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A GENERAL FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE HYBRID PAIRS IN QUASI - METRIC SPACES AND APPLICATIONS

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Abstract. In this paper a general fixed point theorem for two pairs of occasionally weakly compatible hybrid pairs in quasi - metric spaces is proved. As an application we reduce the study of fixed points for occasionally weakly compatible pairs in G - metric spaces to the study of occasionally weakly compatible pairs in quasi - metric spaces.

1. Introduction

Let A and S be self mappings of a metric space (X, d). Jungck [12] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Sx = Ax$. We denote by $C(A, S)$ the set of all coincidence points of A and S. In [22], Pant defined A and S to be pointwise R - weakly commuting if for each $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \leq Rd(Ax, Sx)$.

It is proved in [23] that pointwise R - weakly commuting is equivalent with the commuting at coincidence points.

Definition 1.1. A and S is said to be weakly compatible [13] if $ASu = SAu$ for $u \in C(A, S)$.

Definition 1.2. A and S are occasionally weakly compatible [4] (owc) if $ASu = SAu$ for $u \in C(A, S)$.

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Remark 1.3. *If $C(A,S) \neq \emptyset$ and A and S are weakly compatible then A and S are occasionally weakly compatible, but the converse is not true (see Example* [4]*).*

Some fixed point theorems for owc mappings are proved in [14], [2], [27] and in other papers.

Recently, Abbas and Rhoades [1] extended Definition 1.2 for pairs of hybrid mappings.

Definition 1.4. Let $f : X \to X$ and $F : X \to 2^X$.

1) A point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. The set of coincidence points of f and F is denoted by $C(f,F)$.

2) A point $x \in X$ is a fixed point of F if $x \in Fx$.

Definition 1.5. A hybrid pair $f : X \to X$ and $F : X \to 2^X$ is occasionally weakly compatible (owc) [1] if there exists $x \in C(f,F)$ such that $fFx \subset Ffx$.

Some results of fixed points for hybrid points of owc hybrid mappings are proved in [1], [5] and in other papers. In [3] some general results for occasionally weakly compatible hybrid pairs in symmetric spaces are proved.

2. Preliminaries

Definition 2.1. Let X be a nonempty set. A quasi - metric on X [30] is a nonnegative real function Q on $X \times X$ such that:

$(Q_1) : Q(x,y) = 0$ if and only if $x = y$,

$(Q_2) : Q(x,y) \leq Q(x,z) + Q(z,y)$ for all $x,y,z \in X$.

A quasi - metric space is a nonempty set X with a quasi - metric Q on X and is denoted by (X,Q).

Some theorems for self mappings in a quasi - metric space are proved in [9], [10], [11], [28], [29] and in other papers. Recently, some fixed points theorems for multivalued mappings in quasi - metric spaces are proved in [6], [15] and in other papers.

In [7], [8] Dhage introduced a new class of generalized metric spaces named D - metric spaces. Mustafa and Sims [17], [18] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric spaces, named G - metric spaces. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [16] - [21], [26] and other papers.

Definition 2.2. Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $y \neq x$,

$(G_3) : G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$ and $y \neq z$,

$(G_4) : G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The function G is called a G - metric on X and the pair (X, G) is called a G - metric space, [17], [18].

Remark 2.3. *If* $G(x, y, z) = 0$, *then* $x = y = z$ [18].

Lemma 2.4. *Let* (X, G) *be a* G *- metric space and* $Q(x, y) = G(x, y, y)$. *Then* $Q(x, y)$ *is a quasi - metric on* X.

Proof. (Q_1) : By (G_1) and Remark 2.3, $Q(x, y) = 0$ if and only if $x = y$.

$(Q_2) : Q(x, y) = G(x, y, y) \leq G(x, z, z) + G(z, y, y) = Q(x, z) + Q(z, y)$.

Hence, $Q(x, y)$ is a quasi - metric space. □

In the following we denote by

$$D(A, B) = \inf\{Q(a, b) : a, b \in X\}, \text{ where } A, B \in 2^X.$$

The study of fixed points for mappings satisfying an implicit relation is initiated by Popa in [24], [25].

In this paper a general fixed point theorem for two pairs of occasionally weakly compatible hybrid pairs in quasi - metric spaces satisfying an implicit relation is proved.

As an application we reduce the study of fixed points for occasionally weakly compatible pairs in G - metric spaces to the study of fixed points of occasionally weakly compatible pairs in quasi - metric spaces.

3. Implicit relations

Definition 3.1. Let $\mathcal{F}_a$ be the set of all functions $\phi(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1): ϕ is nonincreasing in variables t_2, t_5, t_6,

(ϕ_2): $\phi(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

Example 3.2. $\phi(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3, ..., t_6\}$, *where* $k \in (0,1)$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-k) > 0$, $\forall t > 0$.

Example 3.3. $\phi(t_1, ..., t_6) = t_1 - h\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, *where* $h \in (0,1)$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-h) > 0$, $\forall t > 0$.

Example 3.4. $\phi(t_1, ..., t_6) = t_1 - k\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$, *where* $k \in (0,1)$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-k) > 0$, $\forall t > 0$.

Example 3.5. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $a + c < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-(a+c)) > 0$, $\forall t > 0$.

Example 3.6. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $a + c < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-(a+c)) > 0$, $\forall t > 0$.

Example 3.7. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\sqrt{t_5 t_6}$, *where* $a, b, c \geq 0$ *and* $a + c < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-(a+c)) > 0$, $\forall t > 0$.

Example 3.8. $\phi(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1-\alpha)(at_5 + bt_6)$, *where* $0 < \alpha < 1$, $a, b \geq 0$ *and* $a + b < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t(1-\alpha)(1-(a+b)) > 0$, $\forall t > 0$.

Example 3.9. $\phi(t_1, ..., t_6) = t_1^2 - at_2^2 - b\dfrac{\min\{t_5, t_6\}}{1 + t_3 + t_4}$, *where* a, $b \geq 0$ *and* $a + b < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t,t,0,0,t,t) = t^2(1-(a+b)) > 0$, $\forall t > 0$.

Example 3.10. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\dfrac{t_5 + t_6}{1 + t_3 + t_4}$, *where* a, $b \geq 0$ *and* $a + 2b < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - (a + 2b)) > 0$, $\forall t > 0$.

Example 3.11. $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $0 < c < 1$, a, $b \geq 0$ *and* $\max\{c, a + b\} < 1$.

(ϕ_1): Obviously.
(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - \max\{c, a + b\}) > 0$, $\forall t > 0$.

4. Main results

Theorem 4.1. *Let* f, h *be self mappings of a quasi - metric space and* F, H *be maps of* X *into* 2^X *such that the pairs* (f, F) *and* (h, H) *are owc. If*

$$\begin{gathered}\phi(Q(fx, hy), D(Fx, Hy), D(fx, Fx),\\ D(hy, Hy), D(fx, Hy), D(Fx, hy)) \leq 0,\end{gathered} \tag{4.1}$$

hold for all $x, y \in X$ *for which* $fx \neq hy$ *and* $\phi \in \mathcal{F}_a$, *then* f, h, F *and* H *have a unique common fixed point.*

Proof. Because (f, F) and (h, H) are owc, there exist $x, y \in X$ such that $fx \in Fx$ and $hy \in Hy$ and $fFx \subset Ffx$ and $hHy \subset Hhy$. First we prove that $fx = hy$. Suppose that $fx \neq hy$. Then $0 \neq Q(fx, hy) \geq D(Fx, Hy)$. By (4.1) and (ϕ_1) we have

$$\phi(Q(fx, hy), Q(fx, hy), 0, 0, Q(fx, hy), Q(fx, hy)) \leq 0,$$

a contradiction of (ϕ_2). Hence $fx = hy$. Next we prove that $fx = f^2x$. Suppose that $fx \neq f^2x$. Then

$$0 \neq Q(f^2x, fx) \geq D(Ffx, fx) = D(Ffx, hy) \geq D(Ffx, Hy).$$

By (4.1) and (ϕ_1) we have successively

$$\phi(Q(f^2x, hy), D(Ffx, Hy), 0, 0, D(f^2x, Hy), D(Ffx, hy)) \leq 0,$$

$$\phi(Q(f^2x, hy), Q(f^2x, hy), 0, 0, Q(f^2x, hy), Q(f^2x, hy)) \leq 0,$$

$$\phi(Q(f^2x, fx), Q(f^2x, fx), 0, 0, Q(f^2x, fx), Q(f^2x, fx)) \leq 0,$$

a contradiction of (ϕ_2). Hence $fx = f^2x$ and fx is a fixed point of f. Similarly, $hy = h^2y$. Therefore $fx = f^2x = hy = h^2y = hfx$ and fx is a fixed point of h. On the other hand $fx = f^2x \in fFx \subset Ffx$, hence fx is a fixed point of F. Similarly, $fx = f^2x = hy = h^2y \subset$

$hHy \in Hhy = Hfx$ and fx is a fixed point of H. Hence $w = fx$ is a common fixed point of f, h, F and H.

Suppose that $w' \neq w$ is an other common fixed point of f, h, F and H. Then by (4.1) and (ϕ_1) we have successively

$$\begin{gathered}\phi(Q(fw, gw'), D(Fw, Hw'), D(fw, Fw), \\ D(hw', Hw'), D(fw, Hw'), D(Fw, hw')) \leq 0,\end{gathered}$$

$$\phi(Q(w, w'), Q(w, w'), 0, 0, Q(w, w'), Q(w, w')) \leq 0,$$

a contradiction of (ϕ_2). Hence $w = w'$ and w is the unique common fixed point of f, h, F and H. □

If f, F, h, H are single valued mappings, then by Theorem 4.1 we obtain

Theorem 4.2. *Let f, h, F and H be self mappings of a quasi - metric space (X, Q) such that (f, F) and (h, H) are owc. If*

$$\begin{gathered}\phi(Q(fx, hy), Q(Fx, Hy), Q(fx, Fx), \\ Q(hy, Hy), Q(fx, Hy), Q(Fx, hy)) \leq 0,\end{gathered} \tag{4.2}$$

for all $x, y \in X$ for which $fx \neq hy$ and $\phi \in \mathcal{F}_a$, then f, h, F and H have a unique common fixed point.

By Theorem 4.1 and Examples 3.2 - 3.11 we obtain the following theorem

Theorem 4.3. *Let f, h be self mappings of a quasi - metric space (X, Q) and F, H be maps of X into 2^X. If one of the following inequalities holds for all $x, y \in X$ with $fx \neq hy$:*

1)

$$\begin{aligned}Q(fx, hy) \leq\ & k \max\{D(Fx, Hy), D(fx, Fx), \\ & D(hy, Hy), D(fx, Hy), D(Fx, hy)\},\end{aligned}$$

where $k \in (0, 1)$,

2)

$$\begin{aligned}Q(fx, hy) \leq\ & h \max\{D(Fx, Hy), D(fx, Fx), D(hy, Hy), \\ & \frac{1}{2}[D(fx, Hy) + D(Fx, hy)]\},\end{aligned}$$

where $h \in (0, 1)$,

3)

$$\begin{aligned} Q(fx,hy) &\leq k\max\{D(Fx,Hy),\frac{D(fx,Fx)+D(hy,Hy)}{2}, \\ &\frac{D(fx,Hy)+D(Fx,hy)}{2}\}, \end{aligned}$$

where $k \in (0,1)$,

4)

$$\begin{aligned} Q(fx,hy) &\leq aD(Fx,Hy)+b\max\{D(fx,Fx),D(hy,Hy)\}+ \\ &+c\max\{D(Fx,Hy),D(fx,Hy),D(Fx,hy)\}, \end{aligned}$$

where $a,b,c \geq 0$ *and* $a+c<1$,

5)

$$\begin{aligned} Q(fx,hy) &\leq aD(Fx,Hy)+b(D(fx,Fx)+D(hy,Hy))+ \\ &+c\min\{D(fx,Hy),D(Fx,hy)\}, \end{aligned}$$

where $a,b,c \geq 0$ *and* $a+c<1$,

6)

$$\begin{aligned} Q(fx,hy) &\leq aD(Fx,Hy)+b(D(fx,Fx)+D(hy,Hy))+ \\ &+c\sqrt{D(fx,Hy)\cdot D(Fx,hy)}, \end{aligned}$$

where $a,b,c \geq 0$ *and* $a+c<1$,

7)

$$\begin{aligned} Q(fx,hy) &\leq \alpha\max\{D(Fx,Hy),D(fx,Fx),D(hy,Hy)\}+ \\ &+(1-\alpha)(aD(fx,Hy)+bD(Fx,hy)), \end{aligned}$$

where $0<\alpha<1$, $a,b \geq 0$ *and* $a+b<1$,

8)

$$[Q(fx,hy)]^2 \leq a[D(Fx,Hy)]^2+b\frac{\min\{D(fx,Hy),D(Fx,hy)\}}{1+D(fx,Fx)+D(hy,Hy)},$$

where $a,b \geq 0$ *and* $a+b<1$,

9)

$$Q(fx,hy) \leq aD(Fx,Hy)+b\frac{D(fx,Hy)+D(Fx,hy)}{1+D(fx,Fx)+D(hy,Hy)},$$

where $a,b \geq 0$ *and* $a+2b<1$,

10)

$$\begin{aligned} Q(fx,hy) &\leq \max\{cD(Fx,Hy),cD(fx,Fx), \\ &cD(hy,Hy),aD(fx,Hy)+bD(Fx,hy)\}, \end{aligned}$$

where $0 < c < 1$ *and* $\max\{c, a+b\} < 1$,

and (f, F) *and* (h, H) *are owc, then* f, h, F *and* H *have a unique common fixed point.*

By Theorem 4.2 and Examples 3.2 - 3.11 we obtain

Theorem 4.4. *Let* f, h, F *and* H *be self mappings of a quasi - metric space* (X, Q) *such that* (f, F) *and* (h, H) *are owc. If one of the following inequalities holds for all* $x, y \in X$ *for which* $fx \neq hy$:

1)

$$\begin{aligned} Q(fx, hy) &\leq k \max\{Q(Fx, Hy), Q(fx, Fx), \\ &\quad Q(hy, Hy), Q(fx, Hy), Q(Fx, hy)\}, \end{aligned}$$

where $k \in (0, 1)$,

2)

$$\begin{aligned} Q(fx, hy) &\leq k \max\{Q(Fx, Hy), Q(fx, Fx), Q(hy, Hy), \\ &\quad \frac{1}{2}[Q(fx, Hy) + Q(Fx, hy)]\}, \end{aligned}$$

where $k \in (0, 1)$,

3)

$$\begin{aligned} Q(fx, hy) &\leq k \max\{Q(Fx, Hy), \frac{Q(fx, Fx) + Q(hy, Hy)}{2}, \\ &\quad \frac{Q(fx, Hy) + Q(Fx, hy)}{2}\}, \end{aligned}$$

where $k \in (0, 1)$,

4)

$$\begin{aligned} Q(fx, hy) &\leq aQ(Fx, Hy) + b \max\{Q(fx, Fx), Q(hy, Hy)\} + \\ &\quad + c \max\{Q(Fx, Hy), Q(fx, Hy), Q(Fx, hy)\}, \end{aligned}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

5)

$$\begin{aligned} Q(fx, hy) &\leq aQ(Fx, Hy) + b(Q(fx, Fx) + Q(hy, Hy)) + \\ &\quad + c \min\{Q(fx, Hy), Q(Fx, hy)\}, \end{aligned}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

6)

$$\begin{aligned} Q(fx, hy) &\leq aQ(Fx, Hy) + b(Q(fx, Fx) + Q(hy, Hy)) + \\ &\quad + c\sqrt{Q(fx, Hy) \cdot Q(Fx, hy)}, \end{aligned}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

7)

$$\begin{aligned} Q(fx,hy) &\leq \alpha \max\{Q(Fx,Hy), Q(fx,Fx), Q(hy,Hy)\} + \\ &+(1-\alpha)(aQ(fx,Hy)+bQ(Fx,hy)), \end{aligned}$$

where $0<\alpha<1$*,* $a,b\geq 0$ *and* $a+b<1$*,*

8)

$$[Q(fx,hy)]^2 \leq a[Q(Fx,Hy)]^2 + b\frac{\min\{Q(fx,Hy),Q(Fx,hy)\}}{1+Q(fx,Fx)+Q(hy,Hy)},$$

where $a,b\geq 0$ *and* $a+b<1$*,*

9)

$$Q(fx,hy) \leq aQ(Fx,Hy) + b\frac{Q(fx,Hy)+Q(Fx,hy)}{1+Q(fx,Fx)+Q(hy,Hy)},$$

where $a,b\geq 0$ *and* $a+2b<1$*,*

10)

$$\begin{aligned} Q(fx,hy) &\leq \max\{cQ(Fx,Hy), cQ(fx,Fx), \\ &\quad cQ(hy,Hy), aQ(fx,Hy)+bQ(Fx,hy)\}, \end{aligned}$$

where $0<c<1$ *and* $\max\{c,a+b\}<1$*,*

then f,h,F *and* H *have a unique common fixed point.*

5. Applications

Theorem 5.1. *Let* f,h,F *and* H *be self mappings of a* G *- metric space* (X,G) *such that* (f,F) *and* (h,H) *are owc. If*

$$\begin{gathered} \phi(G(fx,hy,hy), G(Fx,Hy,Hy), G(fx,Fx,Fx), \\ G(hy,Hy,Hy), G(fx,Hy,Hy), G(Fx,hy,hy)) \leq 0, \end{gathered} \tag{5.1}$$

for all $x,y\in X$ *for which* $fx\neq hy$ *and* $\phi\in\mathcal{F}_a$*, then* f,h,F *and* H *have a unique common fixed point.*

Proof. As in Lemma 2.4, $Q(x,y)=G(x,y,y)$ is a quasi - metric on X. Then

$$\begin{aligned} G(fx,hy,hy) &= Q(fx,hy), G(Fx,Hy,Hy)=Q(Fx,Hy), \\ G(fx,Fx,Fx) &= Q(fx,Fx), G(hy,Hy,Hy)=Q(hy,Hy), \\ G(fx,Hy,Hy) &= Q(fx,Hy), G(Fx,hy,hy)=Q(Fx,hy). \end{aligned}$$

Then in (X,Q) by (5.1) we have

$$\begin{gathered} \phi(Q(fx,hy), Q(Fx,Hy), Q(fx,Fx), \\ Q(hy,Hy), Q(fx,Hy), Q(Fx,hy)) \leq 0, \end{gathered} \tag{5.2}$$

which is the inequality (4.2), Hence, the conditions of Theorem 4.2 are satisfied and f, h, F and H have a unique common fixed point. $\square$

By Theorem 5.1 and Examples 3.2 - 3.11 we obtain

Theorem 5.2. *Let f, h, F and H be self mappings of a G - metric space such that (f, F) and (h, H) are owc. If one of the following inequalities holds for all $x, y \in X$ for which $fx \neq hy$:*

1)

$$\begin{aligned} G(fx, hy, hy) \leq\ & k \max\{G(Fx, Hy, Hy), G(fx, Fx, Fx), \\ & G(hy, Hy, Hy), G(fx, Hy, Hy), G(Fx, hy, hy)\}, \end{aligned}$$

where $k \in (0, 1)$,

2)

$$\begin{aligned} G(fx, hy, hy) \leq\ & k \max\{G(Fx, Hy, Hy), G(fx, Fx, Fx), \\ & G(hy, Hy, Hy), \frac{1}{2}[G(fx, Hy, Hy) + G(Fx, hy, hy)]\}, \end{aligned}$$

where $k \in (0, 1)$,

3)

$$\begin{aligned} G(fx, hy, hy) \leq\ & k \max\{G(Fx, Hy, Hy), \\ & \frac{G(fx, Fx, Fx) + G(hy, Hy, Hy)}{2}, \\ & \frac{Q(fx, Hy, Hy) + G(Fx, hy, hy)}{2}\}, \end{aligned}$$

where $k \in (0, 1)$,

4)

$$\begin{aligned} G(fx, hy, hy) \leq\ & aG(Fx, Hy, Hy) + \\ & + b \max\{G(fx, Fx), G(gy, Hy, Hy)\} + \\ & + c \max\{G(Fx, Hy, Hy), G(fx, Hy, Hy), \\ & G(Fx, hy, hy)\}, \end{aligned}$$

where $a, b, c \geq 0$ and $a + c < 1$,

5)

$$\begin{aligned} G(fx, hy, hy) \leq\ & aG(Fx, Hy, Hy) + \\ & + b(G(fx, Fx, Fx) + G(hy, Hy, Hy)) + \\ & + c \min\{G(fx, Hy, Hy), G(Fx, hy, hy)\}, \end{aligned}$$

where $a, b, c \geq 0$ and $a + c < 1$,

6)

$$\begin{aligned} G(fx,hy,hy) &\le aG(Fx,Hy,Hy)+ \\ &+b(G(fx,Fx,Fx)+G(hy,Hy,Hy))+ \\ &+c\sqrt{G(fx,Hy,Hy)\cdot G(Fx,hy,hy)}, \end{aligned}$$

where $a,b,c \ge 0$ *and* $a+c<1$,

7)

$$\begin{aligned} G(fx,hy,hy) &\le \alpha \max\{G(Fx,Hy,Hy), \\ &G(fx,Fx,Fx), G(hy,Hy,Hy)\}+ \\ &+(1-\alpha)(aG(fx,Hy,Hy)+bG(Fx,hy,hy)), \end{aligned}$$

where $0<\alpha<1$, $a,b \ge 0$ *and* $a+b<1$,

8)

$$[G(fx,hy,hy)]^2 \le a[G(Fx,Hy,Hy)]^2 + b\frac{\min\{G(fx,Hy,Hy),G(Fx,hy,hy)\}}{1+G(fx,Fx,Fx)+G(hy,Hy,Hy)},$$

where $a,b \ge 0$ *and* $a+b<1$,

9)

$$G(fx,hy,hy) \le aG(Fx,Hy,Hy) + b\frac{G(fx,Hy,Hy)+G(Fx,hy,hy)}{1+G(fx,Fx,Fx)+G(hy,Hy,Hy)},$$

where $a,b \ge 0$ *and* $a+2b<1$,

10)

$$\begin{aligned} G(fx,hy,hy) &\le \max\{cG(Fx,Hy,Hy), cG(fx,Fx,Fx), \\ &cG(hy,Hy,Hy), aG(fx,Hy,Hy)+bG(Fx,hy,hy)\}, \end{aligned}$$

where $a,b \ge 0$, $0<c<1$ *and* $\max\{c,a+b\}<1$,
then f,h,F *and* H *have a unique common fixed point.*

Remark 5.3. *In the proof of the existence of fixed points in the papers* [18] - [21] *and in other papers is used "*$G(x,y,y)$*" instead "*$G(x,y,z)$*". Hence, the study of fixed points in* G *- metric spaces can be reduced in this cases at the study of fixed points in quasi - metric spaces.*

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