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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 2, 23 - 33

ON α-S-CLOSED CRISP SUBSETS OF A FUZZY TOPOLOGICAL SPACE

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Abstract. In this paper, we introduce a new type of covering property in a fuzzy topological space X, called the property of α-s-closedness of subsets of X. We characterize α-s-closed subsets in many ways, e.g. by means of ordinary nets and power-set filterbases.

1. Introduction and preliminaries

It is well known from literature that in 1987 Di Maio and Noiri [4] proposed a certain type of covering axiom in topological spaces and named them s-closed spaces. Afterwards, many mathematicians have endeavoured to study the said property from many directions. In this connection the papers [5, 6] by Ganguly and Basu need be mentioned. Gantner et al. [7] paved a new direction towards the study of covering property in fuzzy setting in terms of a novel concept viz. α-shading. We resort to the same concept here to define the proposed idea of α-s-closedness in a fuzzy topological space (henceforth to be abbreviated as fts).

In this paper, our aim is to study α-s-closedness of crisp subsets (i.e., ordinary subsets) of an fts. We then characterize it via different ways which are also true in α-s-closedness of X if one puts $A = X$. In the next section such characterizations are done while in the latter section we investigate a class of crisp subsets of X which inherit α-s-closedness of X. In the last section, we introduce a type of functions keeping α-s-closedness invariant.

Keywords and phrases: α-s-closed space, s-θ_α-adherent point of net and filterbase, α-s-interiorly finite intersection property.
(2010)Mathematics Subject Classification: 54A40, 54D20

Throughout the paper, by (X, τ) or simply by X we mean an fts in the sense of Chang [3]. The closure and interior of a fuzzy set A [11] in X will be denoted by clA and $intA$ respectively. A fuzzy set A in X is called fuzzy regular open (semiopen) if $A = intclA$ (respectively, $U \leq A \leq clU$, for some fuzzy open set U)[1]. The complement $(1-A)$ of a fuzzy regular open (semiopen) set is called fuzzy regular closed (respectively, semiclosed).

The semiclosure of a fuzzy set A in X, to be denoted by $sclA$, is defined by the union of all those fuzzy points x_t (where x is the singleton support and t the value of the fuzzy point x_t, $0 < t \leq 1$) such that for any fuzzy semiopen set U with $U(x)+t > 1$, there exists $y \in X$ with $U(y) + A(y) > 1$ [8]. The semi interior of a fuzzy set A in X, written as $sintA$, is defined by $sintA = 1 - scl(1 - A)$ [10]. It is known [10] that a fuzzy set A in X is fuzzy semiclosed (semiopen) iff $A = sclA$ (respectively, $A = sintA$).

2. α-s-CLOSEDNESS: CHARACTERIZATIONS

Definition 2.1. [7] Let A be a crisp subset of an fts X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0 < \alpha < 1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$. If, in addition, the members are fuzzy open (semiopen) then $\mathcal{U}$ is called a fuzzy open (resp. semiopen) α-shading of A.

Definition 2.2. Let X be an fts and A be a crisp subsets of X. A is said to be α-s-closed if for every semiopen α-shading $(0 < \alpha < 1)$ $\mathcal{U}$ of A, there is a finite semiproximate α-subshading of A, i.e., there is a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{sclU : U \in \mathcal{U}_0\}$ is again an α-shading of A. If $A = X$ in addition, then X is called an α-s-closed space.

Mashhour et al. [9] defined a fuzzy set A in an fts X to be fuzzy regular semiopen if there is a fuzzy regular open set U such that $U \leq A \leq clU$, and they proved that a fuzzy regular semiopen set is fuzzy semiopen but not conversely.

Theorem 2.3. A subset A of X is α-s-closed iff every α-shading of A by fuzzy regular semiopen sets in X has a finite proximate α-subshading.

Proof. The proof follows from the definition of α-s-closedness and the fact that whenever $\{V_i : i \in \Lambda\}$ is a fuzzy semiopen α-shading of A, then $\{(intclV_i) \bigcup V_i : i \in \Lambda\}$ is an α-shading of A by fuzzy regular semiopen sets.

Theorem 2.4. Let X be an fts. A crisp subset A of X is α-s-closed iff every family of fuzzy semiopen sets, the semi-interiors of whose semi-closures form an α-shading of A, contains a finite subfamily, the semi-closures of whose members form an α-shading of A.

Proof. It is sufficient to observe that for a fuzzy semiopen set U, $U \leq sint(sclU) \leq scl(sint(sclU)) = sclU$.

Theorem 2.5. A crisp subset A of an fts X is α-s-closed iff for every collection $\{F_i : i \in \Lambda\}$ of fuzzy semiopen sets with the property that for each finite subset Λ_0 of Λ, there is $x \in A$ such that $\inf\limits_{i\in\Lambda_0} F_i(x) \geq 1-\alpha$, one has $\inf\limits_{i\in\Lambda}(sclF_i)(y) \geq 1-\alpha$, for some $y \in A$.

Proof. Let A be α-s-closed, and if possible, let for a collection $\{F_i : i \in \Lambda\}$ of fuzzy semiopen sets in X with the stated property, $(\bigcap\limits_{i\in\Lambda} sclF_i)(x) < 1-\alpha$, for each $x \in A$. Then $\alpha < (1 - \bigcap\limits_{i\in\Lambda} sclF_i)(x) = [\bigcup\limits_{i\in\Lambda}(1 - sclF_i)](x)$, for each $x \in A$ which shows that $\{1 - sclF_i : i \in \Lambda\}$ is a semiopen α-shading of A. By α-s-closedness of A, there is a finite subset Λ_0 of Λ such that $\{scl(1 - sclF_i) : i \in \Lambda_0\} = \{1 - sint(sclF_i) : i \in \Lambda_0\}$ is an α-shading of A. Hence $\alpha < [\bigcup\limits_{i\in\Lambda_0}(1 - sint(sclF_i))](x)$ $= [1 - (\bigcap\limits_{i\in\Lambda_0} sint(sclF_i))](x)$, for each $x \in A$. Then $(\bigcap\limits_{i\in\Lambda_0} F_i)(x) \leq [\bigcap\limits_{i\in\Lambda_0} sint(sclF_i)](x) < 1-\alpha$, for each $x \in A$, which contradicts the stated property of the collection $\{F_i : i \in \Lambda\}$.

Conversely, let under the given hypothesis, A be not α-s-closed. Then there is a semiopen α-shading $\mathcal{U} = \{U_i : i \in \Lambda\}$ of A having no finite semi-proximate α-subshading, i.e., for every finite subset Λ_0 of Λ, $\{sclU_i : i \in \Lambda_0\}$ is not an α-shading of A, i.e., there exists $x \in A$ such that $\sup\limits_{i\in\Lambda_0}(sclU_i)(x) \leq \alpha$, i.e., $1 - \sup\limits_{i\in\Lambda_0}(sclU_i)(x) = \inf\limits_{i\in\Lambda_0}(1 - sclU_i)(x) \geq 1-\alpha$. Hence $\{1 - sclU_i : i \in \Lambda\}$ is a family of fuzzy semiopen sets with the stated property.

Consequently, there is some $y \in A$ such that $\inf\limits_{i\in\Lambda}[scl(1 - sclU_i)](y) \geq 1-\alpha$. Then $\sup\limits_{i\in\Lambda} U_i(y) \leq \sup\limits_{i\in\Lambda}(sintsclU_i)(y)$ $= 1 - \inf\limits_{i\in\Lambda}(1 - sintsclU_i)(y) = 1 - \inf\limits_{i\in\Lambda}[scl(1 - sclU_i)]((y) \leq \alpha$. This shows that $\{U_i : i \in \Lambda\}$ fails to be an α-shading of A, a contradiction.

Let us now introduce the following definition:

Definition 2.6. Let $\{S_n : n \in (D, \geq)\}$ (where $(D, \geq)$ is a directed set) be an ordinary net in X and $\mathcal{F}$ be a power-set filterbase on X, and $x \in X$ be any crisp point in X. Then x is called an s-θ_α-adherent point of:

(a) the net $\{S_n\}$ if for each fuzzy semiopen set U in X with $U(x) > \alpha$ and for each $m \in D$, there exists $k \in D$ such that $k \geq m$ in D and $(sclU)(S_k) > \alpha$,

(b) the filterbase $\mathcal{F}$ if for each fuzzy semiopen set U with $U(x) > \alpha$ and for each $F \in \mathcal{F}$, there exists a crisp point x_F in F such that $(sclU)(x_F) > \alpha$.

Theorem 2.7. A crisp subset A of an fts X is α-s-closed iff every net in A has an s-θ_α-adherent point in A.

Proof. Let us suppose that A be α-s-closed, but there be a net $\{S_n : n \in (D, \geq)\}$ in A ($(D, \geq)$ being a directed set, as usual) having no s-θ_α-adherent point in A. Then for each $x \in A$, there is a fuzzy semiopen set U_x with $U_x(x) > \alpha$, and there is an $m_x \in D$ such that $(sclU_x)(S_n) \leq \alpha$, for all $n \geq m_x$ $(n \in D)$. Now, $\mathcal{U} = \{1 - sclU_x : x \in X\}$ is a collection of fuzzy semiopen sets such that for any finite subcollection $\{1 - sclU_{x_1}, ..., 1 - sclU_{x_k}\}$ (say) of $\mathcal{U}$, there exists $m \in D$ with $m \geq m_{x_1}$,..., m_{x_k} in D such that $(\bigcup_{i=1}^{k} sclU_{x_i})(S_n) \leq \alpha$, for all $n \geq m$ $(n \in D)$, i.e., $\inf_{1 \leq i \leq k} (1 - sclU_{x_i})(S_n) \geq 1 - \alpha$, for all $n \geq m$.

Hence by Theorem 2.5, there exists some $y \in A$ such that $\inf_{x \in A} [scl(1- sclU_x)(y)] \geq 1 - \alpha$, i.e.,$(\bigcup_{x \in A} U_x)(y) \leq [\bigcup_{x \in A} sint(sclU_x)](y) = 1 - [1 - (\bigcup_{x \in A} sintsclU_x)(y)] = 1 - \inf_{x \in A} [scl(1 - sclU_x)](y) \leq 1 - 1 + \alpha = \alpha$. We have, in particular, $U_y(y) \leq \alpha$, contradicting the definition of U_y. Hence the result is proved.

Conversely, let every net in A have s-θ_α-adherent point in A and suppose $\{F_i : i \in \Lambda\}$ be an arbitrary collection of fuzzy semiopen sets in X. Let Λ_f denote the collection of all subsets of Λ, then $(\Lambda_f, \geq)$ is a directed set, where for $\mu, \lambda \in \Lambda_f$, $\mu \geq \lambda$ iff $\mu \supseteq \lambda$. For each $\mu \in \Lambda_f$, put $F_\mu = \bigcap \{F_i : i \in \mu\}$.

Let for each $\mu \in \Lambda_f$, there be a point $x_\mu \in A$ such that

$$\inf_{i \in \mu} F_i(x_\mu) \geq 1 - \alpha \quad (1).$$

Then by Theorem 2.5 it is enough to show that $\inf_{i\in\Lambda}(sclF_i)(z) \geq 1-\alpha$ for some $z \in A$. If possible, let

$$\inf_{i\in\Lambda}(sclF_i)(z) < 1-\alpha, \text{ for each } z \in A \quad (2).$$

Now, $S=\{x_\mu : \mu \in (\Lambda_f, \geq)\}$ is clearly a net of points in A. By hypothesis, there is an s-θ_α-adherent point z in A of this net. By (2), $\inf_{i\in\Lambda}(sclF_i)(z) < 1-\alpha$ implies that there exists $i_0 \in \Lambda$ such that $(sclF_{i_0})(z) < 1-\alpha$, i.e., $1-sclF_{i_0}(z) > \alpha$. Since z is an s-θ_α-adherent point of S, for the index $\{i_0\} \in \Lambda_f$, there is $\mu_0 \in \Lambda_f$ with $\mu_0 \geq \{i_0\}$ (i.e., $i_0 \in \mu_0$) such that $scl(1-sclF_{i_0})(x_{\mu_0}) > \alpha$, i.e., $sintsclF_{i_0}(x_{\mu_0}) < 1-\alpha$. Since $i_0 \in \mu_0$, $\inf_{i\in\mu_0} F_i(x_{\mu_0}) \leq F_{i_0}(x_{\mu_0}) \leq sintsclF_{i_0}(x_{\mu_0}) < 1-\alpha$, which contradicts (1). This completes the proof.

Theorem 2.8. A crisp subset A of an fts X is α-s-closed iff every filterbase $\mathcal{F}$ on A has an s-θ_α-adherent point in A.

Proof. Let A be α-s-closed and let there exist, if possible, a filterbase $\mathcal{F}$ on A having no s-θ_α-adherent point in A. Then for each $x \in A$, there exist a fuzzy semiopen set U_x with $U_x(x) > \alpha$, and an $F_x \in \mathcal{F}$ such that $(sclU_x)(y) \leq \alpha$, for each $y \in F_x$. Then $\mathcal{U} = \{U_x : x \in A\}$ is a fuzzy semiopen α-shading of A. Thus there exist finitely many points x_1,x_2,..., x_n in A such that $\mathcal{U}_0 = \{sclU_{x_i} : i = 1, 2, ..., n\}$ is an α-shading of A. Choose $F \in \mathcal{F}$ such that $F \leq F_{x_1} \bigcap F_{x_2} \bigcap ... \bigcap F_{x_n}$. Then $(sclU_{x_i})(y) \leq \alpha$, for all $y \in F$ and for $i = 1, 2, ..., n$. Thus $\mathcal{U}_0$ fails to be an α-shading of A, a contradiction.

Conversely, let the condition hold and suppose, if possible, $\{y_n : n \in (D, \geq)\}$ be a net in A having no s-θ_α-adherent point in A. Then for $x \in A$, there are a fuzzy semiopen set U_x with $U_x(x) > \alpha$ and an $m_x \in D$ such that $(sclU_x)(y_n) \leq \alpha$, for all $n \geq m_x$ ($n \in D$). Thus $\mathcal{F} = \{F_x : x \in A\}$, where $F_x = \{y_n : n \geq m_x\}$ generates a filterbase $\mathcal{F}^*$ on A. By hypothesis, $\mathcal{F}^*$ has an s-θ_α-adherent point z (say) in A. But there are a fuzzy semiopen set U_z with $U_z(z) > \alpha$ and an $m_z \in D$ such that $(sclU_z)(y_n) \leq \alpha$, for all $n \geq m_z$, i.e., for all $p \in F_z$ ($\in \mathcal{F} \subseteq \mathcal{F}^*$), $(sclU_z)(p) \leq \alpha$. Hence z cannot be an s-θ_α-adherent point of the filterbase $\mathcal{F}^*$, a contradiction. Hence by Theorem 2.7, A is α-s-closed.

Definition 2.9. A family $\{F_i : i \in \Lambda\}$ of fuzzy sets in an fts X is said to have α-s-interiorly finite intersection property or simply α-s-IFIP in a subset A of X, if for each finite subset Λ_0 of Λ, there exists $x \in A$ such that $[\bigcap_{i\in\Lambda_0} sintF_i](x) \geq 1-\alpha$.

Theorem 2.10. A crisp subset A of an fts X is α-s-closed iff for every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy semiclosed sets in X with α-s-IFIP in A, there exists $x \in A$ such that $\inf\limits_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

Proof. Assuming A ($\subseteq X$) to be α-s-closed, let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a family of fuzzy semiclosed sets with α-s-IFIP in A. If possible, let for each $x \in A$, $\inf\limits_{i\in\Lambda} F_i(x) < 1-\alpha$, i.e., $(\bigcap\limits_{i\in\Lambda} F_i)(x) < 1-\alpha$ and hence $[\bigcup\limits_{i\in\Lambda}(1-F_i)](x) > \alpha$. Thus $\mathcal{U} = \{1-F_i : i \in \Lambda\}$ is a fuzzy semiopen α-shading of A. By α-s-closedness of A, there is a finite subset Λ_0 of Λ such that $[\bigcup\limits_{i\in\Lambda_0} scl(1-F_i)](x) > \alpha$, i.e., $1-(\bigcap\limits_{i\in\Lambda_0} sintF_i)(x) > \alpha$, i.e., $(\bigcap\limits_{i\in\Lambda_0} sintF_i)(x) < 1-\alpha$, for each $x \in A$, which shows that $\mathcal{F}$ does not have α-s-IFIP in A, a contradiction.

Conversely, let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a fuzzy semiopen α-shading of A. Then $\mathcal{F} = \{1-U_i : i \in \Lambda\}$ is a family of fuzzy semiclosed sets with $\inf\limits_{i\in\Lambda}(1-U_i)(x) < 1-\alpha$, for each $x \in A$, so that $\mathcal{F}$ does have α-s-IFIP. Hence for some finite subset Λ_0 of Λ, we have for each $x \in A$, $[\bigcap\limits_{i\in\Lambda_0} sint(1-U_i)](x) < 1-\alpha$ imples that $1-(\bigcup\limits_{i\in\Lambda_0} sclU_i)(x) < 1-\alpha$, for each $x \in A$ so that $(\bigcup\limits_{i\in\Lambda_0} sclU_i)(x) > \alpha$, for each $x \in A$. Hence A is α-s-closed.

Putting $A = X$ in the characterization theorems so far, of α-s-closed crisp subset A, we obtain as follows:

Theorem 2.11. For an fts (X, τ), the following are equivalent:
(a) X is α-s-closed.
(b) Every α-shading of X by fuzzy regular semiopen sets has a finite proximate α-subshading.
(c) Every family of fuzzy semiopen sets, the semi-interiors of whose semi-closures form an α-shading of X, contains a finite subfamily, the semi-closures of whose members form an α-shading of X.
(d) For every collection $\{F_i : i \in \Lambda\}$ of fuzzy semiopen sets with the property that for each finite subset Λ_0 of Λ, there is $x \in X$ such that $\inf\limits_{i\in\Lambda_0} F_i(x) \geq 1-\alpha$, one has $\inf\limits_{i\in\Lambda}(sclF_i)(y) \geq 1-\alpha$, for some $y \in X$.
(e) Every net in X has an s-θ_α-adherent point in X.
(f) Every filterbase on X has an s-θ_α-adherent point in X.

(g) For every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy semiclosed sets in X with α-s-IFIP in X, there exists $x \in X$ such that $\inf_{i\in\Lambda} F_i(x) \geq 1 - \alpha$.

3. $\theta^{\alpha}_{s^*}$-CLOSED SET, $\theta^{\alpha}_{s^*}$-CONTINUITY AND α-s-CLOSEDNESS

In this section we introduce a class of crisp subsets of an fts X, which inherit α-s-closedness of X, and also we try to ascertain a class of α-s-closed fts's for which the former class of subsets is precisely the family of subsets inheriting the α-s-closedness of the spaces concerned. Finally, we search for a type of function under which α-s-closedness remains invariant.

Definition 3.1. Let (X, τ) be an fts and $A \subseteq X$. A point $x \in X$ is said to be a $\theta^{\alpha}_{s^*}$-limit point of A if for every fuzzy semiopen set U in X with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(sclU)(y) > \alpha$. The set of all $\theta^{\alpha}_{s^*}$-limit points of A will be denoted by $[A]^{\alpha}_{s^*}$.
The $\theta^{\alpha}_{s^*}$-closure of A, to be denoted by $\theta^{\alpha}_{s^*}$-clA, is defined by $\theta^{\alpha}_{s^*}$-clA= $A \cup [A]^{\alpha}_{s^*}$.

Definition 3.2. A crisp subset A of an fts X is said to be $\theta^{\alpha}_{s^*}$-closed if it contains all its $\theta^{\alpha}_{s^*}$-limit points. Any subset B of X is called $\theta^{\alpha}_{s^*}$-open if $X \setminus B$ is $\theta^{\alpha}_{s^*}$-closed.

Remark 3.3. For any $A \subseteq X$ it is clear that $A \subseteq \theta^{\alpha}_{s^*}$-$clA$, and $\theta^{\alpha}_{s^*}$-clA $= A$ iff $[A]^{\alpha}_{s^*} \subseteq A$. Then in view of Definition 3.1, it follows that A is $\theta^{\alpha}_{s^*}$-closed iff $\theta^{\alpha}_{s^*}$-$clA = A$. It is also clear that $A \subseteq B \subseteq X$ implies that $[A]^{\alpha}_{s^*} \subseteq [B]^{\alpha}_{s^*}$.

Theorem 3.4. A $\theta^{\alpha}_{s^*}$-closed subset A of an α-s-closed space X is α-s-closed.

Proof. Let A $(\subseteq X)$ be $\theta^{\alpha}_{s^*}$-closed in X. Then for any $x \notin A$, there is a fuzzy semiopen set U_x such that $U_x(x) > \alpha$, and $(sclU_x)(y) \leq \alpha$ for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. Now to prove that A is α-s-closed, consider a fuzzy semiopen α-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \cup \mathcal{V}$ is a fuzzy semiopen α-shading of X. Since X is α-s-closed, there exists a finite subcollection $\{V_1, V_2, ..., V_n\}$ of $\mathcal{U} \cup \mathcal{V}$ such that for every $t \in X$, there exists V_i $(1 \leq i \leq n)$ with $sclV_i(t) > \alpha$. For every member U_x of $\mathcal{U}$, $sclU_x(y) \leq \alpha$ for every $y \in A$. So if this subcollection contains any member of $\mathcal{U}$, we omit it and hence we get the result.

To achieve the converse of Theorem 3.4, we define the following:

Definition 3.5. An fts (X, τ) is said to be α_s-Urysohn if for any two distinct points x, y of X, there exist U, $V \in \tau$ with $U(x) > \alpha$, $V(y) > \alpha$ and $\min(sclU(z), sclV(z)) \leq \alpha$ for each $z \in X$.

Theorem 3.6. An α-s-closed set in an α_s-Urysohn space X is $\theta^{\alpha}_{s^*}$-closed.

Proof. Let A be α-s-closed and $x \in X \setminus A$. Then for each $y \in A$, $x \neq y$. By α_s-Urysohn property of X, there exist fuzzy open sets U_y and V_y such that $U_y(x) > \alpha$, $V_y(y) > \alpha$ and $\min((sclU_y)(z), (sclV_y)(z)) \leq \alpha$, for all $z \in X$ (1).

Then $\mathcal{U} = \{V_y : y \in A\}$ is a fuzzy open and hence fuzzy semiopen α-shading of the α-s-closed set A. Then by α-s-closedness of A, there are finitely many points y_1, y_2, ..., y_n in A such that $\mathcal{U}_0 = \{sclV_{y_1}, sclV_{y_2}, ..., sclV_{y_n}\}$ is again an α-shading of$A$. Now, $U = U_{y_1} \cap ... \cap U_{y_n}$ is a fuzzy open set and hence a fuzzy semiopen set such that $U(x) > \alpha$. In order to show that A to be $\theta^{\alpha}_{s^*}$-closed, it now suffices to show that $(sclU)(y) \leq \alpha$ for each $y \in A$. In fact, if for some $z \in A$, we assume $(sclU)(z) > \alpha$ then as $z \in A$, we have $(sclV_{y_k})(z)) > \alpha$ for some k $(1 \leq k \leq n)$. Also, $(sclU_{y_k})(z) > \alpha$. Hence $\min[(sclU_{y_k})(z), (sclV_{y_k})(z)] > \alpha$, contradicting (1).

Corollary 3.7. In an α-s-closed, α_s-Urysohn space X, a subset A of X is α-s-closed iff it is $\theta^{\alpha}_{s^*}$-closed.

Theorem 3.8. In an α-s-closed space X, every cover of X by $\theta^{\alpha}_{s^*}$-open sets has a finite subcover.

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of X by $\theta^{\alpha}_{s^*}$-open sets. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Now, as $X \setminus U_x$ is $\theta^{\alpha}_{s^*}$-closed, there exists a fuzzy semiopen set V_x in X such that

$V_x(x) > \alpha$, and $(sclV_x)(y) \leq \alpha$ for each $y \in X \setminus U_x$ (1)

Then $\{V_x : x \in X\}$ forms a fuzzy semiopen α-shading of the α-s-closed space X. Thus there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that

$\{sclV_{x_i} : i = 1, 2, ..., n\}$ is an α-shading of X (2)

We claim that $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \cup_{i=1}^{n} U_{x_i} = \cap_{i=1}^{n}(X \setminus U_{x_i})$. Then by (1), $sclV_{x_i}(y) \leq \alpha$ for $i = 1, 2, ..., n$. Therefore, $(\cup_{i=1}^{n} sclV_{x_i})(y) \leq \alpha$, contradicting (2).

Theorem 3.9. Let (X, τ) be an fts. If X is α-s-closed then every collection of $\theta^{\alpha}_{s^*}$-closed sets in X with finite intersection property has nonempty intersection.

Proof. Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a collection of $\theta^{\alpha}_{s^*}$-closed sets in X having finite intersection property. If possible, let $\bigcap_{i \in \Lambda} F_i = \phi$. Then $X \setminus \bigcap_{i \in \Lambda} F_i = X$ implies that $\bigcup_{i \in \Lambda}(X \setminus F_i) = X$ which shows that

$\mathcal{U} = \{X \setminus F_i : i \in \Lambda\}$ is an θ^{α}_{s*}-open cover of X. Then by Theorem 3.8, there is a finite subset Λ_0 of Λ such that $\bigcup\limits_{i\in\Lambda_0}(X \setminus F_i) = X$ so that $\bigcap\limits_{i\in\Lambda_0} F_i = \phi$, a contradiction.

For achieving the converse of the above theorem, we need to introduce the following notation:

Notation 3.10. For any fuzzy set A in an fts X, the subset $\overline{A}^{\alpha}_{s}$ of X is defined by $\overline{A}^{\alpha}_{s} = \{x \in X : (sclA)(x) \leq \alpha\}$.

Remark 3.11. For any fuzzy semiopen set U in an fts (X, τ), the set $\overline{U}^{\alpha}_{s} = \{x \in X : (sclU)(x) \leq \alpha\}$ is $\overline{\theta}^{\alpha}_{s*}$-closed. In fact, if $y \notin \overline{U}^{\alpha}_{s}$, then U is a fuzzy semiopen set in X such that $(sclU)(y) > \alpha$, and $(sclU)(z) \leq \alpha$, for all $z \in \overline{U}^{\alpha}_{s}$. Then $V = sclU$ is a fuzzy semiopen set such that $V(y) > \alpha$ and $(sclV)(z) \leq \alpha$, for all $z \in \overline{U}^{\alpha}_{s}$.

The next theorem serves as a weak converse of Theorem 3.9.

Theorem 3.12. Let (X, τ) be an fts. Then X is α-s-closed if every collection of θ^{α}_{s*}-closed sets in X satisfying the finite intersection property, has nonempty intersection.

Proof. Let $\mathcal{U}$ be a fuzzy semiopen α-shading of X. Consider the collection $\mathcal{C} = \{\overline{V}^{\alpha}_{s} : V \in \mathcal{U}\}$ where $\overline{U}^{\alpha}_{s}$ stands for the expression given in Remark 3.11. Then $\mathcal{C}$ is a collection of θ^{α}_{s*}-closed sets.

Since $\mathcal{U}$ is an α-shading of X, for every $x \in X$ there exists $V \in \mathcal{U}$ with $V(x) > \alpha$, so that $(sclV)(x) > \alpha$. Hence $x \notin \overline{V}^{\alpha}_{s}$, for some $V \in \mathcal{U}$. Thus $\cap\{\overline{V}^{\alpha}_{s} : V \in \mathcal{U}\} = \phi$. Then by hypothesis $\mathcal{C}$ does not satisfy the finite intersection property. Hence there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\cap\{\overline{V}^{\alpha}_{s} : V \in \mathcal{U}_0\} = \phi$. Thus for every $x \in X$, there exists $V \in \mathcal{U}_0$ such that $x \notin \overline{V}^{\alpha}_{s}$ and hence $sclV(x) > \alpha$, proving that $\mathcal{U}_0$ is a finite α-subshading of $\mathcal{U}$. Thus X is α-s-closed.

Let us now introduce a class of functions under which α-s-closedness remains invariant.

Definition 3.13. Let X, Y be fts's. A function $f : X \to Y$ is said to be θ^{α}_{s*}-continuous if $f^{-1}(A)$ is θ^{α}_{s*}-closed in X for every θ^{α}_{s*}-closed set A in Y.

Theorem 3.14. Let (X, τ) and (Y, τ_1) be fts's and let $f : X \to Y$ be a θ^{α}_{s*}-continuous function. If A $(\subseteq X)$ is α-s-closed in X, then so is $f(A)$ in Y.

Proof. Consider a fuzzy semiopen α-shading $\mathcal{V}$ of $f(A)$ in Y. For every $x \in A$, $f(x) \in f(A)$ and hence there exists $U_{f(x)} \in \mathcal{V}$ such that $U_{f(x)}(f(x)) > \alpha$. Clearly $(sclU_{f(x)})^{-1}[0, \alpha]$ is θ^{α}_{s*}-closed in Y. For, $z \notin$

$(sclU_{f(x)})^{-1}[0,\alpha]$ implies that $U_{f(x)} \in SO(Y)$ (where $SO(Y)$ denotes the set of all fuzzy semiopen sets in Y) and $(sclU_{f(x)})(z) > \alpha$, and $(sclU_{f(x)})(y) \leq \alpha$ for every $y \in (sclU_{f(x)})^{-1}[0,\alpha]$.

Clearly, $x \notin f^{-1}((sclU_{f(x)})^{-1}[0,\alpha])$. Thus x is not a $\theta^{\alpha}_{s^*}$-limit point of $f^{-1}((sclU_{f(x)})^{-1}[0,\alpha])$ (since $(sclU_{f(x)})^{-1}[0,\alpha]$ is $\theta^{\alpha}_{s^*}$-closed and f is $\theta^{\alpha}_{s^*}$-continuous imply that $f^{-1}((sclU_{f(x)})^{-1}[0,\alpha])$ is $\theta^{\alpha}_{s^*}$-closed). Then there is a fuzzy semiopen set V_x with $V_x(x) > \alpha$, but $(sclV_x)(z) \leq \alpha$ for every $z \in f^{-1}((sclU_{f(x)})^{-1}[0,\alpha])$. Then $\mathcal{U} = \{V_x : x \in A\}$ is a fuzzy semiopen α-shading of A.

As A is α-s-closed, there exist finitely many members V_{x_1}, ..., V_{x_n} of $\mathcal{U}$ such that for every $t \in A$, $[sclV_{x_i}](t) > \alpha$ for some i $(i = 1, 2, ..., n)$.

It suffices to prove that $\{sclU_{f(x_i)} : i = 1, 2, ..., n\}$ is an α-shading of $f(A)$.

Indeed, let $s \in f(A)$. Then there exists $t \in A$ such that $f(t) = s$. Then $sclV_{x_j}(t) > \alpha$ for some j $(1 \leq j \leq n)$. Thus $t \notin f^{-1}((sclU_{f(x_j)})^{-1}[0,\alpha])$ implies that $f(t) \notin (sclU_{f(x_j)})^{-1}[0,\alpha]$ so that $sclU_{f(x_j)}(f(t)) > \alpha$. Consequently, $sclU_{f(x_j)}(s) > \alpha$. Hence the theorem.

In [2], θ^{α}_{s}-limit point of a crisp subset A in an fts X is defined as follows:

Definition 3.15. Let (X, τ) be an fts and $A \subseteq X$. A point $x \in X$ is said to be a θ^{α}_{s}-limit point of A if for every fuzzy semiopen set U in X with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(clU)(y) > \alpha$.

It is clear from Definition 3.1 and Definition 3.15 that $\theta^{\alpha}_{s^*}$-limit point of a crisp subset A of an fts X is a θ^{α}_{s}-limit point of A. But the converse may not be true as seen from the following example.

Example 3.16. Let $X = \{a, b\}$, $A = \{b\}$, $\tau = \{0_X, 1_X, B\}$ where $B(a) = 0.5$, $B(b) = 0.4$. Then (X, τ) is an fts. Now fuzzy semiopen sets in X are 0_X, 1_X and U where $U(a) = 0.5$, $0.4 \leq U(b) \leq 0.6$. We claim that a is a θ^{α}_{s}-limit point of A but not a $\theta^{\alpha}_{s^*}$-limit point of A.

Now consider the fuzzy semiopen set V given by $V(a) = 0.5$, $V(b) = 0.4$. Let $\alpha = 0.47$. Then $V(a) = 0.5 > \alpha$ but $(sclV)(b) = V(b) = 0.4 \not> \alpha$ (as V is fuzzy semiclosed set in X also). Therefore, a is not a $\theta^{\alpha}_{s^*}$-limit point of A.

But for any fuzzy semiopen set U in X other than 1_X with $U(a) > \alpha$, $(clU)(b) = (1_X \setminus B)(b) = 0.6 > \alpha$ and so a is a θ^{α}_{s}-limit point of A.

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