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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 2, 13 - 22

$\mathcal{A}$-EXPANSION CONTINUOUS MAPS AND $(\mathcal{A}, \mathcal{B})$-WEAKLY CONTINUOUS MAPS IN HEREDITARY GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study $\mathcal{A}$-expansion continuous maps, closed-$\mathcal{B}$-continuous maps, $(\mathcal{A}, \mathcal{B})$-weakly continuous maps and closed $(\mathcal{A}, \mathcal{B})$-continuous maps in hereditary generalized topological spaces. We, also present several results including decompositions of (μ, λ)-continuity and $(\mathcal{A}, \mathcal{I}d)$-weakly continuity.

1. Introduction and Preliminaries

Let (X, μ) be a generalized topological space (GTS) [2]. A GTS (X, μ) is called a quasi topological space [5] if μ is closed under finite intersection. A nonempty family $\mathcal{H}$ of subsets of X is said to be a *hereditary class* [4], if, from $A \in \mathcal{H}$ and $B \subset A$, it follows that $B \in \mathcal{H}$. Given a generalized topological space (X, μ) with a hereditary class $\mathcal{H}$, for each $A \subseteq X$, $A^*(\mathcal{H}, \mu)$=$\{x \in X$: for every μ-open set V containing x, $V \cap A \notin \mathcal{H}\}$ and $c^*_\mu(A) = A \cup A^*(\mathcal{H}, \mu)$[4]. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space. If there is no confusion, we simply write A^* instead of $A^*(\mathcal{H}, \mu)$. A hereditary class $\mathcal{H}$ is μ-codense [4] iff $\mu \cap \mathcal{H} = \emptyset$.

Definition 1.1. [2] *A map $f : (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-continuous iff $U \in \lambda$ implies that $f^{-1}(U) \in \mu$.*

Keywords and phrases: generalized topology, $\mathcal{A}$-expansion continuity and $\mathcal{B}$-expansion continuity, $(\mathcal{A}, \mathcal{B})$-weakly continuity, closed $(\mathcal{A}, \mathcal{B})$-continuity.
(2010)Mathematics Subject Classification: 54A05.

Definition 1.2. [6] *A map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is said to be weakly (μ, λ)-$\mathcal{H}$-continuous (briefly $w_{(\mu,\lambda)}$-$\mathcal{H}$-c), if for each $x \in X$ and each λ-open neighbourhood V of $f(x)$, there exists a μ-open neighbourhood U of x such that $f(U) \subset c^*_\lambda(V)$.*

Theorem 1.3. [6] *A map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is weakly (μ, λ)-$\mathcal{H}$-continuous if and only if for each λ-open set $V \subset Y$, $f^{-1}(V) \subset i_\mu(f^{-1}(c^*_\lambda(V)))$.*

Definition 1.4. [6] *A map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is said to be weak* (μ, λ)-$\mathcal{H}$-continuous (briefly $w^*_{(\mu,\lambda)}$-$\mathcal{H}$-c), if for each $V \in \lambda$, $f^{-1}(f^*_r(V))$ is μ-closed in (X, μ), where $f^*_r(V) = V^* - i_\lambda(V)$ is λ-closed in $(Y, \lambda, \mathcal{H})$.*

A generalized topological space (X, μ) is called a quasi-topological space [3] if μ is closed under finite intersections. For $A \subset X$ we denote by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [1], [2], [5]).

Lemma 1.5. [5] *If (X, μ) is a quasi-topological space, then $i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$ for all subsets A and B of X.*

2. Expansion of μ-open sets in hereditary generalized topological spaces

Definition 2.1. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. A map $\mathcal{A} : \mu \to 2^X$ is said to be an expansion on $(X, \mu, \mathcal{H})$ if $U \subseteq \mathcal{A}U$ for each $U \in \mu$.*

Remark 2.2. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space.*

1. *Define $cl : \mu \to 2^X$ by $clU = c^*_\mu U$ for each $U \in \mu$. Then cl is an expansion on $(X, \mu, \mathcal{H})$ because $U \subseteq c^*_\mu U = clU$ for each $U \in \mu$.*
2. *Define $\mathcal{F} : \mu \to 2^X$ by $\mathcal{F}U = (f^*_r U)^c$ for each $U \in \mu$. Then $\mathcal{F}$ is an expansion on $(X, \mu, \mathcal{H})$ because $\mathcal{F}U = (f^*_r U)^c = (U^* - i_\mu U)^c = (U^* - U)^c = (U^* \cap U^c)^c = (U^*)^c \cup U \supseteq U$ for each $U \in \mu$.*

Definition 2.3. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. A pair of expansions $\mathcal{A}$ and $\mathcal{B}$ on $(X, \mu, \mathcal{H})$ is said to be mutually dual if $\mathcal{A}U \cap \mathcal{B}U = U$ for each $U \in \mu$.*

Example 2.4. *Let $X = \mathbb{R}$ be the set of all real numbers, $\mu = \{\emptyset, \mathbb{Q}\}$ and $\mathcal{H} = \{\emptyset, \sqrt{2}\}$. Let $\mathcal{A}\emptyset = \emptyset$, $\mathcal{A}\mathbb{Q} = \mathbb{Q} \cup \sqrt{2}$. Let $\mathcal{F}U = (f_r^* U)^c$ for each $U \in \mu$. Now $\mathcal{F}\emptyset = X$, $\mathcal{F}\mathbb{Q} = (f_r^*\mathbb{Q})^c = (\mathbb{Q}^* - i_\mu \mathbb{Q})^c = (\mathbb{R} - \mathbb{Q})^c = \mathbb{Q}$. Then $\mathcal{A}$ and $\mathcal{F}$ are expansions on $(X, \mu, \mathcal{H})$ and $\mathcal{F}$ is mutually dual to $\mathcal{A}$.*

Proposition 2.5. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. Then the expansions cl and $\mathcal{F}$ are mutually dual.*

Proof. Let $U \in \mu$. Now,

$$
\begin{aligned}
clU \cap \mathcal{F}U =& c_\mu^* U \cap (f_r^* U)^c = c_\mu^* U \cap (U^* - U)^c \\
=& c_\mu^* U \cap (U^* \cap U^c)^c = c_\mu^* U \cap ((U^*)^c \cup U) \\
=& (c_\mu^* U \cap (U^*)^c) \cup (c_\mu^* U \cap U) \\
=& ((U \cup U^*) \cap (U^*)^c) \cup ((U \cup U^*) \cap U) \\
=& ((U \cap (U^*)^c) \cup (U^* \cap (U^*)^c)) \cup ((U \cap U) \cup (U^* \cap U)) \\
=& (U \cap (U^*)^c) \cup U \cup (U^* \cap U) = U.
\end{aligned}
$$

□

Definition 2.6. *Let (X, μ) be a generalized topological space. Let $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space. Let $\mathcal{A}$ be an expansion on Y. A map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is said to be $\mathcal{A}$-expansion continuous if $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{A}V))$, for each $V \in \lambda$.*

Remark 2.7. *A weakly (μ, λ)-$\mathcal{H}$-continuous map is $\mathcal{A}$-expansion continuous by Theorem* 1.3 *and 1 of Remark* 2.2.

Theorem 2.8. *Let (X, μ) be a quasi topological space and $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space. Let $\mathcal{A}$ and $\mathcal{B}$ be two mutually dual expansions on $(Y, \lambda, \mathcal{H})$. Then a map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is (μ, λ)-continuous if and only if f is $\mathcal{A}$-expansion continuous and $\mathcal{B}$-expansion continuous.*

Proof. **Necessity:** Since $\mathcal{A}$ and $\mathcal{B}$ are mutually dual expansions on $(Y, \lambda, \mathcal{H})$, $\mathcal{A}V \cap \mathcal{B}V = V$ for each $V \in \lambda$. Let $V \in \lambda$, then $f^{-1}(V) = f^{-1}(\mathcal{A}V) \cap f^{-1}(\mathcal{B}V)$. Since f is (μ, λ)-continuous, $f^{-1}(V) = i_\mu(f^{-1}(V))$. So, $f^{-1}(V) = i_\mu(f^{-1}(\mathcal{A}V) \cap f^{-1}(\mathcal{B}V)) = i_\mu(f^{-1}(\mathcal{A}V)) \cap i_\mu(f^{-1}(\mathcal{B}V))$ by Lemma 1.5. Thus $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{A}V))$ and $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$, for each $V \in \lambda$. Hence f is $\mathcal{A}$-expansion continuous and $\mathcal{B}$-expansion continuous.

Sufficiency: Let $\mathcal{A}$, $\mathcal{B}$ be two mutually dual expansions on $(Y, \lambda, \mathcal{H})$ and f be $\mathcal{A}$-expansion continuous and $\mathcal{B}$-expansion continuous. Then $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{A}V))$, for each $V \in \lambda$ and $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$, for each $V \in \lambda$. Also $\mathcal{A}V \cap \mathcal{B}V = V$. Therefore, $f^{-1}(\mathcal{A}V) \cap f^{-1}(\mathcal{B}V)$. Hence, $i_\mu(f^{-1}(V)) = (i_\mu(f^{-1}(\mathcal{A}V)) \cap i_\mu(f^{-1}(\mathcal{B}V))) \supseteq (f^{-1}(V) \cap f^{-1}(V)) = f^{-1}(V)$. So, $i_\mu(f^{-1}(V)) \supseteq f^{-1}(V)$. But, $i_\mu(f^{-1}(V)) \subseteq f^{-1}(V)$ always. Therefore, $f^{-1}(V) = i_\mu(f^{-1}(V))$. This implies that $f^{-1}(V) \in \mu$ for each $V \in \lambda$. Therefore f is (μ, λ)-continuous. □

Definition 2.9. *Let (X, μ) be a generalized topological space and $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space. Let $\mathcal{B}$be an expansion on $(Y, \lambda, \mathcal{H})$. A map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is said to be closed $\mathcal{B}$-continuous if $f^{-1}((\mathcal{B}V)^c)$ is μ-closed in (X, μ) for each $V \in \lambda$.*

Remark 2.10. *A weak* (μ, λ)-$\mathcal{H}$-continuous map is a closed-$\mathcal{B}$-continuous map.*

Proposition 2.11. *A closed $\mathcal{B}$-continuous map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $\mathcal{B}$-expansion continuous.*

Proof. We first prove $(f^{-1}((\mathcal{B}V)^c))^c{=}f^{-1}(\mathcal{B}V)$. Let $x \in (f^{-1}((\mathcal{B}V)^c))^c$. Then $x \notin (f^{-1}((\mathcal{B}V)^c))$. Hence $f(x) \notin (\mathcal{B}V)^c$, so $f(x) \in \mathcal{B}V$ and $x \in f^{-1}(\mathcal{B}V)$. So, $(f^{-1}(\mathcal{B}V)^c)^c \subseteq f^{-1}(\mathcal{B}V)$. Conversely, let $x \in f^{-1}(\mathcal{B}V)$. Then $f(x) \in \mathcal{B}V$ and $f(x) \notin (\mathcal{B}V)^c$. Now $x \notin f^{-1}((\mathcal{B}V)^c)$, implies $x \in (f^{-1}((\mathcal{B}V)^c))^c$. So, $f^{-1}(\mathcal{B}V) \subseteq (f^{-1}((\mathcal{B}V)^c))$. Therefore, $(f^{-1}((\mathcal{B}V)^c))^c{=}f^{-1}(\mathcal{B}V)$. Since $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is a closed $\mathcal{B}$-continuous map, $f^{-1}((\mathcal{B}V)^c)$ is a μ-closed set in (X, μ). This implies $(f^{-1}(\mathcal{B}V)^c)^c \in \mu$. Hence $f^{-1}(\mathcal{B}V) \in \mu$ and so $f^{-1}(\mathcal{B}V){=}i_\mu(f^{-1}(\mathcal{B}V))$. Also, note that $V \subseteq \mathcal{B}V$ and this implies $f^{-1}(V) \subseteq f^{-1}(\mathcal{B}V) = i_\mu(f^{-1}(\mathcal{B}V))$. Therefore $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$ for each $V \in \lambda$. Hence f is $\mathcal{B}$-expansion continuous. □

Example 2.12. *Let $X = Y = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\lambda = \{\emptyset, \{a\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Let $\mathcal{B}(\{a\}) = \{a, c\}$. Then the identity map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is (μ, λ)-continuous but not closed $\mathcal{B}$-continuous, because $f^{-1}(\mathcal{B}(\{a\})^c) = f^{-1}(\{b\}) = \{b\}$ is not μ-closed in (X, μ).*

Definition 2.13. *An expansion $\mathcal{A}$ on $(X, \mu, \mathcal{H})$ is said to be open if $\mathcal{A}V \in \mu$ for all $V \in \mu$.*

Definition 2.14. *An open expansion $\mathcal{A}$ on $(X, \mu, \mathcal{H})$ is said be idempotent if $\mathcal{A}(\mathcal{A}V) = \mathcal{A}V$ for all $V \in \mu$.*

Remark 2.15. (1) *The expansion* cl=c^*_μ *is idempotent .*
(2) *If* $\mathcal{H}$ *is* μ*-codense, then the expansion* $\mathcal{F}V = (f^*_r V)^c$ *for all* $V \in \mu$ *is idempotent. In fact the expansion* $\mathcal{F}$ *is open also.*

Proposition 2.16. *Let* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$*, where* $\mathcal{H}$ *is* λ*-codense and* $\mathcal{B}$ *be an open and idempotent expansion on* $(Y, \lambda, \mathcal{H})$*. Then* f *is* $\mathcal{B}$*-expansion continuous if and only if closed* $\mathcal{B}$*-continuous.*

Proof. The Sufficiency follows from Proposition 2.11.
Necessity: Let f be $\mathcal{B}$-expansion continuous, where $\mathcal{B}$ is idempotent and $V \in \lambda$. Since $\mathcal{B}V \in \lambda$ and $\mathcal{B}(\mathcal{B}V) = \mathcal{B}V$, then $f^{-1}(\mathcal{B}V) \subseteq i_\mu(f^{-1}(\mathcal{B}(\mathcal{B}V))) = i_\mu(f^{-1}(\mathcal{B}V))$. Thus $f^{-1}(\mathcal{B}V) \in \mu$ and therefore f is closed $\mathcal{B}$-continuous. □

Theorem 2.17. *Let* (X, μ) *be a quasi topological space and* $(Y, \lambda, \mathcal{H})$ *be a hereditary generalized topological space, where* $\mathcal{H}$ *is* λ*-codense. Let* $\mathcal{A}$*,* $\mathcal{B}$ *be two mutually dual expansions on* $(Y, \lambda, \mathcal{H})$*,* $\mathcal{B}$ *is open and idempotent. Then a map* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ *is* (μ, λ)*-continuous if and only if* f *is* $\mathcal{A}$*-expansion continuous and closed* $\mathcal{B}$*-continuous.*

Proof. Follows from Theorem 2.8 and Proposition 2.16. □

Corollary 2.18. *Let* (X, μ) *be a quasi topological space and* $(Y, \lambda, \mathcal{H})$ *be a hereditary generalized topological space, where* $\mathcal{H}$ *is* λ*-codense. Then a map* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ *is* (μ, λ)*-continuous if and only if* f *is weakly* (μ, λ)*-*$\mathcal{H}$*-continuous and weak**(μ, λ)*-*$\mathcal{H}$*-continuous.*

Proof. Follows from Remark 2.7 and Remark 2.10. □

3. $(\mathcal{A}, \mathcal{B})$-weak continuity

Definition 3.1. *Let* $\mathcal{A}$ *be an expansion on* $(X, \mu, \mathcal{H})$*. A subset* A *of* X *is said to be* $\mathcal{A}$*-open if for each* $x \in A$ *there exist* $U \in \mu(x)$ *such that* $\mathcal{A}U \subseteq A$*. A subset* B *of* X *is* $\mathcal{A}$*-closed if its complement* $X - B$ *is* $\mathcal{A}$*-open.*

Note that $\mathcal{A}$-open sets are also μ-open sets in $(X, \mu, \mathcal{H})$.
Definition 3.2. *Let* (X, μ) *be a GTS,* $(Y, \lambda, \mathcal{H})$ *be a hereditary generalized topological space,* $\mathcal{A}$ *and* $\mathcal{B}$ *be expansions on* (X, μ) *and* $(Y, \lambda, \mathcal{H})$ *respectively. We say that a map* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ *is* $(\mathcal{A}, \mathcal{B})$*-weakly continuous if and only if* $\mathcal{A}f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$ *for every* $V \in \lambda$*.*

The expansion $\mathcal{A}V = V$ denoted by $\mathcal{A} = \mathcal{I}d$ is called the identity expansion. It is clear that the (μ, λ)-continuity is equivalent to $\mathcal{I}d$-expansion continuity.

Remark 3.3. *Let (X, μ) be a GTS $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space, $\mathcal{A}$ and $\mathcal{B}$ be expansions on (X, μ) and $(Y, \lambda, \mathcal{H})$ respectively. Let $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ be a map.*

(1) *Observe that when $\mathcal{A} = \mathcal{I}d$ and $\mathcal{B} = \mathcal{I}d$, f is $(\mathcal{A}, \mathcal{B})$-weakly continuous if and only if f is (μ, λ)-continuous in the usual sense* [2].
(2) *Let $\mathcal{A} = \mathcal{I}d$ and $\mathcal{B} = c^*_\lambda$. Then f is $(\mathcal{A}, \mathcal{B})$-weakly continuous if and only if f is weakly (μ, λ)-$\mathcal{H}$-continuous in the sense* [6].
(3) *Let $\mathcal{A} = \mathcal{I}d$ and $\mathcal{B}$ be any expansion. Then f is $(\mathcal{A}, \mathcal{B})$-weakly continuous if and only if f is $\mathcal{B}$-expansion continuous.*

Definition 3.4. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. An expansion $\mathcal{A}$ on $(X, \mu, \mathcal{H})$ is said to be subadditive if for every collection of μ-open set $\{U_\alpha : \alpha \in \Delta\}$, $\mathcal{A}(\cup_{\alpha\in\Delta} U_\alpha) \subseteq \cup_{\alpha\in\Delta} \mathcal{A}U_\alpha$.*

Lemma 3.5. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and expansion $\mathcal{A}$ on $(X, \mu, \mathcal{H})$ is subadditive. Then for every $\mathcal{A}$-open set U we have $\mathcal{A}U = U$.*

Proof. Let U be an $\mathcal{A}$-open set. Then for every $x \in U$ there exists a μ-open set U_x such that $x \in U_x \subseteq \mathcal{A}U_x \subseteq U$. Therefore, $\cup U_x \subseteq \cup \mathcal{A}U_x \subseteq U$, so $\cup U_x \subseteq \mathcal{A}(\cup U_x) \subseteq U$. Therefore, $U \subseteq \mathcal{A}U \subseteq U$ and hence $\mathcal{A}U = U$. □

Theorem 3.6. *Let (X, μ) be a quasi topological space and $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space and $\mathcal{A}$ be an expansion on (X, μ). If $\mathcal{B}$ and $\mathcal{B}^*$ are mutually dual expansions on $(Y, \lambda, \mathcal{H})$, then a map $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $(\mathcal{A}, \mathcal{I}d)$-weakly continuous if and only if f is both $(\mathcal{A}, \mathcal{B})$-weakly continuous and $(\mathcal{A}, \mathcal{B}^*)$-weakly continuous.*

Proof. Suppose f is $(\mathcal{A}, \mathcal{I}d)$-weakly continuous. Then for every $V \in \lambda$ we have $\mathcal{A}f^{-1}(V) \subseteq i_\mu(f^{-1}(V)) \subseteq i_\mu(f^{-1}(\mathcal{B}V)) \cap i_\mu(f^{-1}(\mathcal{B}^*V))$. This implies that f is $(\mathcal{A}, \mathcal{B})$-weakly continuous and $(\mathcal{A}, \mathcal{B}^*)$-weakly continuous. Conversely, Suppose f is $(\mathcal{A}, \mathcal{B})$-weakly continuous and $(\mathcal{A}, \mathcal{B}^*)$-weakly continuous. Then for every $V \in \lambda$ we have $\mathcal{A}f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$ and $\mathcal{A}f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}^*V))$. Thus $\mathcal{A}f^{-1}(V) \subseteq$
$i_\mu(f^{-1}(\mathcal{B}V)) \cap i_\mu(f^{-1}(\mathcal{B}^*V)) = i_\mu(f^{-1}(\mathcal{B}V \cap \mathcal{B}^*V))$. Since $\mathcal{B}$ and $\mathcal{B}^*$ are mutually dual expansions on $(Y, \lambda, \mathcal{H})$, we get that $\mathcal{A}f^{-1}(V) \subseteq i_\mu(f^{-1}(V))$, which implies that f is $(\mathcal{A}, \mathcal{I}d)$-weakly continuous. □

Corollary 3.7. [4] *Let (X,μ) be a quasi topological space and $(Y,\lambda,\mathcal{H})$ be a hereditary generalized topological space. A map $f:(X,\mu)\to(Y,\lambda,\mathcal{H})$ is weakly (μ,λ)-$\mathcal{H}$-continuous if and only if for each $V\in\lambda$, $f^{-1}(V)\subseteq i_\mu(f^{-1}(c^*_\lambda(V)))$.*

Proof. According to Remark 3.3(2), f is weakly (μ,λ)-$\mathcal{H}$-continuous if and only if $(\mathcal{I}d,c^*_\lambda)$-weakly continuous. Now since the $\mathcal{I}d$ and c^*_λ are mutually dual expansions on $(Y,\lambda,\mathcal{H})$, the conclusion follows from Theorem 3.6. □

Corollary 3.8. *Let (X,μ) be a quasi topological space and $(Y,\lambda,\mathcal{H})$ be a hereditary generalized topological space. Let $\mathcal{B}$ and $\mathcal{B}^*$ be two mutually dual expansions on $(Y,\lambda,\mathcal{H})$. Then a map $f:(X,\mu)\to(Y,\lambda,\mathcal{H})$ is (μ,λ)-continuous if and only if f is $\mathcal{B}$-expansion continuous and $\mathcal{B}^*$-expansion continuous.*

Proof. According to Remark 3.3(1), f is (μ,λ)-continuous if and only if $(\mathcal{I}d,\mathcal{I}d)$-weakly continuous. Also, Remark 3.3(3) assures f is $\mathcal{B}$-expansion continuous and $\mathcal{B}^*$-expansion continuous if and only if f is $(\mathcal{I}d,\mathcal{B})$-weakly continuous and $(\mathcal{I}d,\mathcal{B}^*)$-weakly continuous. Now since the $\mathcal{B}$ and $\mathcal{B}^*$ are mutually dual expansions on $(Y,\lambda,\mathcal{H})$, then the conclusion follows from Theorem 3.6. □

Definition 3.9. *Let (X,μ) be a GTS and $\mathcal{A}$ be an expansion on (X,μ). Let $(Y,\lambda,\mathcal{H})$ be a hereditary generalized topological space and $\mathcal{B}$ be an expansion on $(Y,\lambda,\mathcal{H})$. A map $f:(X,\mu)\to(Y,\lambda,\mathcal{H})$ is said to be closed $(\mathcal{A},\mathcal{B})$-continuous if $f^{-1}((\mathcal{B}V)^c)$ is $\mathcal{A}$-closed in (X,μ) for each $V\in\lambda$.*

Theorem 3.10. *Let (X,μ) be a GTS and $(Y,\lambda,\mathcal{H})$ be a hereditary generalized topological space and $\mathcal{A}$ and $\mathcal{B}$ be expansions on (X,μ) and $(Y,\lambda,\mathcal{H})$ respectively. If $\mathcal{A}$ is subadditive and monotone expansion then every closed $(\mathcal{A},\mathcal{B})$-continuous function is $(\mathcal{A},\mathcal{B})$-weakly continuous.*

Proof. Let $f:(X,\mu)\to(Y,\lambda,\mathcal{H})$ be closed $(\mathcal{A},\mathcal{B})$-continuous and let $V\in\lambda$. We first prove $(f^{-1}((\mathcal{B}V)^c))^c=f^{-1}(\mathcal{B}V)$. Let $x\in(f^{-1}((\mathcal{B}V)^c))^c$. Then $x\notin(f^{-1}((\mathcal{B}V)^c))$. Hence $f(x)\notin(\mathcal{B}V)^c$, so $f(x)\in\mathcal{B}V$ and $x\in f^{-1}(\mathcal{B}V)$. So, $(f^{-1}(\mathcal{B}V)^c)^c\subseteq f^{-1}(\mathcal{B}V)$. Conversely, let $x\in f^{-1}(\mathcal{B}V)$. Then $f(x)\in\mathcal{B}V$ and $f(x)\notin(\mathcal{B}V)^c$. Now $x\notin f^{-1}((\mathcal{B}V)^c)$, implies $x\in(f^{-1}((\mathcal{B}V)^c))^c$. So, $f^{-1}(\mathcal{B}V)\subseteq(f^{-1}((\mathcal{B}V)^c))$. Therefore, $(f^{-1}((\mathcal{B}V)^c))^c=f^{-1}(\mathcal{B}V)$. Since $f:(X,\mu)\to(Y,\lambda,\mathcal{H})$ is a closed $(\mathcal{A},\mathcal{B})$-continuous function, $f^{-1}((\mathcal{B}V)^c)$ is a $\mathcal{A}$-closed set in (X, μ). This implies $(f^{-1}(\mathcal{B}V)^c)^c$ is $\mathcal{A}$-open in (X, μ). Hence $f^{-1}(\mathcal{B}V) \in \mu$ and so $f^{-1}(\mathcal{B}V) = i_\mu(f^{-1}(\mathcal{B}V))$. Also, note that $V \subseteq \mathcal{B}V$ and this implies $f^{-1}(V) \subseteq f^{-1}(\mathcal{B}V) = i_\mu(f^{-1}(\mathcal{B}V))$. Since $\mathcal{A}$ is sub-additive $\mathcal{A}(i_\mu(f^{-1}(\mathcal{B}V))) = i_\mu(f^{-1}(\mathcal{B}V))$ and $\mathcal{A}$ is monotone, then $\mathcal{A}(f^{-1}(V)) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$ for each $V \in \lambda$. Hence f is $(\mathcal{A}, \mathcal{B})$-weakly continuous. □

Let Γ be the set of all expansion on GTS (X, μ), a partial order " $<$ " can be defined by the relation $\mathcal{A} < \mathcal{B}$ if and only if $\mathcal{A}V < \mathcal{B}V$ for all $V \in \mu$. It is clear that $\mathcal{I}d < \mathcal{A}$ for any expansion $\mathcal{A}$ on (X, μ), thus the set $(\Gamma, <)$ has a minimum element.

Theorem 3.11. *Let (X, μ) be a GTS and $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space and $\mathcal{A}$ be an expansion on $(Y, \lambda, \mathcal{H})$. Let $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ be $\mathcal{A}$-expansion continuous. Then f is $\mathcal{B}$-expansion continuous for any expansion $\mathcal{B}$ on $(Y, \lambda, \mathcal{H})$ such that $\mathcal{A} < \mathcal{B}$.*

Proof. Let f be $\mathcal{A}$-expansion continuous and $\mathcal{A} < \mathcal{B}$. Then $\mathcal{A}V < \mathcal{B}V$ for all $V \in \mu$ and $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{A}V))$, then $f^{-1}(V) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$. Thus $\mathcal{A}$-expansion continuous implies $\mathcal{B}$-expansion continuous for any expansion $\mathcal{B}$ on $(Y, \lambda, \mathcal{H})$ such that $\mathcal{A} < \mathcal{B}$. □

Corollary 3.12. *Let (X, μ) be a GTS and $(Y, \lambda, \mathcal{H})$ be a hereditary generalized topological space and let $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$. Then every (μ, λ)-continuity implies $\mathcal{A}$-expansion continuity for any expansion $\mathcal{A}$ on $(Y, \lambda, \mathcal{H})$.*

Proof. Since (μ, λ)-continuity is equivalent to $\mathcal{I}d$-expansion continuity, the result follows from Theorem 3.11 and the fact that $\mathcal{I}d < \mathcal{A}$ for any expansion $\mathcal{A}$ on $(Y, \lambda, \mathcal{H})$. □

Theorem 3.13. *Let $\mathcal{A}$ be any expansion on GTS (X, μ). Then the expansion $\mathcal{B}V = V \cup (\mathcal{A}V)^c$ is the maximal expansion on (X, μ) which is mutually dual to $\mathcal{A}$.*

Proof. Let $\mathcal{B}_\mathcal{A}$ be the set of all expansion on GTS (X, μ) which are mutually dual to $\mathcal{A}$. Since $V \subseteq \mathcal{A}V$, for any $V \in \mu$, $\mathcal{A}V$ can be written as $\mathcal{A}V = V \cup (\mathcal{A}V - V)$. Let $\mathcal{B}V = V \cup (\mathcal{A}V)^c = (\mathcal{A}V - V)^c$. It is obvious that $\mathcal{B}$ is an expansion on (X, μ) and $\mathcal{A}V \cap \mathcal{B}V = V$ for any $V \in \mu$. Thus $\mathcal{B} \in \mathcal{B}_\mathcal{A}$. Given any expansion $\mathcal{B}^*$ on (X, μ), write $\mathcal{B}^*V = V \cup (\mathcal{B}^*V - V)$. If $\mathcal{B}^* \in \mathcal{B}_\mathcal{A}$, then $(\mathcal{A}V - V) \cap (\mathcal{B}^*V - V) = \emptyset$, thus $\mathcal{B}^*V - V \subseteq (\mathcal{A}V - V)^c = \mathcal{B}V$. Therefore, $\mathcal{B}^*V \subseteq \mathcal{B}V$ and we have that $\mathcal{B}^* < \mathcal{B}$. Hence $\mathcal{B}$ is the maximal element of $\mathcal{B}_{\mathcal{A}}$. □

Proposition 3.14. *Let $f : (X,\mu) \to (Y,\lambda,\mathcal{H})$, where $\mathcal{H}$ is λ-codense and $\mathcal{B}$ be an open and idempotent expansion on $(Y,\lambda,\mathcal{H})$ and if $\mathcal{A}$ is subadditive and monotone expansion on (X,μ). Then f is $(\mathcal{A},\mathcal{B})$-weakly continuous if and only if closed $(\mathcal{A},\mathcal{B})$-continuous.*

Proof. The Sufficiency follows from Theorem 3.10.
Let f be $(\mathcal{A},\mathcal{B})$-weakly continuous, where $\mathcal{B}$ is open and idempotent, then $\mathcal{A}(f^{-1}(V)) \subseteq i_\mu(f^{-1}(\mathcal{B}V))$ for each $V \in \lambda$. Since $\mathcal{B}V \in \lambda$ and $\mathcal{B}(\mathcal{B}V) = \mathcal{B}V$, we have $f^{-1}(\mathcal{B}V) \subseteq \mathcal{A}(f^{-1}(\mathcal{B}V)) \subseteq i_\mu(f^{-1}(\mathcal{B}(\mathcal{B}V))) = i_\mu(f^{-1}(\mathcal{B}V))$. Thus $f^{-1}(\mathcal{B}V) \in \mu$ and $\mathcal{A}(f^{-1}(\mathcal{B}V)) = f^{-1}(\mathcal{B}V)$ and hence $(f^{-1}(\mathcal{B}V))^c = f^{-1}((\mathcal{B}V)^c)$ is $\mathcal{A}$-closed therefore f is closed $(\mathcal{A},\mathcal{B})$-continuous. □

Theorem 3.15. *Let (X,μ) be a GTS and $(Y,\lambda,\mathcal{H})$ be a hereditary generalized topological space and $\mathcal{A}$ and $\mathcal{B}$ be expansions on $\{f^{-1}(V) : V \in \mu\}$ and $(Y,\lambda,\mathcal{H})$ respectively. Let $f : (X,\mu) \to (Y,\lambda,\mathcal{H})$ be $(\mathcal{A},\mathcal{B})$-weakly continuous. Then f is $(\mathcal{A}^*,\mathcal{B}^*)$-weakly continuous for any expansions $\mathcal{A}^*$ and $\mathcal{B}^*$ on $\{f^{-1}(V) : V \in \mu\}$ and $(Y,\lambda,\mathcal{H})$ respectively such that $\mathcal{A}^* < \mathcal{A}$ and $\mathcal{B} < \mathcal{B}^*$.*

Proof. This follows easily from Definition 3.2. □

ACKNOWLEDGEMENT

The authors wish to thank the referee for useful comments and suggestions.

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