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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 2, 77 - 85

NONEXPANSIVE OPERATORS ASSOCIATED TO A SYSTEM OF INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. In this paper we shall establish two results on the existence in $C_L([a,b];[a,b]^2)$ of the solutions of a system of iterative integral equations. The main tools used in our study are the nonexpansive operator technique and Schauder's fixed point theorem.

1. Introduction

Systems of integral equations with deviating arguments appear in a large number of physical, biological and economical researches. The iterative functional-integral equations form a class of integral equations with modified argument, such as the equation

$$x(t) = \int_a^b K(t;s;x(s);x(g(s)))ds + f(t) \quad (1.1)$$

considered by *Dobriţoiu* (see [6]). Here $t \in [a,b]$, $K \in C([a,b] \times [a;b] \times \mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$, $f \in C([a,b]; \mathbb{R}^m)$, $g \in C([a,b];[a,b])$. Some results concerning Picard operator associated with the problem (1.1) can be found in [1],[6],[7],[12],[14].

Before to present our main results, we will give some useful definitions and theorems.

Keywords and phrases: system of integral equations, existence of solutions, nonexpansive mapping.
(2010) Mathematics Subject Classification: 45B05, 45D05, 47H10

Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an $\alpha-$contraction if there exists $\alpha\in[0,1)$ such that

$$d(Tx,Ty)\le \alpha d(x,y)\ \text{ for all } x,y\in X.$$

If the above condition is satisfied with $\alpha=1$, the mapping T is said to be nonexpansive. Let K be a nonempty subset of a real normed linear space E and $T:K\to K$ be a map. In this setting, T is nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|\,,\ \text{ for all } x,y\in K.$$

The nonexpansive mappings are generalizations of α- contractions that do not inherit properties of contractive mappings. More precisely, if K is a nonempty closed subset of a Banach space E and $T:K\to K$ is a nonexpansive mapping which is not an α- contraction, then, as it is shown by the following example, T could not admit fixed points.

Example 1.1. ([8], *Example 3.3, pp. 30)*
In the space $c_0(N)$ the isometry defined by

$$T(x_1,x_2,...)=(1,x_1,x_2,...)$$

maps the unit ball into its boundary but T has not fixed points.

We will use the following variant of Schauder's fixed point theorem.

Theorem 1.2. (Schauder, [13]) *Let X be a Banach space, $K\subset X$ a nonempty, compact convex set and let $T:K\to K$ be a continuous operator. Then T has at least one fixed point.*

The aim of this paper is to obtain existence results for some systems of integral equations with deviating arguments, using the technique of nonexpansive operators introduced in [3] and used in [4], [10],[11],[9].

2. Main results

We consider a system of iterative integral equations of the form

$$\begin{cases} y_1(x)=f_1(x)+\int\limits_{x_0}^{x} K_1(s,y_1(s),y_2(s),y_1(y_1(s)))ds \\ y_2(x)=f_2(x)+\int\limits_{x_0}^{x} K_2(s,y_1(s),y_2(s),y_2(y_2(s)))ds \end{cases} \tag{2.1.1}$$

We consider $C_L([a,b];[a,b]^2)$, the Banach space $(C([a,b],\mathbb{R}^2);\|\cdot\|_C)$, endowed with the norm $\|x\|_C=(\|x_1\|,\|x_2\|)$, where $\|x_i\|=$

$\max\limits_{t\in[a,b]} |x_i(t)|$, $i = 1, 2$. We define the set

$$C_L([a,b],[a,b]^2) = \{(y_1,y_2) \in C([a,b],[a,b]^2) : |y_i(t_1) - y_i(t_2)| \leq \\ \leq L \cdot |t_1 - t_2|\,, \text{ for all } t_1,t_2 \in [a,b], i = 1,2\}$$

where $L > 0$ is fixed and denote

$$C_x = \max\{x - a, b - x\}$$

for all $x \in [a,b]$.

Theorem 2.1. *Assume that*

(i) $f \in C([a,b];[a,b]^2)$

(ii) $K \in C([a,b]^4;\mathbb{R}^2)$

(iii) There exists $L_k > 0$ *such that*

$$|K_i(s,u_1,v_1,w_1) - K_i(s,u_2,v_2,w_2)| \leq L_k \cdot (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|),$$

for every $s, u_i, v_i, w_i \in [a,b], i = 1,2$; *and*

there is $L' > 0$ *such that*

$$|f_i(t_1) - f_i(t_2)| \leq L' \cdot |t_1 - t_2| \quad \textit{for all } t_1,t_2 \in [a,b],\ i = 1,2\,.$$

(iv) $M + L' \leq L$, *where we denote*

$$M = \max_{1\leq i\leq 2} \sup\{|K_i(s,u_i,v_i,w_i)| : s,u_i,v_i,w_i \in [a,b]\}$$

(v) We have $|f_i(x)| \leq |f_i(x_0)|$, *for every* $x \in [a,b]$, $i = 1,2$ *and one of the following conditions holds:*

a) $M \cdot C_{x_0} \leq C_{y_0}$, $y_0 = \max(f_1(x_0), f_2(x_0))$;

b) $x_0 = a$, $M(b-a) \leq b - C_{y_0}$, $K(s,u_i,v_i,w_i) \geq 0$, *for all* $s,u_i,v_i,w_i \in [a,b], i = 1,2$;

c) $x_0 = b$, $M(b-a) \leq C_{y_0} - a$, $K(s,u_i,v_i,w_i) \geq 0$, *for all* $s,u_i,v_i,w_i \in [a,b], i = 1,2$;

(vi) $L_k \cdot (L+1) \cdot C_{x_0} \leq 1$.

Then the system (2.1.1) has at least one solution in $C_L([a,b];[a,b]^2)$.

Proof. It follows from [5, Lemma 1] that $C_L([a,b];[a,b]^2)$ is a nonempty, compact and convex subset of the Banach space $(C([a,b],\mathbb{R}^2);\|\cdot\|_C)$. For any $y \in C_L([a,b];[a,b]^2)$ and $t \in [a,b]$ we consider the integral operator $F : C_L([a,b];[a,b]^2) \to C([a,b];[a,b]^2)$ defined by

$$(Fy)(t) = ((F_1y)(t),(F_2y)(t)) = (f_1(t) + \int_{x_0}^{t} K(s,y_1(s),y_2(s),y_1(y_1(s)))ds, \\ f_2(t) + \int_{x_0}^{t} K(s,y_1(s),y_2(s),y_2(y_2(s)))ds).$$

The set of fixed points of F is the set of solutions of the system (2.1.1). We prove that $C_L([a,b],[a,b]^2)$ is an invariant set with respect to F, which means that $F(C_L([a,b],[a,b]^2)) \subset C_L([a,b],[a,b]^2)$. By $x \leq y$, $x, y \in \mathbb{R}^2$ we understand that $x_i \leq y_i$, for every $i = 1, 2$. We have

$$\begin{gathered}
|(Fy)(t)| = (|(F_1y)(t)|, |(F_2y)(t)|) \leq \\
\leq \left(|f_1(t)| + \int_{x_0}^{t} |K_1(s, y_1(s), y_2(s), y_1(y_1(s)))| \, ds, \right. \\
\left. |f_2(t)| + \int_{x_0}^{t} |K_2(s, y_1(s), y_2(s), y_2(y_2(s)))| \, ds \right) \\
\leq (|f_1(x_0)| + M \cdot |t - x_0|, |f_2(x_0)| + M \cdot |t - x_0|) \\
\leq (|f_1(x_0)| + M \cdot C_{x_0}, |f_2(x_0)| + M \cdot C_{x_0}) \\
\text{and} \\
|(Fy)(t)| = (|(F_1y)(t)|, |(F_2y)(t)|) \geq \\
\geq \left(|f_1(t)| - \int_{x_0}^{t} |K_1(s, y_1(s), y_2(s), y_1(y_1(s)))| \, ds, \right. \\
\left. |f_2(t)| - \int_{x_0}^{t} |K_2(s, y_1(s), y_2(s), y_2(y_2(s)))| \, ds \right) \\
\geq (|f_1(x_0)| - M \cdot |t - x_0|, |f_2(x_0)| - M \cdot |t - x_0|) \\
\geq (|f_1(x_0)| - C_{y_0}, |f_2(x_0)| - C_{y_0})
\end{gathered}$$

By hypothesis (iv), we conclude that each component of the vector $|(Fy)(t)|$ is in $[a,b]$. So $Fy \in C([a,b],[a,b]^2)$.
We shows that $Fy \in C_L([a,b],[a,b]^2)$, for every $y = (y_1, y_2) \in C_L([a,b],[a,b]^2)$. For any $t_1, t_2 \in [a,b]$ we have

$$\begin{gathered}
|(Fy)(t_1) - (Fy)(t_2)| = (|(F_1y)(t_1) - (F_1y)(t_2)|, |(F_2y)(t_1) - (F_2y)(t_2)|) \leq \\
(|f_1(t_1) - f_1(t_2)| + \textstyle\int_{t_1}^{t_2} |K_1(s, y_1(s), y_2(s), y_1(y_1(s)))| \, ds, \\
|f_2(t_1) - f_2(t_2)| + \textstyle\int_{t_1}^{t_2} |K_2(s, y_1(s), y_2(s), y_2(y_2(s)))| \, ds) \leq \\
((L_1' + M) \cdot |t_1 - t_2|, (L_2' + M) \cdot |t_1 - t_2|) \leq (L \cdot |t_1 - t_2|, L \cdot |t_1 - t_2|).
\end{gathered}$$

which proves by hypotesis (iii), that $Fy \in C_L([a,b],[a,b]^2)$, for every $y \in C_L([a,b],[a,b]^2)$.

For any $y, z \in C_L([a,b],[a,b]^2)$, $y = (y_1, y_2), z = (z_1, z_2)$ we have

$$\begin{aligned}
&|(Fy)(t) - (Fz)(t)| = (|(F_1y)(t) - (F_1z)(t)|\,, |(F_2y)(t) - (F_2z)(t)|) \leq \\
&\left(\int_{x_0}^{t} |K_1(s, y_1(s), y_2(s), y_1(y_1(s))) - K_1(s, z_1(s), z_2(s), z_1(z_1(s)))|\, ds,\right. \\
&\left.\int_{x_0}^{t} |K_2(s, y_1(s), y_2(s), y_2(y_2(s))) - K_2(s, z_1(s), z_2(s), z_2(z_2(s)))|\, ds\right) \leq \\
&\left(L_{k_1} \int_{x_0}^{t} (|y_1(s) - z_1(s)| + |y_2(s) - z_2(s)| + |y_1(y_1(s)) - z_1(z_1(s))|)\, ds,\right. \\
&\left.L_{k_2} \int_{x_0}^{t} (|y_1(s) - z_1(s)| + |y_2(s) - z_2(s)| + |y_2(y_2(s)) - z_2(z_2(s))|)\, ds\right) \leq \\
&\qquad \left(L_{k_1} \int_{x_0}^{t} [(1 + L_1)\, |y_1(s) - z_1(s)| + |y_2(s) - z_2(s)|]\, ds,\right. \\
&\qquad \left.L_{k_2} \int_{x_0}^{t} [(1 + L_2)\, |y_2(s) - z_2(s)| + |y_1(s) - z_1(s)|]\, ds\right) \leq \\
&\qquad \left(L_{k_1}(1 + L_1) \int_{x_0}^{t} (|y_1(s) - z_1(s)| + |y_2(s) - z_2(s)|) ds,\right. \\
&\qquad \left.L_{k_2}(1 + L_2) \int_{x_0}^{t} (|y_1(s) - z_1(s)| + |y_2(s) - z_2(s)|) ds\right)
\end{aligned}$$

Applying the maximum in last inequality, we obtain

$$\|Fy - Fz\| \leq L_f(1 + L) \cdot C_{x_0} \|y - z\|\,.$$

which, by hypotesis (v), shows that F is nonexpansiv operator, hence continuous, defined on the bounded, closed and convex subset $C_L([a,b],[a,b]^2)$ of Banach's space $(C([a,b],\mathbb{R}^2), \|\cdot\|_C)$ and applying the Schauder's fixed point theorem, F has at least one fixed point in $C_L([a,b];[a,b]^2)$. Consequently, the system (2.1.1) has at least one solution $y \in C_L([a,b];[a,b]^2)$. □

If the system is system of Fredholm integral equations with modified argument, of iterative type, then it is necessary to set new conditions of kernel.

We consider the following system of Fredholm integral equations with modified argument, of iterative type:

$$\text{(2.1.2)}\qquad \begin{cases} y_1(x) = f_1(x) + \int\limits_a^b K_1(x, y_1(s), y_2(s), y_1(y_1(s)))ds \\ y_2(x) = f_2(x) + \int\limits_a^b K_2(x, y_1(s), y_2(s), y_2(y_2(s)))ds, \end{cases}$$

where $x, s \in [a, b], y = (y_1, y_2) \in C([a, b]; [a, b]^2), f = (f_{1,}, f_2) \in C([a, b]; \mathbb{R}^2), K = (K_1, K_2) \in C([a, b]^4; \mathbb{R}^2)$.

The next theorem is a result on the existence of solutions in $C_L([a, b]; [a, b]^2)$ of the system (2.1.2).

Theorem 2.2. *Assume that*

(i) $K \in C([a, b]^4; \mathbb{R}^2)$ and $f \in C([a, b]; [a, b]^2)$;

(ii) There exists $L_k > 0$ such that

$$|K_i(t_1, u, v, w) - K_i(t_2, u, v, w)| \leq L_k \cdot |t_1 - t_2|,$$

for all$t_1, t_2 \in [a, b], i = 1, 2$ and

there exists $L' > 0$ such that

$$|f_i(t_1) - f_i(t_2)| \leq L' \cdot |t_1 - t_2| \;\; i = 1, 2, \text{ for every } t_1, t_2 \in [a, b];$$

(iii) There exists $L_k > 0$ such that

$$|K_i(s, u_1, v_1, w_1) - K_2(s, u_2, v_2, w_2)| \leq L'_k \cdot (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|),$$

for all $s, u_i, v_i, w_i \in [a, b]$, $i = 1, 2$;

(iv) $L' + L_k(b - a) \leq L$,

(v) We have $|f_i(x)| \leq |f_i(x_0)|$, for every $x \in [a, b]$, $i = 1, 2$ and one of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$, where $y_0 = \max(f_1(x_0), f_2(x_0))$and

$$M = \max_{1 \leq i \leq 2} \sup\{|K_i(s, u_i, v_i, w_i)| : s, u_i, v_i, w_i \in [a, b]\}.$$

b) $x_0 = a$, $M(b - a) \leq b - C_{y_0}$, $K(s, u_i, v_i, w_i) \geq 0$, for all $s, u_i, v_i, w_i \in [a, b], i = 1, 2$;

c) $x_0 = b$, $M(b - a) \leq C_{y_0} - a$, $K(s, u_i, v_i, w_i) \geq 0$, for all $s, u_i, v_i, w_i \in [a, b], i = 1, 2$;

(vi) $L'_k \cdot (L + 1) \cdot (b - a) \leq 1$.

Then the system (2.1.2) has at least one solution in $C_L([a, b]; [a, b]^2)$.

Proof. We consider the integral operator $F : C_L([a,b];[a,b]^2) \to C([a,b];[a,b]^2)$ defined by

$$(Fy)(t) = ((F_1y)(t),(F_2y)(t)) = (f_1(t) + \int\limits_a^b K(s,y_1(s),y_2(s),y_1(y_1(s)))ds,$$
$$f_2(t) + \int\limits_a^b K(s,y_1(s),y_2(s),y_2(y_2(s)))ds)$$

for all $y = (y_1, y_2) \in C_L([a,b];[a,b]^2)$ and $t \in [a,b]$.
The set of fixed point of the operator F coincides with the set of solutions of system (2.1.2). First we show that the invariance property $F(C_L([a,b],[a,b]^2)) \subset C_L([a,b],[a,b]^2)$ holds. We have

$$\begin{gathered}|(Fy)(t)| = (|(F_1y)(t)|, |(F_2y)(t)|) \leq \\ \leq \Big(|f_1(t)| + \textstyle\int_a^b |K_1(s,y_1(s),y_2(s),y_1(y_1(s)))|\, ds, \\ |f_2(t)| + \textstyle\int_a^b |K_2(s,y_1(s),y_2(s),y_2(y_2(s)))|\, ds\Big) \\ \leq (|f_1(x_0)| + M\cdot(b-a), |f_2(x_0)| + M\cdot(b-a))\end{gathered}$$

and

$$\begin{gathered}|(Fy)(t)| = (|(F_1y)(t)|, |(F_2y)(t)|) \geq \\ \geq \Big(|f_1(t)| - \textstyle\int_a^b |K_1(s,y_1(s),y_2(s),y_1(y_1(s)))|\, ds, \\ |f_2(t)| - \textstyle\int_a^b |K_2(s,y_1(s),y_2(s),y_2(y_2(s)))|\, ds\Big) \\ \geq (|f_1(x_0)| - M\cdot(b-a), |f_2(x_0)| - M\cdot(b-a)) \\ \geq (|f_1(x_0)| - C_{y_0}, |f_2(x_0)| - C_{y_0})\end{gathered}$$

By hypothesis (v) we deduce that each component of vector $|(Fy)(t)|$ is in $[a,b]$. So, $Fy \in C([a,b],[a,b]^2)$.
We show that $Fy \in C_L([a,b],[a,b]^2)$, for every $y \in C_L([a,b],[a,b]^2)$. For all $t_1, t_2 \in [a,b]$ we have

$$\begin{gathered}|(Fy)(t_1) - (Fy)(t_2)| = (|(F_1y)(t_1) - (F_1y)(t_2)|, |(F_2y)(t_1) - (F_2y)(t_2)|) \leq \\ (|f_1(t_1) - f_1(t_2)| + \textstyle\int_a^b |K_1(t_1,y_1(s),y_2(s),y_1(y_1(s))) - K_1(t_2,y_1(s),y_2(s),y_1(y_1(s)))|\, ds, \\ |f_2(t_1) - f_2(t_2)| + \textstyle\int_a^b |K_2(t_1,y_1(s),y_2(s),y_2(y_2(s))) - K_2(t_2,y_1(s),y_2(s),y_2(y_2(s)))|\, ds) \leq \\ ((L_1' + L_{k_1}(b-a))\cdot|t_1 - t_2|, (L_2' + L_{k_2}(b-a))\cdot|t_1 - t_2|)\end{gathered}$$

which shows by hypothesis (iv) that $Fy \in C_L([a,b],[a,b]^2)$, for every $y \in C_L([a,b],[a,b]^2)$.
On the other hand, for all $y, z \in C_L([a,b],[a,b]^2)$, $y = (y_1, y_2), z =$

(z_1, z_2) we have

$$\begin{gathered}
|(Fy)(t)-(Fz)(t)| = (|(F_1y)(t)-(F_1z)(t)|\,, |(F_2y)(t)-(F_2z)(t)|) \leq \\
(\int_a^b |K_1(t,y_1(s),y_2(s),y_1(y_1(s)))-K_1(t,z_1(s),z_2(s),z_1(z_1(s)))|\,ds, \\
\int_a^b |K_2(t,y_1(s),y_2(s),y_2(y_2(s)))-K_2(t,z_1(s),z_2(s),z_2(z_2(s)))|\,ds) \leq \\
(L'_{k_1}\int_a^b (|y_1(s)-z_1(s)|+|y_2(s)-z_2(s)|+|y_1(y_1(s))-z_1(z_1(s))|)\,ds, \\
L'_{k_2}\int_a^b (|y_1(s)-z_1(s)|+|y_2(s)-z_2(s)|+|y_2(y_2(s))-z_2(z_2(s))|)\,ds) \leq \\
(L'_{k_1}\int_a^b [(1+L_1)\,|y_1(s)-z_1(s)|+|y_2(s)-z_2(s)|]\,ds, \\
L'_{k_2}\int_a^b [(1+L_2)\,|y_2(s)-z_2(s)|+|y_1(s)-z_1(s)|]\,ds) \leq \\
(L'_{k_1}(1+L_1)\int_{x_0}^t (|y_1(s)-z_1(s)|+|y_2(s)-z_2(s)|)ds, \\
L'_{k_2}(1+L_2)\int_{x_0}^t (|y_1(s)-z_1(s)|+|y_2(s)-z_2(s)|)ds)
\end{gathered}$$

Applying the maximum in last inequality, we obtain

$$\|Fy - Fz\| \leq L'_k(1+L)\cdot(b-a)\,\|y-z\|\,.$$

which, by hypothesis (vi), shows that the operator F, defined on the compact and convex subset $C_L([a,b],[a,b]^2)$ of the Banach space $(C([a,b],\mathbb{R}^2),\|\cdot\|_C)$ is nonexpansive, hence continuous. Applying the Schauder's fixed point theorem, it follows that the operator F has at least one fixed point in $C_L([a,b];[a,b]^2)$. Consequently, the system (2.1.2) has at least one solution $y \in C_L([a,b];[a,b]^2)$. □

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