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A GENERALIZATION OF LOCALLY MINKOWSKI GL^n - SPACE

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Abstract. We continue the investigation of the generalized Lagrange spaces [4] with the case when F^n is a locally Minkowski space. Moreover, we generalize the GL - metric $g_{ij}(y)$ and give the canonical metric connections.

1. Introduction

Recently, the authors of this paper studied in [4] the generalized Lagrange spaces, shortly $GL-$spaces, $GL^n = (M, g_{ij}(y))$ with the fundamental tensor

(1.1) $g_{ij}(y) = \gamma_{ij} + (1/c^2)y_i y_j,\ y_i = \gamma_{ij}(y)y^j,$

where M is a smooth manifold and $\gamma_{ij}(y)$ is the fundamental tensor of a Finsler space $F^n = (M, F(x,y))$.

They investigated the spaces in the cases when F^n is a locally Minkowski space.

The case when $\gamma_{ij}(x,y)$ do not depend on the directional variable, that is when the $GL-$metric (g_{ij}) from (1.1) reduces to

(1.2) $g_{ij}(x,y) = \gamma_{ij}(x) + (1/c^2)y_i y_j,\ y_i = \gamma_{ij}(x,y)y^j,$

was studied by R. Miron and T. Kawaguchi.

The same authors consider in an excellent paper, [3], the $GL-$metric

(1.3) $g_{ij}(x,y) = \gamma_{ij}(x) + \left(1 - \dfrac{1}{n^2(x,y)}\right) y_1 y_j,\ y_i = \gamma_{ij}(x,y)y^j,$

where the function $n > 1$ is the so-called refractive index. It is obvious that for

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(1.4) $\frac{1}{h^2} = 1 - \frac{1}{c^2}$,

the $GL-$metric (1.3) coincides with the $GL-$metric (1.2).

Let $V^i(x)$ be a local vector field on M and $S_V : M \to TM$ be the local section, $S_V(x) = (x, V(x))$, of the tangent bundle $\pi : TM \to M$.

The restriction of the tensor field g_{ij} from (1.3) to the image of mapping S_V gives the Synge metric from the Relativistic Optics:

(1.5) $g_{ij}(x, V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, V(x))}\right) V_i V_j, V_i(x) = \gamma_{ij}(x)V^j(x)$.

This fact shows the importance of the $GL-$metric (1.3) for Relativistic Optics.

2. The locally Minkowski $\mathbf{GL^n}$ – space

Let $GL^n = (M, g_{ij}(y))$ be a local Minkowski space [4] with the fundamental tensor

(2.1.) $g_{ij}(y) = \gamma_{ij} + (1 - \frac{1}{n^2\ (\mathrm{y})})y_i y_j$, where $y_i = \gamma_{ij}(y)y^j$

Theorem 2.1.

a) $GL^n = (M, \gamma_{ij})$ is a generalized Lagrange space.

b) The GL^n space is not reductible to a Finsler space.

Proof : a) We consider

(2.2.) $g^{ij} = \gamma^{ij} - \frac{1}{a_{(y)}}(1 - \frac{1}{n^2\ (\mathrm{y})})y^i y^j$

with

(2.3.) $a(y) = 1 + (1 - \frac{1}{n^2\ (\mathrm{y})}\ ) \left\|y\right\|^2$,

where

(2.4.) $\|y\| = \gamma_{ij}y^i y^j$

and we obtain the g^{ij} is a contravariant symetric d-tensor field on TM , that verifies

(2.5.) $g_{ij}g^{jk} = \delta_i^k$

b) The Cartan tensor

$C_{ijk} = \frac{1}{2}\ \frac{\partial g_{ij}}{\partial y^k}$

is expressed as:

(2.6.) $C_{ijk} = \frac{1}{n^3}\frac{\partial n}{\partial y^k}y_i y_j + \frac{1}{2}\ (1 - \frac{1}{n^2\ (\mathrm{y})})(\gamma_{ik}y_j + \gamma_{jk}y_i)$

The necessary condition for the GL^n – space to be reductible to a Finsler space is that C_{ijk} is a symmetric tensor for all (i, j, k) .

We get

(2.7.) $\frac{1}{n^3}\frac{\partial n}{\partial y^k}y_i y_j + \frac{1}{2}\ (1 - \frac{1}{n^2\ (\mathrm{y})})(\gamma_{ik}y_j + \gamma_{jk}y_i) =$

$= \frac{1}{n^3}\frac{\partial n}{\partial y^i}y_k y_i + \frac{1}{2}\ (1 - \frac{1}{n^2\ (\mathrm{y})})(\gamma_{ik}y_j + \gamma_{jk}y_i)$

Now , transvecting with γ^{js} , it results

(2.8.) $\frac{1}{n^3}[\frac{\partial n}{\partial y^k}y_i + \frac{\partial n}{\partial y^i}y_k]y^s + \frac{1}{2}\ (1 - \frac{1}{n^2\ (\mathrm{y})})(\delta^s_k y_i - \delta^s_i y_k) = 0$

For s=k , we have

(2.9.) $\frac{1}{n^3}[(\frac{\partial n}{\partial y^k}y^k)y_i - \frac{\partial n}{\partial y^i}\ \|y\|^2] + \frac{1}{2}\ (1 - \frac{1}{n^2\ (\mathrm{y})})(ny_i - y_i) = 0$

Transvecting by y^i we obtain

(2.10.) (n-1) $\|y\|2 = 0$

This relation is impossible for n>1 , so it results the conclusion of the theorem .

Let as consider the nonlinear connection $\dot{N}$ with the coefficients

(2.11.) $\dot{N}^i_j = \frac{1}{2}\ \frac{\partial}{y^j}\ \{\gamma^i_{rs}\ (x,y)y^r y^s\}$

Here, all the coefficients γ^i_{jk} (y) of γ^i_{ij} (y) are vanishing, so $\dot{N}^i_j = 0$

This result imply that the autoparallel curves of $\dot{N}$ are given by

(2.12.) $\frac{dx^i}{dt} = y^i\ ,\ \frac{dy^i}{dt} = 0$

and the adapted bases to the horizontal distribution $\dot{N}$ and the vertical distribution V coincides to the natural basis on TM ($\frac{\partial}{x^i}, \frac{\partial}{y^i}$)

The adapted cobasis to $\dot{N}$ is (dx^i , dy^i) .

Now we can state

Theorem 2.2_ The $GL^n = (M, g_{ij}(y))$ space has a canonical metrical connection CΓ ($\dot{N}$) = (L$^i_{jk}$, C$^i_{jk}$) with the coefficients

(2.13.) $\begin{cases} \mathrm{L}^i_{jk} = 0 \\ \mathrm{C}^i_{jk} = \dot{C}^i_{jk} + \hat{C}^i_{jk} \end{cases}$

where

(2.14.) $\dot{C}^i_{jk} = \frac{1}{2}\ \gamma^{is}(\frac{\partial\gamma_{\mathrm{sk}}}{\partial y^j} + \frac{\partial\gamma_{\mathrm{js}}}{\partial y^k} - \frac{\partial\gamma_{\mathrm{jk}}}{\partial y^s}),\ \hat{C}^i_{jk} = g^{is}\hat{C}_{jsk},$

$$\hat{C}_{ijk} = -n(y)\{y_i y_j \frac{\partial n}{\partial y^k} + y_j y_k \frac{\partial n}{\partial y^i} - y_k y_i \frac{\partial n}{\partial y^j}\}$$

Properties .

a) The h-paths of the GL^n - space with respect to CΓ ($\dot{N}$) connexion are given by

(2.15.) $\begin{cases} \frac{dx^i}{dt} = \mathrm{y^i} \\ \frac{d^2\ x^i}{dt^2} = 0 \end{cases}$

b) The v- paths of the GL^n - space with respect to CΓ ($\dot{N}$) connexion in a point x^i_0 are expressed by

(2.16.) $\begin{cases} x^i = x^i_0 \\ \frac{d^2\ y^i}{dt^2} + \left(\dot{C}^i_{jk} + \hat{C}^i_{jk}\right)(x_0,\ y)\frac{dy^j}{dt}\frac{dy^k}{dt} = 0 \end{cases}$

c) The Einstein equations of the GL^n - space with respect to CΓ ($\dot{N}$) connexion are as follows :

(2.17.) $\begin{cases} S_{ij} = \frac{1}{2}\ S\ g_{ij} = n\ T_{ij} \\ T_j\ /^i_j = 0 \end{cases}$

where S^i_{jkh} is the v-curvature tensor $S_{ij} = S^h_{ihh}$ and $S = g^{ij} S_{ij}$

3. A generalization of locally Minkowski GL^n - space

Let F^n= (M , F(x,y)) a Finsler space with the fundamental function F(x,y) .

One consider the metric

(3.1.) $g_{ij}(x, y) = \gamma_{ij}(x, y) + (1 - \frac{1}{n^2 \text{ (x, y)}})y_i y_j$

where

$$\gamma_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is the fundamental tensor of F^n .

We can extend the theory from the paragraph 2.

Theorem 3.1.

a) The space $GL^n = (M, g_{ij}(x, y))$ with $g_{ij}(x, y)$ given by (3.1.) is a generalized Lagrange space.

b) The space GL^n is not reductible to a Finsler space.

Proof : a) We consider $g_{ij}(x, y) = g_{ji}(x, y)$ a d - tensor field on TM and

(3.2.) g^{ij} (x,y) = γ^{ij} (x,y) - $\frac{1}{\text{a (x, y)}}$ $(1 - \frac{1}{n^2 \text{ (x, y)}}$) $y^i y^j$

where

(3.3) a (x,y) = 1+ $(1 - \frac{1}{n^2 \text{ (x, y)}}$) F^2(x,y)

and γ^{ij} (x,y) is the contravariant tensor of the fundamental tensor $\gamma_{ij}(x, y$ of the Finsler space F^n.

Using $y_i = \frac{1}{2}$ $\frac{\partial F^2}{\partial y^i}$ and $y_i y^j$= F^2(x,y) we get

(3.4.) $g_{ij}(x, y)g^{jk}(x, y) = \delta^k_i$

Consequently, we have

$$\text{rang } ||g_{ij}|| = \text{rang } ||g^{ij}|| = \text{n}$$

b) We know that the GL^n - space is reductible to a Finsler space F^n= (M , F) if there exists a positive function F (x,y), 1- homogeneous in y^i which is the solution of the system

(3.5.) $\frac{1}{2}$ $\frac{\partial^2 F^2}{\partial y^i \partial y^j} = g_{ij}(x, y)$

In this case

$\frac{1}{2}$ $\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = 2C_{ijk} = \frac{\partial g_{ij}}{\partial y^k}$

is a symmetric d- tensor field

When the system (3.5.) doesn ' t have any solution, the GL^n - space is not reductible to a Finsler space F^n .

The tensor C_{ijk} is given by

(3.6.) $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{2}\frac{\partial}{\partial y^k}[\gamma_{ij} + (1 - n^2)y_i y_j]$

with $u(x,y) = \frac{1}{\mathrm{n\ (x,\ y)}}$

We get

(3.7.) $C_{ijk} = \dot{C}_{ijk} - u\frac{\partial u}{\partial y^k} y_i y_j + (1-u^2)(\gamma_{ik} y_j + \gamma_{jk} y_i)$

where $\dot{C}_i jk$ is the Cartan tensor of the space F^n.

(3.8.) $\dot{C}_{ijk} = \frac{1}{2}\frac{\partial \gamma_{\mathrm{ij}}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$

The symmetry of C_{ijk} it results from $C_{ijk} = C_{ikj}$

So, we obtain

(3.9.) $-u y_i (\frac{\partial u}{\partial y^k}\ y_j - \ \frac{\partial u}{\partial y^j}\ y_k\ ) + (1-u^2)(\gamma_{ik} y_j - \gamma_{ij} y_k) = 0$

Transvecting by γ^{js} it results

(3.10.) $-u y_i (\frac{\partial u}{\partial y^k}\ y_s - \ \frac{\partial u}{\partial y^j}\ \gamma^{\mathrm{js}} y_k\ ) + (1-u^2)(\gamma_{ik} y^s - \delta_i^s y_k) = 0$

For s = i we get

(3.11.) $-u(\frac{\partial u}{\partial y^k}\ F^2 - \ \frac{\partial u}{\partial y^s}\ y_k\ \mathrm{y^j}) + (1-u^2)(y_k - n y_k) = 0$

A new contraction with $\mathrm{y^k}$ give us :

(3.12.) $(1-u^2)(1-n)\ F^2 = 0$

It results $F^2 = 0$. This is impossible because F > 0.

We consider the generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$ with the fundamental tensor g_{ij} given by (3.1.) equipped with the nonlinear connection $\dot{N}$ which is the Cartan nonlinear connection of the Finsler space F^n .

The coefficients are:

(3.13.) $\dot{N}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\ \{\ \gamma^{\mathrm{i}}{}_{\mathrm{rs}}(\mathrm{x,y})\ \mathrm{y^r y^s}\ \}$

where $\gamma^{\mathrm{i}}{}_{\mathrm{jk}}(x,y)$ are the Christoffel symbols of the fundamental tensor $\gamma_{ij}(x,y)$ of the space F^n.

The adapted basis to the horizontal distribution $\dot{N}$ and the vertical distribution V is $(\frac{\dot{\delta}}{\delta x^i},\ \frac{\partial}{\partial y^i}\ )$.

where

(3.14.) $\frac{\dot{\delta}}{\delta x^i} = \frac{\partial}{\partial y^i}$ - $\dot{N}^j_i(\mathrm{x,y})\ \frac{\partial}{\partial y^j}$

The adapted cobasis is $(dx^i,\ \dot{\delta} y^i\ )$ with

(3.15.) $\dot{\delta} y^i = dy^i + \dot{N}^i_j(\mathrm{x,y})\ dx^j$

The following important result holds

Theorem 3.2. The local coefficients of the canonical metrical connection $C\Gamma\ (\dot{N})$ of generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$ are given by

(3.16.) $\mathrm{L}^i_{jk} = \dot{\mathrm{F}}^i_{jk} + \overset{i}{\underset{jk}{\wedge}}\ ,\ \mathrm{C}^i_{jk} = \dot{C}^i_{jk} + \hat{C}^i_{jk}$

where

(3.17.) $\dot{\mathrm{F}}^i_{jk} = \frac{1}{2}\ \gamma^{\mathrm{is}}(\ \frac{\dot{\delta}\gamma_{\mathrm{sk}}}{\delta x^j} + \frac{\dot{\delta}\gamma_{\mathrm{js}}}{\delta x^k} - \frac{\dot{\delta}\gamma_{\mathrm{jk}}}{\delta x^s}\ )$

$$\dot{C}^{i}_{jk} = \frac{1}{2}\gamma^{\mathrm{is}}\left(\frac{\partial\gamma_{\mathrm{sk}}}{\partial y^{j}} + \frac{\partial\gamma_{\mathrm{js}}}{\partial y^{k}} - \frac{\delta\gamma_{\mathrm{jk}}}{\delta y^{s}}\right)$$

with $\frac{\dot{\delta}}{\delta x^{k}} = \frac{\partial}{\partial x^{k}} - N^{j}_{k}(\mathrm{x},\mathrm{y})\ \frac{\partial}{\partial y^{j}}$ are the generalized Cristoffel symbols of the canonical Cartan metrical connection $C\dot{\Gamma} = \left(\dot{\mathrm{F}}^{i}_{jk}\ ,\dot{C}^{i}_{jk}\right)$ of the Finsler space F^{n} and

(3.18.) $$\overset{i}{\underset{jk}{\wedge}} = \mathrm{g}^{\mathrm{ih}}\ \underset{jhk}{\wedge}$$

$$\hat{C}^{i}_{jk} = \mathrm{g}^{\mathrm{ih}}\ \widehat{\mathrm{C}}_{\mathrm{jhk}}$$

(3.19.) $$\wedge_{ijk} = -u\left(y_i y_j\ \frac{\dot{\delta}u}{\delta x^{k}} + y_j y_k\ \frac{\dot{\delta}u}{\delta x^{i}} - y_k y_i\ \frac{\dot{\delta}u}{\delta x^{j}}\right)$$

$$\widehat{\mathrm{C}}_{\mathrm{jhk}} = u(y_i y_j \frac{\partial u}{\partial y^{k}} + y_j y_k \frac{\partial u}{\partial y^{i}} - y_k y_i \frac{\partial u}{\partial y^{j}})$$

Conclusion. Using the connections $\dot{N}$ and $\mathrm{C}\Gamma\ (\dot{N})$ we can study the geometry of generalized Lagrange space $GL^{n} = (M, g_{ij}(x,y))$.

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