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INVARIANT APPROXIMATION FOR NONCOMMUTING PAIRS OF SELF-MAPPINGS

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Abstract. The existence of common fixed points of best approximation for noncommuting pairs with different types of nonexpansive mappings have been proved. We also obtain some results on common fixed points from the set of best simultaneous approximation for a map T which is asymptotically (G, S)-nonexpansive where (T, G) and (T, S) are not necessarily commuting pairs. The proved results generalize and extend several known results on the subject.

1. Introduction and Preliminaries

Approximation theory has gained impetus due to its wide range of applications. Fixed point theorems have been extensively applied in approximation theory and in the last five decades interesting results have been obtained in this direction (see [1] - [8], [10] - [18] and references cited therein). Common fixed points of two commuting mappings satisfying some contractive or nonexpansive type conditions have been studied by many researchers. The introduction of different types of noncommuting mappings such as R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting was a turning point in the fixed point arena.

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In this paper, the existence of common fixed points of best approximation for noncommuting pairs with different types of nonexpansive mappings have been obtained. We also obtain some results on common fixed points from the set of best simultaneous approximation for a map T which is asymptotically (G,S)-nonexpansive where (T,G) and (T,S) are not necessarily commuting pairs. First, we give some notations and recall few definitions.

For a nonempty subset M of a metric space (X,d) and $x \in X$, an element $y \in M$ is said to be a **best approximation** of x to M or a **best M-approximants** to x if $d(x,y) = d(x,M) \equiv \inf\{d(x,z) : z \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$ and is called the set of best M-approximants to x.

An element $g_\circ \in M$ is said to be a **best simultaneous approximation** of the pair $y_1, y_2 \in X$ if

$$\max\{d(y_1,g_\circ), d(y_2,g_\circ)\} = \inf_{g\in M} \max\{d(y_1,g), d(y_2,g)\}.$$

i.e. if Y denotes the product space $X \times X$ equipped with metric d^* defined by,

$$d^*\{(x_1,x_2),(y_1,y_2)\} = \max\{d(x_1,y_1), d(x_2,y_2)\}$$

and $D(K) = \{(k,k) : k \in K\}$, then $k_0 \in K$ ia a best simultaneous approximation to y_1 and y_2 if and only if $(k_0,k_0) \in D(K)$ is a best approximation to $(y_1,y_2) \in Y$ (see Narang [14]).

For a metric space (X,d), a continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

holds for all $u \in X$. The metric space (X,d) together with a convex structure is called a **convex metric space** [19].

A subset K of a convex metric space (X,d) is said to be a **convex set** [19] if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$. A set K is said to be **p-starshaped** [9] where $p \in K$, provided $W(x,p,\lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$ i.e. the segment $[p,x] = \{W(x,p,\lambda) : \lambda \in [0,1]\}$ joining p to x is contained in K for all $x \in K$. K is said to be **starshaped** if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X,d) is said to satisfy **Property (I)** [9] if for all $x,y,q \in X$ and $\lambda \in [0,1]$,

$$d(W(x,q,\lambda), W(y,q,\lambda)) \leq \lambda d(x,y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [9], [19]). Property (I) is always satisfied in a normed linear space.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is said to be **affine** if for all $x, y \in M$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0, 1]$. g is said to be **affine with respect to** $p \in M$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M a nonempty subset of X, and G, S, T are self mappings of M. T is said to be

(i) S**-contraction** if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$,

(ii) S**-nonexpansive** if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in M$.

(iii) (G, S)**-asymptotically nonexpansive** if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(Gx, Sy)$, for all $x, y \in M$.

If $G = S$, then T is called S**-asymptotically nonexpansive** and if $G = S =$ identity mapping, then T is called an **asymptotically nonexpansive** mapping.

(iv) **uniformly asymptotically regular** on M if, for each $\epsilon > 0$, there exists a positive integer N such that $d(T^n(x), T^n(y)) < \epsilon$ for all $n \geq N$ and for all $x, y \in M$.

A point $x \in M$ is a **common fixed (coincidence) point** of S and T if $x = Sx = Tx$ $(Sx = Tx)$. The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$). The pair (S, T) is said to be

i) **commuting** on M if $STx = TSx$ for all $x \in M$.

ii) R**-weakly commuting** on M if there exists $R > 0$ such that $d(TSx, STx) \leq R\, d(Tx, Sx)$ for all $x \in M$.

iii) **compatible** if $\lim d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M.

iv) **weakly compatible** if S and T commute at their coincidence points,i.e., if $STx = TSx$ whenever $Sx = Tx$.

Suppose (X, d) is a convex metric space, M a q-starshaped subset with $q \in F(S) \cap M$ and is both T- and S-invariant. Then T and S are called

i) R**-subweakly commuting** on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R\, dist(Sx, W(Tx, q, k))$, $k \in [0, 1]$;

ii) C_q**-commuting** if $STx = TSx$ for all $x \in C_q(S,T)$, where $C_q(S,T) = \cup\{C(S,T_k) : 0 \leq k \leq 1\}$ and $T_k x = \{W(Tx,q,k) : 0 \leq k \leq 1\}$.

Example 1.1. [2] *Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0,\infty)$. Define $T,S : M \to M$ by $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$; and $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x \geq 1$. Then M is q-starshaped with $q = 1$, and $C_q(T,S) = [1,\infty)$. Moreover S and T are C_q-commuting but neither R-weakly commuting nor R-subweakly commuting for all $R > 0$.*

The ordered pair (T,I) of two self maps of a metric space (X,d) is called a **Banach operator pair** [8], if the set $F(I)$ of fixed points of I is T-invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, commuting pair (T,I) is a Banach operator pair but not conversely (see [8]). If (T,I) is a Banach operator pair then (I,T) need not be a Banach operator pair (see [8]). If the self maps T and I of X satisfy $d(ITx,Tx) \leq kd(Ix,x)$, for all $x \in X$ and for some $k \geq 0$, $ITx = TIx$ whenever $x \in F(I)$ i.e. $Tx \in F(I)$, then (T,I) is a Banach operator pair. This class of non-commuting mappings is different from the known classes of non-commuting mappings viz. R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting etc. existing in the literature. Hence the concept of Banach operator pair is of basic importance for the study of common fixed points.

Example 1.2. *Let $X = \mathbb{R}$ with usual metric and $K = [1,\infty)$. Let $T(x) = x^3$ and $I(x) = 2x - 1$, for all $x \in K$. Then $F(I) = \{1\}$. Here (T,I) is a Banach operator pair but T and I are not commuting.*

Example 1.3. *Let $X = \mathbb{R}$ with the usual metric d and $M = [1,\infty)$. Define $T,I : M \to M$ by $Tx = x^2$ and $Ix = 2x - 1$, for all $x \in M$. As $F(I) = \{1\}$, M is q-starshaped with $q = 1 \in F(I)$ and $C_q(I,T) = [1,\infty)$, (T,I) is a Banach operator pair on M since $T(F(I)) \subseteq F(I)$ but (T,I) is not C_q-commuting pair and hence not commuting.*

We shall denote $G_\circ$ by the class of closed convex subsets containing a point $x_\circ$ of a convex metric space (X,d) with property (I). For $M \in G_\circ$ and $p \in X$, let $M_p = \{x \in M : d(x,x_\circ) \leq 2d(p,x_\circ)\}$. Then $P_M(p) \subset M_p \in G_\circ$ as $x \in P_M(p) \Rightarrow d(p,x) = dist(p,M) \Rightarrow d(x,x_\circ) \leq d(x,p) + d(p,x_\circ) \leq 2d(p,x_\circ) \Rightarrow x \in M_p$. For a self mapping $g : X \to X$, let $C^g_M(x) = \{u \in M : g(u) \in P_M(x)\}$.

2. Main Results

In this section, we extend and generalize some recent common fixed point and invariant approximation results of Al-Thagafi [1], Chandok and Narang [4], Habiniak [10], Hussain and Jungck [12], Khan and Akbar [13], Narang and Chandok [15] [16], Shahzad [18], Vijayaraju [20] and of few others to convex metric spaces. We begin the section with the following result.

Proposition 2.1. *If C is a convex subset of a convex metric space (X,d) then the set $P_C(x)$ is convex and so starshaped.*

Proof. Let $y,z \in P_C(x)$ and $\lambda \in [0,1]$. Consider

$$\begin{aligned} d(x,W(y,z,\lambda)) &\leq \lambda d(x,y) + (1-\lambda)d(x,z) \\ &= \lambda d(x,C) + (1-\lambda)d(x,C) \\ &= d(x,C) \\ &\leq d(x,W(y,z,\lambda)) \text{ as } W(y,z,\lambda) \in C. \end{aligned}$$

Therefore, $d(x,W(y,z,\lambda)) = d(x,C)$ and so $W(y,z,\lambda) \in P_C(x)$.

Aliter. Since $P_C(x) = C \cap B(x,d(x,C))$, where $B(x,r)$ denotes a closed ball in X with center x and radius r, and in a convex metric space every ball is convex and intersection of convex sets is convex (see [19]), the result follows. This proof also implies that $P_C(x)$ is closed if C is closed. □

We shall be using the following result of Hussain [11] to prove our next theorem.

Lemma 2.2. [11] *Let M be a subset of a metric space (X,d) and (f,g) be a Banach operator pair on M. Assume that $cl(T(M))$ is complete, and T and g satisfy for all $x,y \in M$ and $0 \leq h < 1$,*

$$\begin{aligned} d(Tx,Ty) \leq\ & h \max\{d(fx,gy), d(Tx,fx), d(Ty,gy), d(Tx,gy), \\ & d(Ty,fx)\}. \end{aligned}$$

If f and g are continuous, $F(f) \cap F(g)$ is nonempty, then there is a unique common fixed point of T, f and g.

Theorem 2.3. *Let C be a q-starshaped subset of a convex metric space (X,d) with Property (I), and T,g and h be self maps of C. Suppose that g and h are continuous, and $F(g)$ and $F(h)$ are q-starshaped with $q \in F(g) \cap F(h)$. If $\overline{T(C)}$ is compact, T is continuous and pairs (T,g),*

(T,h) are Banach operator pairs and satisfy

$$\begin{aligned} d(Tx,Ty) \leq & \max\{d(hx,gy), dist(hx,[q,Tx]), dist(gy,[q,Ty]), \\ & dist(hx,[q,Ty]), dist(gy,[q,Tx])\} \end{aligned}$$

for all $x,y \in C$, then T,g and h have a common fixed point.

Proof. For each $n \geq 1$, define $T_n : C \to C$ by $T_n(x) = W(Tx,q,k_n)$, $x \in C$ where $< k_n >$ is a sequence in $(0,1)$ such that $k_n \to 1$. Then each T_n is a self mapping of C. Since (T,g) is a Banach operator pair and $F(g)$ is q-starshaped, for each $x \in F(g)$, $T_n(x) = W(Tx,q,k_n) \in F(g)$, since $Tx \in F(g)$. Thus (T_n,g) is a Banach operator pair for each n. Similarly, (T_n,h) is a Banach operator pair on C. Consider

$$\begin{aligned} d(T_nx,T_ny) &= d(W(Tx,q,k_n), W(Ty,q,k_n)) \\ &\leq d(Tx,Ty) \\ &\leq k_n \max\{d(hx,gy), dist(hx,[q,Tx]), dist(gy,[q,Ty]), \\ &\quad dist(hx,[q,Ty]), dist(gy,[q,Tx])\} \\ &\leq k_n \max\{d(hx,gy), d(hx,T_nx), d(gy,T_ny), d(hx,T_ny), \\ &\quad d(gy,T_nx)\} \end{aligned}$$

for all $x,y \in C$. As $\overline{T(C)}$ is compact, $\overline{T_n(C)}$ is compact for each n and hence complete. So by Lemma 2.2, there exists $x_n \in C$ such that x_n is a common fixed point of g,h and T_n for each n. The compactness of $\overline{T(C)}$ implies the existence of a subsequence $< Tx_{n_i} >$ of $< Tx_n >$ such that $Tx_{n_i} \to y \in C$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i}x_{n_i} = W(Tx,q,k_{n_i}) \to y,$$

and the result follows by using the continuity of T,h and g. □

Theorem 2.4. *Let f,g,T be self mappings of a convex metric space (X,d) with property (I), $M \in G_\circ$ such that $T(M_u) \subset f(M) \subset M = g(M)$ with $u \in F(T) \cap F(f) \cap F(g)$. Suppose that $d(fx,u) \leq d(x,u)$, $d(gx,u) = d(x,u)$ and $d(Tx,u) \leq d(fx,gu)$ for all $x \in M$, $\overline{f(M_u)}$ is compact, then*

(i) $P_M(u)$ is nonempty, closed and convex;

(ii) $T(P_M(u)) \subset f(P_M(u)) \subset P_M(u) = g(P_M(u))$;

(iii) $P_M(u) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$, provided T, f and g are continuous, $F(f)$ and $F(g)$ are p-starshaped with $p \in F(f) \cap F(g) \cap P_M(u)$, the pairs (T,f) and (T,g) are Banach operator pairs on $P_M(u)$

and satisfy

$$\begin{aligned} d(Tx,Ty) &\le max\{d(fx,gy), dist(fx,[p,Tx]), dist(gy,[p,Ty]), \\ &\quad dist(gx,[p,Ty]), dist(gy,[p,Tx])\} \end{aligned}$$

for all $x, y \in P_M(u)$, $p \in F(f) \cap F(g)$.

Proof. If $u \in M$ then all the three results are obvious. So assume that $u \notin M$. If $x \in M \backslash M_u$ then $d(x, x_\circ) > 2d(u, x_\circ)$ and so $d(u,x) \ge d(x,x_\circ) - d(u,x_\circ) > \underline{d(u,xo)} \ge dist(u,M)$. Thus $\alpha = dist(u,M) \le d(u,x_\circ)$. Since $\overline{T(M_u)}$ is compact, and the distance function is continuous, there exists $z \in \overline{T(M_u)}$ such that $\beta = dist(u, \overline{T(M_u)} = d(u,z)$. Hence

$$\begin{aligned} \alpha = dist(u,M) &\le dist(u, cl(T(M_u)) \text{ as } T(M_u) \subset M \Rightarrow \overline{T(M_u)} \subset M. \\ &= \beta \\ &= dist(u, T(M_u)) \\ &\le d(u,Tx) \\ &\le d(u,x) \end{aligned}$$

for all $x \in M_u$. Therefore $\alpha = \beta = dist(u,M)$ i.e. $dist(u,M) = dist(u, \overline{T(M_u)} = d(u,z)$ i.e. $z \in P_M(u)$ and so $P_M(u)$ is non-empty. The closedness and convexity of $P_M(u)$ follows from that of Proposition 2.1. This proves (i).

To prove (ii), let $z \in P_{M_u}$. Then $d(fz,u) = d(fz,fu) \le d(z,u) = dist(u,M)$. This implies that $fz \in P_M(u)$ and so $f(P_M(u)) \subset P_M(u)$. Similarly $g(P_M(u)) \subset P_M(u)$. For the converse, assume that $y \in P_M(u)$, then $y \in M = g(M)$. Thus there is some $x \in M$ such that $y = gx$. Now $d(x,u) = d(gx,u) = d(y,u) = dist(u,M)$. This implies that $x \in P_M(u)$ and so $g(P_M(u)) = P_M(u)$. Let $y \in T(P_M(u))$. Since $T(M_u) \subset f(M)$ and $P_M(u) \subset M_u$, there exists $z \in P_M(u)$ and $t_1 \in M$ such that $y = Tz = ft_1$. Further, we have

$$d(ft_1,u) = d(Tz,u) \le d(fz,gu) = d(fz,u) \le d(z,u) = dist(u,M).$$

Thus, $t_1 \in C_M^f(u)$. Also $ft_1 \in M$ and $dist(u,M) \le d(ft_1,u)$, it follows that $dist(u,M) = dist(ft_1,u)$. Since $d(t_1,u) = d(ft_1,u) = dist(u,M)$, $t_1 \in P_M(u)$ and $y = ft_1 \in f(P_M(u))$. Hence $T(P_M(u)) \subseteq f(P_M(u))$ and so (ii) holds.

By (ii), compactness of $\overline{f(M_u)}$ implies that $\overline{T(P_M(u))}$ is compact. The result (iii) follows from Theorem 2.3 applied to $P_M(u)$ $\square$

Remark 2.5. *1. Theorems 2.3 and 2.4 generalize and extend the corresponding results of Chandok and Narang* [4], [6], *Hussain* [11] *and of Narang and Chandok* [16].

2. Theorem 2.4 has been proved in normed linear spaces by Hussain [11].

We now prove the following result on the set of best simultaneous approximation using Theorem 2.3.

Theorem 2.6. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), G and S be continuous self-mappings of K such that T is (G, S)-asymptotically nonexpansive and $F(G) \cap F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S) \cap F(G)$. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.1}$$

for all $x \in X$ and $i = 1, 2$. If the pairs (T, G), (T, S) are Banach operator pairs on D, T is uniformly asymptotically regular on D and $F(S)$ and $F(G)$ are starshaped with respect to $z \in F(G) \cap F(S)$, then D contains T-, G- and S- invariant point.

Proof. Since D is the set of best simultaneous approximation to y_1 and y_2 and $d(Tx, y_i) \leq d(x, y_i)$ for all $x \in X$ and $i = 1, 2$, Tx is in D. Thus T maps D into itself. Since T is (G, S)-asymptotically nonexpansive, there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(Gx, Sy)$, for all $x, y \in K$. Define T_n as $T_n(x) = W(T^n x, z, a_n)$ for all $x \in D$ where $a_n = (1 - 1/n)/k_n$. Since (T, S) is a Banach operator pair and $F(S)$ is starshaped with respect to $z \in F(S)$, for each $x \in F(S)$ and $Tx \in F(S)$ we have $T_n(x) = W(T^n x, z, a_n) \in F(S)$ for each n. Thus (T_n, S) is a Banach operator pair for each n. Similarly, we can prove that (T_n, G) is a Banach operator pair for each n.

Since T is (G, S)-asymptotically nonexpansive, we have

$$\begin{aligned} d(T_n x, T_n y) &= d(W(T^n x, z, a_n), W(T^n y, z, a_n)) \\ &\leq a_n d(T^n x, T^n y) \\ &\leq a_n k_n d(Gx, Sy) \\ &= ((1 - (1/n))/k_n) k_n d(Gx, Sy) \\ &= (1 - (1/n)) d(Gx, Sy). \end{aligned}$$

Since, D is compact and T is continuous on D, by Theorem 2.3 there is a point x_n in D such that $x_n = T_n x_n = Sx_n = Gx_n$. Therefore

$$\begin{aligned} d(x_n, T^n x_n) &= d(T_n x_n, T^n x_n) \\ &= d(W(T^n x_n, z, a_n), T^n x_n) \\ &\leq a_n d(T^n x_n, T^n x_n) + (1 - a_n) d(z, T^n x_n) \\ &\to 0. \end{aligned}$$

Since (T, G) is Banach operator pair and $Gx_n = x_n$, $GT^n x_n = T^n G x_n = T^n x_n$, T is uniformly asymptotically regular and (G, S)-asymptotically nonexpansive on D and $x_n = T_n x_n = Sx_n$, it follows that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(G(T^n x_n), S(x_n)) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n (G x_n), S(x_n)) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d((T^n x_n), x_n) \\ &\to 0. \end{aligned}$$

Since D is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x \in D$. Since T is continuous, $T(x_{n_i}) \to T(x)$, and so

$$d(x, Tx) \leq d(x, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx) \to 0,$$

which gives $Tx = x$. Since G and S are continuous and $G(x_{n_i}) = x_{n_i} = S(x_{n_i})$, it follows that $Gx = x = Sx$. Hence $x \in F(T, S, G)$. $\square$

If $y_1 = y_2 = x$, we have

Corollary 2.7. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), G and S are continuous self-mappings of K such that T is (G, S)-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that the set D of best K-approximants to x is nonempty, compact and starshaped with respect to $z \in F(G) \cap F(S)$, and D is invariant under T. If the pairs (T, G), (T, S) are Banach operator pairs on D, T is uniformly asymptotically regular on D and $F(G)$ and $F(S)$ are starshaped with respect to $z \in F(G) \cap F(S)$, then D contains T-, G- and S-invariant point.*

If $G = S$, we have

Corollary 2.8. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty.*

Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$. Suppose that T satisfies

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.2}$$

for all $x \in X$ and $i = 1, 2$. If the pair (T, S) is a Banach operator pair on D, T is uniformly asymptotically regular on D and $F(S)$ is starshaped with respect to $z \in F(S)$, then D contains T- and S-invariant point.

Remark 2.9. *a. Theorem 2.6 has been proved in normed linear spaces by Khan and Akbar* [13].

b. In comparison with a theorem of Narang and Chandok ([15]*-Theorem 3), the uniform R-subweakly commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped.*

c. In comparison with a result of Narang and Chandok ([15]*-Corollary 4.1), the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped.*

d. In comparison with a result of Vijayaraju ([20]*-Corollary 2.4), the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped and the spaces undertaken are convex metric spaces.*

Conclusion. The results proved in this paper represent the strong variants of the corresponding results of Al-Thagafi (Theorem 4.1) [1], Habiniak (Theorem 8) [10], Hussain and Jungck (Theorem 2.14) [12], Narang and Chandok (Theorem 3, Corollary 4.1) [15], Shahzad (Theorem 2.4) [18], Vijayaraju (Corollary 2.4) [20] and of few others in the sense that the commutativity of the maps are replaced by the general hypothesis that the mappings are Banach operator pairs and linearity of mappings are also removed. Moreover, spaces undertaken are convex metric spaces.

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