"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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A GENERAL COMMON FIXED POINT THEOREM FOR WEAKLY COMMUTING PAIRS OF TYPE (KB)

ABDELKRIM ALIOUCHE AND VALERIU POPA

Abstract. We prove a general coincidence and a common fixed point theorem for two pairs of hybrid mappings satisfying an implicit relation using the concept of weak commutativity of type (KB) which generalizes theorem 2 of [14], theorem 3 of [6] and a theorem of [1].

1. Introduction and preliminaries

Fixed point theorems for single-valued and set-valued mappings have several applications in mathematical sciences and engineering, see [7] and [13] .

Let (X, d) a metric space and $B(X)$ the set of all nonempty bounded subsets of X. As in [2] and [3], we define the functions $\delta(A, B)$ and $D(A, B)$ by

$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$,

$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ for all $A, B \in B(X)$.

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B consists also of a single point b, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of δ that

$\delta(A, B) = \delta(B, A) \geq 0$,

$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$,

$\delta(A, B) = 0$ iff $A = B = \{a\}$,

$\delta(A, A) = diam A$ for all $A, B, C \in B(X)$.

Definition 1.1. A sequence $\{A_n\}$, $n = 1, 2...$of sets in $B(X)$ is said to be convergent to the closed set A in $B(X)$ if

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(i) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}$, where $a_n \in A_n$ and

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ. The set A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 1.2. *Let $\{A_n\}$ a sequence in $B(X)$ and $y \in X$ such that $\delta(A_n, y) \to 0$. Then, the sequence $\{A_n\}$ converges to $\{y\}$ in $B(X)$.*

Definition 1.3. Let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point x if whenever $\{\mathrm{x}_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$.

We say that F is a continuous mapping of X into $B(X)$ if F is continuous at each point x in X.

Definition 1.4. Let $f : X \to X$ and $F : X \to B(X)$.

i) A point $x \in X$ is a coincidence point of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.

ii) A point $x \in X$ is a strict coincidence point of f and F if $\{fx\} = Fx$.

iii) A point $x \in X$ is a fixed point of F if $x \in Fx$.

iv) A point $x \in X$ is a strict fixed point of F if $Fx = \{x\}$.

Definition 1.5. The mappings $f : X \to X$ and $F : X \to B(X)$ are weakly commuting if $fFx \in B(X)$ and for all $x \in X$

$$\delta(Ffx, fFx) \le \max\{\delta(fx, Fx), diam(fFx)\}.$$

Remark 1.6. *i) Two commuting mappings f and F are weakly commuting, but the converse is not true as it is shown in* [2].

ii) If F is also a single-valued mapping, then we obtain the definition of weakly commuting, see [12]

Definition 1.7. The mappings $f : X \to X$ and $F : X \to B(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $fx_n \to t$ and $Fx_n \to \{t\}$ as $n \to \infty$ for some $t \in X$.

If F is a single-valued self-mapping on X, then this definition reduces to that of [4].

Definition 1.8. The mappings $f, g : X \to X$ are called R-weakly commuting of type A_g if for all $x \in X$, there exists some $R > 0$ such that

$$d(ffx, gfx) \le Rd(fx, gx).$$

It was shown in [8] that compatible mappings are the R-weakly commuting mappings of type A_g, but the converse is not true in general.

Definition 1.9. The mappings $f : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting of type (KB) [6] at $x \in X$ if there exists some $R > 0$ such that

$$\delta(ffx, Ffx) \leq R\delta(fx, Fx).$$

f and F are weakly commuting of type (KB) on X if the above inequality holds for all $x \in X$.

If f and F are δ-compatible then they are weakly commuting of type (KB), but the converse is not true in general, see [6].

If F is a single-valued self-mapping on X, then this definition reduces to that of [8].

The following theorem was proved by [14].

Theorem 1.10. *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying the following conditions:*

$$\cup F(X) \subset J(X), \ \cup G(X) \subset I(X),$$

$$\begin{aligned}\delta(Fx, Gy) \ \leq \ & \alpha \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), \\ & +(1-\alpha)[aD(Ix, Gy) + bD(Jy, Fx)]\}\end{aligned}$$

for all $x, y \in X$, where $0 \leq \alpha < 1$, $a, b \geq 0$, $a + b < 1$ and $\dfrac{\alpha}{a-b} < 1 - a - b$. Suppose that one of $I(X)$ or $J(X)$ is complete. If both the pairs (F, I) and (G, J) are weakly commuting of type (KB) at coincidence points in X, then there exists a unique fixed point $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$..

The following theorem was proved by [6].

Theorem 1.11. *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying the following conditions:*

$$\cup F(X) \subset J(X), \ \cup G(X) \subset I(X),$$

$$\begin{aligned}\delta(Fx, Gy) \ \leq \ & \max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Gy), \\ & aD(Ix, Gy) + bD(Jy, Fx)\}\end{aligned}$$

for all $x, y \in X$, where $0 \leq c < 1$, $a, b \geq 0$, $a + b < 1$ and $c \max\{\dfrac{a}{1-a}, \dfrac{b}{1-b}\} < 1$. Suppose that one of $I(X)$ or $J(X)$ is complete. If both the pairs (F, I) and (G, J) are weakly commuting of type

(KB) at coincidence points in X, then there exists a unique fixed point $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$..

In [9] and [10], the study of fixed points for mappings satisfying implicit relations was introduced and the study of a pair of hybrid mappings satisfying implicit relations was initiated in [11].

It is our purpose in this paper is to prove a general coincidence and a common fixed point theorem for two pairs of hybrid mappings satisfying an implicit relation using the concept of weak commutativity of type (KB) which generalizes theorem 2 of [14], theorem 3 of [6] and a theorem of [1].

2. Implicit relation

Let Φ_6 the family of all real continuous mappings $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) : ϕ is decreasing in variables t_2, t_3, t_4, t_5 and t_6.

(ϕ_2) : there exists $h_1, h_2 \geq 0$ with $h_1 h_2 < 1$ such that

(ϕ_{2a}) : $\phi(u, v, v, u, u + v, 0) \leq 0$ implies $u \leq h_1 v$.

(ϕ_{2b}) : $\phi(u, v, u, v, 0, u + v) \leq 0$ implies $u \leq h_2 v$.

(ϕ_u) : $\phi(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Example 2.1. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\}$, $0 < a < 1$.

Example 2.2. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - a t_2^2 - b \dfrac{t_5 t_6}{1 + t_3^2 + t_4^2}$, $a > 0$, $b \geq 0$ *and* $a + b < 1$.

Example 2.3. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - a \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, t_4 t_6\} - c_3 t_5 t_6$, $c_1 > 0$, $c_2, c_3 \geq 0$, $c_1 + 2c_2 < 1$ *and* $c_1 + c_3 < 1$.

Example 2.4. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 + \dfrac{1}{1 + t_2} - \dfrac{a t_5 + b t_6}{1 + t_3 + t_4}$, $a, b > 0$ *and* $a + b < 1$.

Example 2.5. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 + \dfrac{1}{1 + t_2^2} - \dfrac{(a t_5 + b t_6)^2}{1 + t_3 + t_4}$, $a, b > 0$ *and* $a + b < 1$.

Example 2.6. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a t_2 - b \min\{t_3, t_4\} - c \min\{t_5, t_6\}$, $a, b, c > 0$, $a + b < 1$ *and* $a + c < 1$.

Example 2.7. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(a t_5 + b t_6)$, $0 \leq \alpha < 1$, $a, b \geq 0$, $a + b < 1$ *and* $\dfrac{\alpha}{a - b} < 1 - a - b$.

Example 2.8. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, $0 \le c < 1$, $a, b \ge 0$, $a + b < 1$ *and* $c \max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1$.

3. Main results

Theorem 3.1. *Let (X, d) be a metric space, $f, g : X \to X$ and $F, G : X \to B(X)$ be mapping satisfying the following conditions:*

$$\cup F(X) \subset g(X), \ \cup G(X) \subset f(X), \tag{3.1}$$

$$\begin{array}{c}\phi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \\ \delta(gy, Gy), D(fx, Gy), D(gy, Fx) \le 0\end{array} \tag{3.2}$$

for all $x, y \in X$ and $\phi \in \Phi_6$. Suppose that one of $f(X)$ or $g(X)$ is complete. Then F and f have a strict coincidence point and G and g have a strict coincidence point.

If the pairs (F, f) and (G, g) are weakly commuting of type (KB) at coincidence points in X, then there exists a unique fixed point $z \in X$ such that $\{z\} = \{fz\} = \{gz\} = Fz = Gz$..

Proof. Let x_0 be an arbitrary point in X. By (3.1), we can define a sequence $\{x_n\}$ in X such that

$$gx_{2n+1} \in Fx_{2n} = Z_{2n}, \ fx_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \ n = 0, 1, 2, ...$$

Using (3.2) and (ϕ_1), we have

$$\begin{aligned} 0 &\ge \phi(\delta(Fx_{2n}, Gx_{2n+1}), d(fx_{2n}, gx_{2n+1}), \delta(fx_{2n}, Fx_{2n}), \\ &\quad \delta(gx_{2n+1}, Gx_{2n+1}), D(fx_{2n}, Gx_{2n+1}), D(Fx_{2n}, gx_{2n+1})) \\ &\ge \phi(\delta(Z_{2n}, Z_{2n+1}), \delta(Z_{2n-1}, Z_{2n}), \delta(Z_{2n-1}, Z_{2n}), \\ &\quad \delta(Z_{2n}, Z_{2n+1}), D(Z_{2n-1}, Z_{2n+1}), 0) \\ &\ge \phi(\delta(Z_{2n}, Z_{2n+1}), \delta(Z_{2n-1}, Z_{2n}), \delta(Z_{2n-1}, Z_{2n}), \\ &\quad \delta(Z_{2n}, Z_{2n+1}), \delta(Z_{2n-1}, Z_{2n}) + \delta(Z_{2n}, Z_{2n+1}), 0) \end{aligned}$$

By (ϕ_{2a}), we obtain

$$\delta(Z_{2n}, Z_{2n+1}) \le h_1 \delta(Z_{2n-1}, Z_{2n}).$$

In the same manner, applying (3.2) we get

$$\begin{aligned} 0 &\ge \phi(\delta(Fx_{2n+2}, Gx_{2n+1}), d(fx_{2n+2}, gx_{2n+1}), \delta(fx_{2n+2}, Fx_{2n+2}), \\ &\quad D(gx_{2n+1}, Gx_{2n+1}), D(fx_{2n+2}, Gx_{2n+1}), D(Fx_{2n+2}, gx_{2n+1})) \\ &\ge \phi(\delta(Z_{2n+2}, Z_{2n+1}), \delta(Z_{2n+1}, Z_{2n}), \delta(Z_{2n+1}, Z_{2n+2}), \\ &\quad \delta(Z_{2n}, Z_{2n+1}), 0, \delta(Z_{2n}, Z_{2n+1}) + \delta(Z_{2n+1}, Z_{2n+2})). \end{aligned}$$

By (ϕ_{2b}), we obtain

$$\delta(Z_{2n+1}, Z_{2n+2}) \leq h_2 \delta(Z_{2n}, Z_{2n+1}).$$

Let $c = h_1 h_2$. Then we get

$$\begin{aligned} \delta(Z_{2n}, Z_{2n+1}) &\leq c^n \delta(Fx_0, Gx_1). \\ \delta(Z_{2n+1}, Z_{2n+2}) &\leq c^n \delta(Gx_1, Fx_2). \end{aligned}$$

Put $M = \max\{\delta(Fx_0, Gx_1), \delta(Gx_1, Fx_2)\}$. It follows from the above inequality that if z_n is an arbitrary point in the set Z_n we obtain

$$\begin{aligned} d(z_n, z_{n+1}) &\leq \delta(Z_n, Z_{n+1}) \\ &\leq c^n M. \end{aligned}$$

Therefore, $\{z_n\}$ is a Cauchy sequence in X. As $gx_{2n+1} \in Fx_{2n} = Z_{2n}$, hence

$$d(gx_{2n+1}, gx_{2m+1}) \leq \delta(Z_{2n}, Z_{2m}) < \epsilon,$$

i.e., $\{gx_{2n+1}\}$ is a Cauchy sequence in $g(X)$. Assume that $g(X)$ is complete. Then, it converges to $z \in g(X)$ and so there exists $v \in X$ such that $z = gv$. Since $fx_{2n} \in Gx_{2n-1} = Z_{2n-1}$ we have

$$d(fx_{2n}, gx_{2n+1}) \leq \delta(Z_{2n-1}, Z_{2n})$$

Therefore, the sequence $\{fx_{2n}\}$ converges to z.. As

$$\begin{aligned} \delta(Fx_{2n}, z) &\leq \delta(Fx_{2n}, fx_{2n}) + d(fx_{2n}, z) \\ &\leq \delta(Z_{2n}, Z_{2n-1}) + d(fx_{2n}, z). \end{aligned}$$

and so $\lim_{n\to\infty} \delta(Fx_{2n}, z) = 0$. In the same manner, we obtain $\lim_{n\to\infty} \delta(Gx_{2n-1}, z) = 0$.

Using (3.2) and (ϕ_1) we have

$$\begin{aligned} &\phi(\delta(Fx_{2n}, Gv), d(fx_{2n}, gv), \delta(fx_{2n}, Fx_{2n}), \\ &\delta(gv, Gv), D(fx_{2n}, Gv), D(Fx_{2n}, gv)) \leq 0 \end{aligned}$$

Letting $n \to \infty$, we get

$$\phi(\delta(z, Gv), 0, 0, \delta(z, Gv), \delta(z, Gv), 0) \leq 0.$$

By (ϕ_{2a}) we obtain $\delta(z, Gv) = 0$ and hence $Gv = \{gv\} = \{z\}$.

Since $\cup G(X) \subset f(X)$, there exists $u \in X$ such that $\{fu\} = Gv = \{gv\} = \{z\}$.

If $Fu \neq \{z\}$, applying (3.2) we have

$$\begin{aligned} 0 &\geq \phi(\delta(Fu, Gv), d(fu, gv), \delta(fu, Fu), \delta(gv, Gv), D(fu, Gv), D(Fu, gv)) \\ &\geq \phi(\delta(Fu, z), 0, \delta(Fu, z), 0, 0, \delta(Fu, z)) \end{aligned}$$

By (ϕ_{2b}) we get $\delta(Fu, z) = 0$ and so $Fu = \{fu\} = Gv = \{gv\} = \{z\}$.

Since $Fu = \{fu\}$ and the pair (F, f) is weakly commuting of type (KB) at coincidence points in X, we obtain $\delta(ffu, Ffu) \leq R\delta(Iu, Fu)$ which gives $Fz = \{fz\}$.

Again since $Gv = \{gv\}$ and the pair (G, g) is weakly commuting of type (KB) at coincidence points in X, we get $\delta(ggv, Ggv) \leq R\delta(gv, Gv)$ which gives $Gz = \{gz\}$.

If $Fz \neq \{z\}$, using (3.2) we have

$$\begin{aligned} 0 &\geq \phi(\delta(Fz, Gv), d(fz, gv), \delta(fz, Fz), \delta(gv, Gv), D(fz, Gv), D(Fz, gv)) \\ &\geq \phi(\delta(Fz, z), \delta(Fz, z), 0, 0, \delta(Fz, z), \delta(Fz, z)) \end{aligned}$$

which is a contradiction of (ϕ_u) and so $Fz = \{fz\} = \{z\}$. Similarly, $Gz = \{gz\} = \{z\}$. Therefore, we have $Fz = \{fz\} = Gz = \{gz\} = \{z\}$. □

Theorem 4 generalizes a theorem of [1].

Corollary 3.2. *Theorem 2.*

Proof. It suffices to take example 7. □

Corollary 3.3. *Theorem 3.*

Proof. It suffices to take example 8. □

4. Conclusion

We proved a general common fixed point theorem for two pairs of hybrid mappings satisfying an implicit relation using the weak commutativity of type (KB). Our theorem generalizes theorem 2 of [14], theorem 3 of [6] and a theorem of [1].

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A. Aliouche

Department of Mathematics, University of Larbi Ben M' Hidi, Oum-El-Bouaghi 04000, ALGERIA, e-mail: alioumath@yahoo.fr

V. Popa

"Vasile Alecsandri" University of Bacău, Calea Mărăşeşti 157, Bacău 600115, ROMANIA, e-mail: vpopa@ub.ro