"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 2, 69 - 76

FIXED POINT THEOREM FOR CYCLIC (μ, ψ, ϕ)-WEAKLY CONTRACTIONS

SUMIT CHANDOK AND VALERIU POPA

Abstract. In this article, we introduce the notion of cyclic (μ, ψ, ϕ)-weakly contraction and derive the existence of fixed point for such mappings in the setup of complete metric spaces. Our result extend and improve some fixed point theorems in the literature.

1. Introduction and Preliminaries

It is well known that the fixed point theorem of Banach, for contraction mappings, is one of the pivotal result in analysis. It has been used in many different fields of mathematics. Fixed point problems involving different type of contractive type inequalities have been studied by many authors (see [1]-[17] and references cited therein).

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [17] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [2] and of Reich types [16].

In [14], Kirk et al. introduced the following notion of cyclic representation and characterized the Banach Contraction Principle in the context of cyclic mapping.

Keywords and phrases: complete metric space; fixed point; cyclic (μ, ψ, ϕ)-weakly contraction.
(2010) Mathematics Subject Classification: 54H25, 47H10.

Definition 1.1. [14] Let X be a non-empty set and $T\colon X \to X$ an operator. By definition, $X = \cup_{i=1}^m X_i$ is a *cyclic representation* of X with respect to T if:

(a) X_i; $i = 1, \dots, m$ are non-empty sets,

(b) $T(X_1) \subset X_2, \dots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Pacurar and Rus [15] proved the following main theorem.

Theorem 1.2. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that $T\colon Y \to Y$ be an operator such that*

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2) $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$ where ϕ is monotone increasing continuous functions $\phi\colon [0, \infty) \to [0, \infty)$, with $\phi(t) > 0$, if $t > 0$; $\phi(t) = 0$ if and only if $t = 0$, $A_{m+1} = A_1$. Then, T has a fixed point $z \in \cap_{i=1}^n A_i$.

It is the aim of this paper to introduce the notion of *cyclic (μ, ψ, ϕ)-weakly contraction* mappings, and then derive a fixed point theorem for such cyclic contractions, in the framework of complete metric spaces.

2. Main Results

To state and prove our main results, we shall introduce our notion of cyclic (μ, ψ, ϕ)-weakly contraction mappings in metric space.

Let θ denote the set of all monotone increasing continuous functions $\mu\colon [0, \infty) \to [0, \infty)$, with $\mu(t) > 0$, if $t > 0$; $\mu(t) = 0$ if and only if $t = 0$ and $\mu(t_1 + t_2) \leq \mu(t_1) + \mu(t_2)$, for all $t_1, t_2 \in [0, \infty)$.

Let Φ denote the set of all continuous functions $\phi\colon [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$, for $t \in (0, \infty)$ and $\phi(0) = 0$.

Let Ψ denote the set of all functions $\psi : [0, \infty)^5 \to [0, \infty)$ such that

(a) ψ is continuous;

(b) ψ is strictly increasing in all the variables;

(c) for all $t \in [0, \infty)\backslash\{0\}$, $\psi(t, t, t, 0, 2t) \leq t$, $\psi(t, t, t, 2t, 0) \leq t$, $\psi(0, 0, t, t, 0) \leq t$, $\psi(0, t, 0, 0, t) \leq t$, and $\psi(t, 0, 0, t, t) \leq t$.

Definition 2.1. Let (X, d) be a metric space, m a natural number, $A_1, A_2 \dots, A_m$ nonempty subsets of X and $Y = \cup_{i=1}^m A_i$. An operator $T\colon Y \to Y$ is called a *cyclic (μ, ψ, ϕ)-weakly contraction* if

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2)

$$\begin{aligned}\mu(d(Tx,Ty)) &\leq \psi(\mu(d(x,y)),\mu(d(x,Tx)),\mu(d(y,Ty)),\\ &\quad \mu(d(x,Ty)),\mu(d(y,Tx)))-\phi(M(x,y)),\end{aligned}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$ where $A_{m+1} = A_1$, $\mu \in \theta$, $\phi \in \Phi$, $\psi \in \Psi$ and $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Theorem 2.2. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that T is a cyclic (μ,ψ,ϕ)-weakly contraction. Then, T has a fixed point $z \in \cap_{i=1}^n A_i$.*

Proof. Let $x_0 \in X$. We can construct a sequence $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, hence the result. Indeed, we can see that $Tx_{n_0} = x_{n_0+1} = x_{n_0}$.

Now, we assume that $x_{n+1} \neq x_n$ for any $n = 0,1,2,\ldots$. As $X = \cup_{i=1}^m A_i$, for any $n > 0$, there exists $i_n \in \{1,2,\ldots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T is a cyclic (μ,ψ,ϕ)-weakly contraction, we have

$$\begin{aligned}\mu(d(x_{n+1},x_n)) &= \mu(d(Tx_n,Tx_{n-1}))\\ &\leq \psi(\mu(d(x_n,x_{n-1})),\mu(d(x_n,Tx_n)),\mu(d(x_{n-1},Tx_{n-1})),\\ &\quad \mu(d(x_n,Tx_{n-1})),\mu(d(x_{n-1},Tx_n)))-\phi(M(x_n,x_{n-1}))\\ &= \psi(\mu(d(x_n,x_{n-1})),\mu(d(x_n,x_{n+1})),\mu(d(x_{n-1},x_n)),\\ &\quad \mu(d(x_n,x_n)),\mu(d(x_{n-1},x_{n+1})))-\phi(M(x_n,x_{n-1})).\end{aligned}$$

where $M(x_n,x_{n-1}) = \max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}$.

If $M(x_n,x_{n-1}) = d(x_n,x_{n+1})$, then

$$\begin{aligned}\mu(d(x_{n+1},x_n)) &\leq \psi(\mu(d(x_n,x_{n+1})),\mu(d(x_n,x_{n+1})),\mu(d(x_n,x_{n+1})),0,\\ &\quad 2\mu(d(x_n,x_{n+1})))-\phi(d(x_n,x_{n+1}))\\ &\leq \mu(d(x_n,x_{n+1}))-\phi(d(x_n,x_{n+1}))\\ &\leq \mu(d(x_n,x_{n+1})),\end{aligned}$$

which is a contradiction. Hence

$$\mu(d(x_{n+1},x_n)) \leq \mu(d(x_n,x_{n-1}))-\phi(d(x_n,x_{n-1})) \tag{2.1}$$

and

$$d(x_{n+1},x_n) \leq d(x_n,x_{n-1}).$$

Thus $\{d(x_{n+1},x_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Therefore, there exists $r \geq 0$ such that $d(x_{n+1},x_n) \to r$.

Letting $n \to \infty$ in (2.1), and using the continuity of μ and ϕ, we obtain that $\mu(r) \leq \mu(r) - \phi(r)$. This implies that $\phi(r) = 0$, hence $r = 0$. Thus we have

$$d(x_{n+1}, x_n) \to 0. \tag{2.2}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. For this purpose, we prove the following result first.

Lemma 2.3. *For every positive ϵ, there exists a natural number n such that if $r, q \geq n$ with $r - q \equiv 1 \pmod m$, then $d(x_r, x_q) < \epsilon$.*

Proof. Assume the contrary. Thus there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1 \pmod m$ satisfying $d(x_{r_n}, x_{q_n}) \geq \epsilon$.

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, we can choose r_n in such that it is a smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1 \pmod m$ and $d(x_{r_n}, x_{q_n}) \geq \epsilon$. Therefore, $d(x_{r_{n-m}}, x_{q_n}) < \epsilon$. By using the triangular inequality, we have

$$\begin{aligned}
\epsilon &\leq d(x_{q_n}, x_{r_n}) \\
&\leq d(x_{q_n}, x_{r_{n-m}}) + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}) \\
&< \epsilon + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}).
\end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we obtain

$$\lim d(x_{q_n}, x_{r_n}) = \epsilon. \tag{2.3}$$

Again, by the triangular inequality,

$$\begin{aligned}
\epsilon &\leq d(x_{q_n}, x_{r_n)} \\
&\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}) \\
&\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}) + \\
&\quad\ d(x_{r_{n+1}}, x_{r_n}).
\end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we get

$$\lim d(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{2.4}$$

Consider

$$\begin{aligned}
d(x_{q_n}, Tx_{r_n}) &= d(x_{q_n}, x_{r_{n+1}}) \\
&\leq d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}),
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned} d(x_{r_n}, Tx_{q_n}) &= d(x_{r_n}, x_{q_{n+1}}) \\ &\leq d(x_{r_n}, x_{q_n}) + d(x_{q_n}, x_{q_{n+1}}). \end{aligned} \tag{2.6}$$

On taking $n \to \infty$ in inequalities (2.5) and (2.6), we have

$$\lim_{n\to\infty} d(x_{q_n}, Tx_{r_n}) = \epsilon, \tag{2.7}$$

and

$$\lim_{n\to\infty} d(x_{r_n}, Tx_{q_n}) = \epsilon. \tag{2.8}$$

As x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact T is a cyclic (μ, ψ, ϕ)-weakly contraction, we obtain

$$\begin{aligned} \mu(\epsilon) &\leq \mu(d(x_{q_{n+1}}, x_{r_{n+1}})) \\ &= \mu(d(Tx_{q_n}, Tx_{r_n})) \\ &\leq \psi(\mu(d(x_{q_n}, x_{r_n})), \mu(d(x_{q_n}, Tx_{q_n})), \mu(d(x_{r_n}, Tx_{r_n})), \\ &\quad \mu(d(x_{q_n}, Tx_{r_n})), \mu(d(x_{r_n}, Tx_{q_n}))) - \phi(M(x_{q_n}, x_{r_n})) \\ &= \psi(\mu(d(x_{q_n}, x_{r_n})), \mu(d(x_{q_n}, x_{q_{n+1}})), \mu(d(x_{r_n}, x_{r_{n+1}})), \\ &\quad \mu(d(x_{q_n}, x_{r_{n+1}})), \mu(d(x_{r_n}, x_{q_{n+1}}))) - \phi(M(x_{q_n}, x_{r_n})), \end{aligned} \tag{2.9}$$

where $M(x_{q_n}, x_{r_n}) = \max\{d(x_{q_n}, x_{r_n}), d(x_{q_n}, Tx_{q_n}), d(x_{r_n}, Tx_{r_n})\}$

On taking $n \to \infty$ in (2.9), using (2.7) and (2.8), continuity of μ, and ϕ and property of ψ, we get that

$$\begin{aligned} \mu(\varepsilon) &\leq \psi(\mu(\varepsilon), \mu(0), \mu(0), \mu(\varepsilon), \mu(\varepsilon)) - \phi(\varepsilon) \\ &\leq \mu(\varepsilon) - \phi(\varepsilon). \end{aligned}$$

Consequently, $\phi(\varepsilon) \leq 0$, which is contradiction with $\varepsilon > 0$. Hence the result is proved. □

Now, using Lemma 2.3, we shall show that $\{x_n\}$ is a Cauchy sequence in Y. Fix $\epsilon > 0$. By Lemma 2.3, we can find $n_0 \in \mathbb{N}$ such that $r, q \geq n_0$ with $r - q \equiv 1 \pmod m$

$$d(x_r, x_q) \leq \frac{\epsilon}{2}. \tag{2.10}$$

Since $\lim d(x_n, x_{n+1}) = 0$, we can also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m}, \tag{2.11}$$

for any $n \geq n_1$.

Assume that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s-r \equiv k \pmod m$. Hence $s-r+t = 1 \pmod m$, for $t = m-k+1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \ldots + d(x_{s+1}, x_s). \tag{2.12}$$

Using (2.10), (2.11) and (2.12), we obtain

$$d(x_r, x_s) \leq \frac{\epsilon}{2} + j \times \frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m \times \frac{\epsilon}{2m} = \epsilon. \tag{2.13}$$

Hence $\{x_n\}$ is a Cauchy sequence in Y. Since Y is closed in X, then Y is also complete and there exists $x \in Y$ such that $\lim x_n = x$.

Now, we shall prove that x is a fixed point of T.

As $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i = \{1, 2, \ldots, m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$ and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_i$. By using the contractive condition, we can obtain

$$\begin{aligned}
\mu(d(x_{n_k+1}, Tx)) &= \mu(d(Tx_{n_k}, Tx)) \\
&\leq \psi(\mu(d(x_{n_k}, x)), \mu(d(x_{n_k}, Tx_{n_k})), \mu(d(x, Tx)), \\
&\quad \mu(d(x_{n_k}, Tx)), \mu(d(x, Tx_{n_k}))) - \phi(M(x_{n_k}, x)) \\
&= \psi(\mu(d(x_{n_k}, x)), \mu(d(x_{n_k}, x_{n_k+1})), \mu(d(x, Tx)), \\
&\quad \mu(d(x_{n_k}, Tx)), \mu(d(x, x_{n_k+1}))) - \phi(M(x_{n_k}, x)),
\end{aligned}$$

where $M(x_{n_k}, x) = \max\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx)\}$. Letting $n \to \infty$ and using continuity of μ and ϕ, we have

$$\begin{aligned}
\mu(d(x, Tx)) &\leq \psi(\mu(0), \mu(0), \mu(d(x, Tx)), \mu(d(x, Tx)), \mu(0)) - \phi(d(x, Tx)) \\
&\leq \mu(d(x, Tx)) - \phi(d(x, Tx)),
\end{aligned}$$

which is a contradiction unless $d(x, Tx) = 0$. Hence x is a fixed point of T.

Now, we shall prove the uniqueness of fixed point.

Suppose that x_1 and x_2 ($x_1 \neq x_2$) are two fixed points of T. Using the contractive condition and continuity of μ and ψ, we have

$$\begin{aligned}
\mu(d(x_1, x_2)) &= \mu(d(Tx_1, Tx_2)) \\
&\leq \psi(\mu(d(x_1, x_2)), \mu(d(x_1, Tx_1)), \mu(d(x_2, Tx_2)), \\
&\quad \mu(d(x_1, Tx_2)), \mu(d(x_2, Tx_1))) - \phi(M(x_1, x_2)) \\
&= \psi(\mu(d(x_1, x_2)), \mu(d(x_1, x_1)), \mu(d(x_2, x_2)), \mu(d(x_1, x_2)), \\
&\quad \mu(d(x_2, x_1))) - \phi(M(x_1, x_2)) \\
&= \psi(\mu(d(x_1, x_2)), \mu(0), \mu(0), \mu(d(x_1, x_2)), \mu(d(x_2, x_1))) - \\
&\quad \phi(M(x_1, x_2)),
\end{aligned}$$

where $M(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, x_1), d(x_2, x_2)\}$.

So we deduce that

$$\begin{aligned} \mu(d(x_1, x_2)) &\leq \psi(\mu(d(x_1, x_2)), 0, 0, \mu(d(x_1, x_2)), \\ &\quad \mu(d(x_2, x_1))) - \phi(d(x_1, x_2)) \\ &\leq \mu(d(x_1, x_2)) - \phi(d(x_1, x_2)) \end{aligned}$$

which is a contradiction unless $x_1 = x_2$. Hence the main result is proved. $\square$

If $\mu(a) = a$, then we have the following result.

Corollary 2.4. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that $T\colon Y \to Y$ be an operator such that*

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2) $d(Tx, Ty) \leq \psi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) - \phi(M(x, y))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$, $\phi \in \Phi$, $\psi \in \Psi$ and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. Then, T has a fixed point $z \in \cap_{i=1}^n A_i$.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, **Principles of weakly contractive maps in Hilbert spaces**, New results in Operator Theory, Advances Appl., Vol. 8 (I. Gohberg and Yu. Lyubich, eds.), Birkhauser, Basel, 1997, 7-22.

[2] D.W. Boyd and T.S.W. Wong, **On nonlinear contractions**, Proc. Am. Math. Soc. 20(1969), 458-464.

[3] S. Chandok, **Some common fixed point theorems for generalized nonlinear contractive mappings**, Comput. Math. Appl. 62(2011), 3692-3699.

[4] S. Chandok, **Common fixed points, invariant approximation and generalized weak contractions**, Int. J. Math. Math. Sci. Vol. 2012 (2012) Article ID 102980.

[5] S. Chandok, **Common fixed points for generalized nonlinear contractive mappings in metric spaces**, Mat. Vesn. 65(2013), 29-34.

[6] S. Chandok, **A fixed point result for weakly Kannan type cyclic contractions**, Int. J. Pure Appl. Math. 82(2) (2013), 253-260.

[7] S. Chandok, **Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces**, J. Nonlinear Analysis Optim. 4(1)(2013), 45-52.

[8] S. Chandok, M. S. Khan and K.P.R. Rao, **Some coupled common fixed point theorems for a pair of mappings satisfying a contractive condition of rational type without monotonicity**, Int. J. Math. Anal. 7(9)(2013), 433-440.

[9] S. Chandok and M. Postolache, **Fixed point theorem for weakly Chatterjea type cyclic contractions**, Fixed Point Theory Appl. 2013(2013):28.
[10] Chi-Ming Chen, **Fixed point theorems for ψ-contractive mappings in ordered metric spaces**, J. Appl. Math. (2012) Article ID 756453.
[11] E. Karapinar and K. Sadarangani, **Fixed point theory for cyclic (ϕ-ψ)-contractions**, Fixed Point Theory Appl. (2011) 2011:69.
[12] E. Karapinar and I.M. Erhan, **Best proximity on different type contractions**, Appl. Math. Inf. Sci. 5 (2011), 342-353.
[13] E. Karapinar, I.M. Erhan, **Cyclic contractions and fixed point theorems**, Filomat 26 (2012), 777-782.
[14] W. A. Kirk, P. S. Srinivasan and P. Veeramani, **Fixed points for mappings satisfying cyclical contractive conditions**, Fixed Point Theory 4(1)(2003), 79- 89.
[15] M. Pacurar and I.A. Rus, **Fixed point theory for cyclic ϕ-contractions**, Nonlinear Anal. 72(2010), 26832693.
[16] S. Reich, **Some fixed point problems**, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Sci. Fis. Nat. 57(1975), 194-198.
[17] B.E. Rhoades, **Some theorems on weakly contractive maps**, Nonlinear Anal. 47(2001), 2683-2693.

S. Chandok
Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar 143001,INDIA, e-mail: chandhok.sumit@gmail.com

V. Popa
“Vasile Alecsandri” University of Bacău, Calea Mărăşeşti 157, Bacău 600115, ROMANIA, e-mail: vpopa@ub.ro