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A NOTE ON KLAMKIN'S INEQUALITY

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Abstract. In this note, we generalize a one variable inequality of Klamkin to the case of two variables.

1. Introduction

In 1974 M. Klamkin [3] proposed the following problem:
Let x be a nonnegative real number, and m, n be integers with $m \geq n \geq 1$. Prove that

$$(m+n)(1+x^m) \geq 2n\frac{1-x^{m+n}}{1-x^n}. \tag{1}$$

Later, M. Klamkin [2] himself solved the problem even in a more general case, assuming that m and n are real numbers. Note that for $x = 1$, the right-hand side of (1) is understood as its limit for $x \to 1$.

This intriguing inequality has some arithmetical applications and has been investigated by several researchers; see e.g. [1, 5, 6, 7]. In this note, we move a further step beyond the case of single variable x by considering two variables x and y. Our main results are shown in the following section.

We remark that there is another famous inequality, which is also called Klamkin's inequality, in the literature of triangle geometry [4], and these two inequalities should not be confused.

2. Generalizations of Klamkin's Inequality

Let $\mathbb{R}$ be the set of real numbers. For $a, b \in \mathbb{R}$, we denote the maximum and minimum of them by $a \vee b$ and $a \wedge b$, respectively. We establish the following inequality.

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Theorem 1. *Let $m \geq k \geq n \geq 1$, $x \geq 0$, $(1/x) \wedge x \leq y \leq (1/x) \vee x$ and $2k \geq m+n$. Then*

$$(m+k+n)(1+x^m)(1+y^k) \geq 3n\frac{1-x^{m+n}y^k}{1-x^n}. \tag{2}$$

When $x = 0$, we interpret $1/x$ as ∞. Therefore, y can be any nonnegative-valued member of $\mathbb{R} \cup \{\infty\}$ when $x = 0$. By convention, for $x = 1$ the right-hand side of (2) is understood as taking limit $x \to 1$.

Proof. We shall divide the proof into four cases: (i) $x = 1$, (ii) $x = 0$, (iii) $x > 1$, and (iv) $0 < x < 1$.

For (i), we have $x = y = 1$. Hence, the left-hand side of (2) is $4(m+k+n)$, while the right-hand side of (2) is $3(m+n)$ by using the L'Hôspital rule. The inequality (2) clearly holds.

For (ii), it suffices to show that

$$m+k+n \geq 3n,$$

which, in turn, is true by our assumptions.

For (iii), the assertion (2) is tantamount to the following form

$$(m+k+n)(x^m+1)(y^k+1)(x^n-1) \geq 3n(x^{m+n}y^k-1). \tag{3}$$

By using Klamkin's inequality (1) and noting that $x > 1$, we obtain

$$\begin{aligned}
E_1 &:= (m+k+n)(x^m+1)(y^k+1)(x^n-1) - 3n(x^{m+n}y^k-1) \\
&= \frac{m+k+n}{m+n}(m+n)(x^m+1)(x^n-1)(y^k+1) - 3n(x^{m+n}y^k-1) \\
&\geq \frac{m+k+n}{m+n} \cdot 2n(x^{m+n}-1)(y^k+1) - 3n(x^{m+n}y^k-1) \\
&= \frac{n}{m+n}\big[2(m+k+n)(x^{m+n}y^k + x^{m+n} - y^k - 1) \\
&\quad -3(m+n)(x^{m+n}y^k-1)\big] \\
&= \frac{n}{m+n}\big[(2k-m-n)(x^{m+n}y^k-1) \\
&\quad +2(m+k+n)(x^{m+n}-y^k)\big].
\end{aligned}$$

Since $x^{m+n} \geq x^k \geq y^k$, we get

$$E_1 \geq \frac{n}{m+n}(2k-m-n)(x^{m+n}y^k-1).$$

Since $2k \geq m+n$, $x > 1$, $y \geq (1/x)$ and $m+n-k \geq 0$, this yields

$$E_1 \geq 0,$$

which proves (3).

For (iv), the assertion (2) is equivalent to the form

$$(m+k+n)(1+x^m)(1+y^k)(1-x^n) \geq 3n(1-x^{m+n}y^k). \tag{4}$$

Utilizing Klamkin's inequality (1) and noting that $0 < x < 1$, we obtain

$$\begin{aligned}
E_2 &:= (m+k+n)(1+x^m)(1+y^k)(1-x^n) - 3n(1-x^{m+n}y^k) \\
&= \frac{m+k+n}{m+n}(m+n)(1+x^m)(1-x^n)(1+y^k) - 3n(1-x^{m+n}y^k) \\
&\geq \frac{m+k+n}{m+n} \cdot 2n(1-x^{m+n})(1+y^k) - 3n(1-x^{m+n}y^k) \\
&= \frac{n}{m+n}\big[2(m+k+n)(1-x^{m+n}y^k - x^{m+n} + y^k) \\
&\quad -3(m+n)(1-x^{m+n}y^k)\big] \\
&= \frac{n}{m+n}\big[(2k-m-n)(1-x^{m+n}y^k) \\
&\quad +2(m+k+n)(y^k - x^{m+n})\big].
\end{aligned}$$

Since $x^{m+n} \leq x^k \leq y^k$, we get

$$E_2 \geq \frac{n}{m+n}(2k-m-n)(1-x^{m+n}y^k).$$

Since $2k \geq m+n$, $x < 1$, $y \leq (1/x)$ and $m+n-k \geq 0$, this gives

$$E_2 \geq 0,$$

which implies (4). Thus the proof of Theorem 1 is complete. □

The following result can be proved analogously.

Theorem 2. *Let $m \geq k \geq n \geq 1$, $x \geq 0$, $(1/x) \wedge x \leq y \leq (1/x) \vee x$ and $k+n \geq m$. Then*

$$(m+k+n)(1+x^k)(1+y^m) \geq 3n\frac{1-x^{k+n}y^m}{1-x^n}. \tag{5}$$

Proof. As above, we separate the proof into four cases: (i) $x=1$, (ii) $x=0$, (iii) $x>1$, and (iv) $0<x<1$.

For (i), we have $x = y = 1$. Hence, the left-hand side of (5) is $4(m+k+n)$, while the right-hand side of (5) is $3(k+n)$ by using the L'Hôspital rule. The inequality (5) holds.

For (ii), we note that

$$m+k+n \geq 3n,$$

holds by our assumptions.

For (iii), the assertion (5) is equivalent to the following form

$$(m+k+n)(x^k+1)(y^m+1)(x^n-1) \geq 3n(x^{k+n}y^m-1). \tag{6}$$

By using Klamkin's inequality (1) and the fact $x > 1$, we have

$$\begin{aligned}
E_3 &:= (m+k+n)(x^k+1)(y^m+1)(x^n-1) - 3n(x^{k+n}y^m-1) \\
&= \frac{m+k+n}{k+n}(k+n)(x^k+1)(x^n-1)(y^m+1) - 3n(x^{k+n}y^m-1) \\
&\geq \frac{m+k+n}{k+n} \cdot 2n(x^{k+n}-1)(y^m+1) - 3n(x^{k+n}y^m-1) \\
&= \frac{n}{k+n}\big[2(m+k+n)(x^{k+n}y^m + x^{k+n} - y^m - 1) \\
&\quad -3(k+n)(x^{k+n}y^m-1)\big] \\
&= \frac{n}{k+n}\big[(2m-k-n)(x^{k+n}y^m-1) \\
&\quad +2(m+k+n)(x^{k+n}-y^m)\big].
\end{aligned}$$

Since $x^{k+n} \geq x^m \geq y^m$, we get

$$E_3 \geq \frac{n}{k+n}(2m-k-n)(x^{k+n}y^m-1).$$

Since $m \geq k \geq n$, $x > 1$, $y \geq (1/x)$ and $k+n-m \geq 0$, this yields

$$E_3 \geq 0,$$

which proves (6).

For (iv), the assertion (5) is equivalent to the form

$$(m+k+n)(1+x^k)(1+y^m)(1-x^n) \geq 3n(1-x^{k+n}y^m). \tag{7}$$

Using Klamkin's inequality (1) and noting that $0 < x < 1$, we have

$$\begin{aligned}
E_4 &:= (m+k+n)(1+x^k)(1+y^m)(1-x^n) - 3n(1-x^{k+n}y^m) \\
&= \frac{m+k+n}{k+n}(k+n)(1+x^k)(1-x^n)(1+y^m) - 3n(1-x^{k+n}y^m) \\
&\geq \frac{m+k+n}{k+n} \cdot 2n(1-x^{k+n})(1+y^m) - 3n(1-x^{k+n}y^m) \\
&= \frac{n}{k+n}\big[2(m+k+n)(1-x^{k+n}y^m - x^{k+n} + y^m) \\
&\quad -3(k+n)(1-x^{k+n}y^m)\big] \\
&= \frac{n}{k+n}\big[(2m-k-n)(1-x^{k+n}y^m) \\
&\quad +2(m+k+n)(y^m-x^{k+n})\big].
\end{aligned}$$

Since $x^{k+n} \leq x^m \leq y^m$, we get

$$E_4 \geq \frac{n}{k+n}(2m-k-n)(1-x^{k+n}y^m).$$

Since $2m \geq k+n$, $x < 1$, $y \leq (1/x)$ and $k+n-m \geq 0$, this yields

$$E_4 \geq 0,$$

which implies (7) and then concludes the proof of Theorem 2. □

We remark that one possible application of this type of inequalities is in probabilistic combinatorics (see e.g. [8]), where $0 \leq x \leq y \leq 1$ may represent normalized probabilities of some appropriate events.

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