(y_n,x_n),F(y,x)) \\
& +G(x_{n+1},x,x) + G(y_{n+1},y,y) \\
\leq\ & G(x_n,x_n,x) + G(y_n,y_n,y) - \theta\,(G(x_n,x_n,x),G(y_n,y_n,y)) \\
& +G(x_{n+1},x,x) + G(y_{n+1},y,y) \\
\leq\ & G(x_n,x_n,x) + G(y_n,y_n,y) + G(x_{n+1},x,x) + G(y_{n+1},y,y)
\end{aligned}$$

Letting $n \to \infty$ in the inequality

$$\begin{aligned}
& G(F(x,y),x,x) + G(F(y,x),y,y) \\
\leq\ & G(x_n,x_n,x) + G(y_n,y_n,y) + G(x_{n+1},x,x) + G(y_{n+1},y,y)
\end{aligned}$$

we obtain

$$G(F(x,y),x,x) + G(F(y,x),y,y) \leq 0$$

which implies $G(F(x,y),x,x) = 0$ and $G(F(y,x),y,y) = 0$. That is, $x = F(x,y)$ and $y = F(y,x)$.
The proof is complete. $\square$

Let Φ denote the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{t\to r} \psi(t) > 0 \text{ for each } r > 0.$$

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\psi \in \Phi$ such that*

$$\begin{aligned} G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w)) \\ \leq G(x,u,w) + G(y,v,z) \\ -\psi\left(\max\{G(x,u,w), G(y,v,z)\}\right) \end{aligned} \tag{3.23}$$

for all $x \succeq u \succeq w$ and $y \preceq v \preceq z$. Suppose that either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}$ is G-convergent to x, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\}$ is G-convergent to y, then $y \preceq y_n$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Proof. By taking $\theta(t_1, t_2) = \psi(\max\{t_1, t_2\})$ in Theorem 3.1 for all $(t_1, t_2) \in [0,\infty)^2$, we get Corollary 3.2, since $\psi \in \Phi$ implies $\theta \in \Theta$. $\square$

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\psi \in \Phi$ such that*

$$\begin{aligned} G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w)) \\ \leq G(x,u,w) + G(y,v,z) - \psi\left(G(x,u,w) + G(y,v,z)\right) \end{aligned} \tag{3.24}$$

for all $x \succeq u \succeq w$ and $y \preceq v \preceq z$. Suppose that either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}$ is G-convergent to x, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\}$ is G-convergent to y, then $y \preceq y_n$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Proof. By taking $\theta(t_1,t_2)=\psi(t_1+t_2)$ in Theorem 3.1 for all $(t_1,t_2)\in[0,\infty)^2$, we obtain Corollary 3.2. $\square$

Corollary 3.4. *Let $(X,\preceq)$ be a partially ordered set and suppose there exists a G-metric G on X such that (X,G) is a complete G- metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\theta\in\Theta$ with $\theta(t_1,t_2)=\theta(t_2,t_1)$ for all $(t_1,t_2)\in[0,\infty)^2$ such that*

$$G(F(x,y),F(u,v),F(w,z)) \le \frac{G(x,u,w)+G(y,v,z)}{2} -\theta\left(G(x,u,w),G(y,v,z)\right) \tag{3.25}$$

for all $x\succeq u\succeq w$ and $y\preceq v\preceq z$. Suppose that either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}$ is G-convergent to x, then $x_n\preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\}$ is G-convergent to y, then $y\preceq y_n$ for all n.
If there exist $x_0,y_0\in X$ such that $x_0\preceq F(x_0,y_0)$ and $y_0\succeq F(y_0,x_0)$, then F has a coupled fixed point in X.

Proof. From (3.25), for all $x\succeq u\succeq w$ and $y\preceq v\preceq z$, we have

$$G(F(x,y),F(u,v),F(w,z)) \le \frac{G(x,u,w)+G(y,v,z)}{2} -\theta\left(G(x,u,w),G(y,v,z)\right)$$

and

$$\begin{aligned} G(F(y,x),F(v,u),F(z,w)) &= G(F(z,w),F(v,u),F(y,x)) \\ &\le \frac{G(z,v,y)+G(w,u,x)}{2} -\theta\left(G(z,v,y),G(w,u,x)\right) \\ &= \frac{G(x,u,w)+G(y,v,z)}{2} -\theta\left(G(x,u,w),G(y,v,z)\right) \end{aligned}$$

Therefore,

$$\begin{aligned} & G(F(x,y),F(u,v),F(w,z))+G(F(y,x),F(v,u),F(z,w)) \\ \le\ & G(x,u,w)+G(y,v,z)-2\theta\left(G(x,u,w),G(y,v,z)\right) \\ \le\ & G(x,u,w)+G(y,v,z)-\theta_1\left(G(x,u,w),G(y,v,z)\right) \end{aligned}$$

for all $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where $\theta_1(t_1, t_2) = 2\theta(t_1, t_2)$ for all $(t_1, t_2) \in [0, \infty)^2$. Since $\theta_1 \in \Theta$, applying Theorem 3.1, we conclude that F has a coupled fixed point in X. □

Remark 3.5. In Corollary 3.4, if we take $\theta(t_1, t_2) = \frac{(1-k)(t_1+t_2)}{2}$, we obtain Theorem 3.1 and 3.2 in [5].

Now we shall prove the uniqueness of the coupled fixed point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation:

$$(x, y), (u, v) \in X \times X, \quad (x, y) \preceq (u, v) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 3.6. *In addition to the hypotheses of Theorem 3.1, suppose that for every (x, y), $(z, t) \in X \times X$, there exists a pair $(u, v) \in X \times X$ such that (u, v) is comparable to (x, y) and (z, t). Then F has a unique coupled fixed point.*

Proof. Suppose (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$. We shall show that $x = z$ and $y = t$.
By the assumption, there exists $(u, v) \in X \times X$ that (u, v) is comparable to (x, y) and (z, t).
We define the sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0 = u, v_0 = v, u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n), \text{ for all } n.$$

Since (u, v) is comparable with (x, y), we may assume that $(x, y) \succeq (u, v) = (u_0, v_0)$ (the other case being similar). By mathematical induction and the mixed monotone property of F, it is easy to prove that

$$(x, y) \succeq (u_n, v_n), \text{ for all } n. \tag{3.26}$$

From (3.1) and (3.26), we have

$$\begin{aligned} G(x, x, u_n) + G(v_n, y, y) &= G(F(x, y), F(x, y), F(u_{n-1}, v_{n-1})) \\ &\quad + G(F(v_{n-1}, u_{n-1}), F(y, x), F(y, x))) \\ &\leq G(x, x, u_{n-1}) + G(v_{n-1}, y, y) \\ &\quad - \theta(G(x, x, u_{n-1}), G(v_{n-1}, y, y)) \end{aligned} \tag{3.27}$$

which implies

$$G(x, x, u_n) + G(v_n, y, y) \leq G(x, x, u_{n-1}) + G(v_{n-1}, y, y)$$

that is, the sequence $\{G(x, x, u_n) + G(v_n, y, y)\}$ is decreasing. Therefore, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} G(x, x, u_n) + G(v_n, y, y) = \alpha$$

We shall show that $\alpha = 0$. Suppose, on the contrary, that $\alpha > 0$. Therefore, $\{G(x, x, u_n)\}$, $\{G(v_n, y, y)\}$ have subsequences converging to α_1, α_2, respectively, where $\alpha_1 + \alpha_2 = \alpha$. Taking the limit, up to subsequences, as $n \to \infty$ in (3.27), we have

$$\alpha \leq \alpha - \lim_{n\to\infty} \theta(G(x, x, u_{n-1}), G(v_{n-1}, y, y)) < \alpha$$

which is a contradiction. Thus, $\alpha = 0$, that is,

$$\lim_{n\to\infty} [G(x, x, u_n) + G(v_n, y, y)] = 0$$

which implies

$$\lim_{n\to\infty} G(x, x, u_n) = \lim_{n\to\infty} G(v_n, y, y) = 0 \tag{3.28}$$

Similarly, we can show that

$$\lim_{n\to\infty} G(z, z, u_n) = \lim_{n\to\infty} G(v_n, t, t) = 0 \tag{3.29}$$

From (3.28) and (3.29), we get $x = z$ and $y = t$, by the uniqueness of the limit of a $G-$ convergent sequence.
Therefore, the coupled fixed point of F is unique. □

Theorem 3.7. *If in addition to the hypotheses of Theorem 3.1 x_0 and y_0 are comparable then F has a fixed point.*

Proof. Following the proof of Theorem 3.1, F hax a coupled fixed point (x, y). We only have to show that $x = y$. Since x_0 and y_0 are comparable, we may assume that $x_0 \succeq y_0$ (the other case being similar). By using mathematical induction and the mixed monotone property of F, one can easily show that

$$x_n \succeq y_n, \quad \text{for all } \; n \geq 0 \tag{3.30}$$

where $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, $n = 0, 1, 2, ...$
By Lemma 2.7, we have

$$\begin{aligned} G(x, x, y) &\leq G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, y) \\ &\leq G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+1}, y) \\ &= G(x, x, x_{n+1}) + G(y_{n+1}, y_{n+1}, y) \\ &\quad + G(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) \end{aligned}$$

Similarly,

$$\begin{aligned} G(y,y,x) &\leq G(y,y,y_{n+1}) + G(x_{n+1},x_{n+1},x) \\ &\quad + G(F(y_n,x_n),F(y_n,x_n),F(x_n,y_n)) \end{aligned}$$

Therefore,

$$\begin{aligned} G(x,x,y)+G(y,y,x) &\leq G(x,x_{n+1},x_{n+1}) + G(y_{n+1},y_{n+1},y) \\ &\quad + G(y,y,y_{n+1}) + G(x_{n+1},x_{n+1},x) \\ &\quad + G(F(x_n,y_n),F(x_n,y_n),F(y_n,x_n)) \\ &\quad + G(F(y_n,x_n),F(y_n,x_n),F(x_n,y_n)) \\ &\leq G(x,x,x_{n+1}) + G(y_{n+1},y_{n+1},y) \\ &\quad + G(y,y,y_{n+1}) + G(x_{n+1},x_{n+1},x) \\ &\quad + G(x_n,x_n,y_n) + G(y_n,y_n,x_n) \\ &\quad - \theta(G(x_n,x_n,y_n),G(y_n,y_n,x_n)) \end{aligned}$$

Suppose that $x \neq y$. Taking $n \to \infty$ in the last inequality, using (3.21) and the continuity of G, we have

$$G(x,x,y)+G(y,y,x) \leq G(x,x,y)+G(y,y,x) - \lim_{n\to\infty} \theta(G(x_n,x_n,y),G(y,y_n,x_n))$$

hence,

$$\lim_{n\to\infty} \theta(G(x_n,x_n,y),G(y,y_n,x_n)) \leq 0,$$

which is false. Indeed, since $\lim_{n\to\infty} G(x_n,x_n,y) = G(x,x,y) > 0$ and $\lim_{n\to\infty} G(y,y_n,x_n) = G(y,y,x)$, we have $\lim_{n\to\infty} \theta(G(x_n,x_n,y),G(y,y_n,x_n)) = \lim_{\substack{r_1\to G(x,x,y)\\ r_2\to G(x,x,y)}} \theta(r_1,r_2) > 0.r_2 \to G(y,y,x)$

Therefore, $x = y$. In other words, we conclude that F has a fixed point in X. □

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