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SOME FIXED POINT THEOREMS FOR T-KANNAN CONTRACTIONS AND WEAKLY COMPATIBLE PAIRS OF MAPPINGS IN G-CONE METRIC SPACES

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Abstract. In the framework of G-cone metric spaces introduced by [9], we prove several fixed point theorems for mappings satisfying T-Kannan, respectively T-Chaterjea contractive conditions, as well as for weakly commuting pair of mappings satisfying contractive conditions. Our results extend and improve similar results that are known in cone metric spaces.

1. Introduction and preliminaries

Different generalizations of the notion of a metric space have been proposed by Gahler [16, 17] and by Dhage [1, 2]. However, Ha et al. [12] have pointed out that the results obtained by Gahler for his 2-metrics are independent, rather than generalizations, of the corresponding results in metric spaces, while in [19, 20], Zead Mustafa and Brailey Sims have pointed out that Dhage's notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid.

Zead Mustafa and Brailey Sims [19] introduced a more appropriate and robust notion of a generalized metric space. Guang and Xian [8] generalized the concept of metric spaces, replacing the set of real

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numbers by an ordered Banach space defining in this way a cone metric space. The notion of G-cone metric space in [9] generalizes the notions of G-metric space and Cone metric space.

Section 1 is dedicated to fixed points of generalized T-Kannan mappings such as TK_1 and TK_2-mappings.

In section 2, Some common fixed point theorems for weakly compatible mappings are proved.

Definition 1.1 ([8]). Let E be a real Banach space and P a subset of E. P is called a cone if and only if

i) P is closed, non-empty and $P \neq \{0\}$;

ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;

iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P ([8]). The cone P is called regular if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Definition 1.2 ([9]). Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to E$ satisfies:

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$; whenever $x \neq y$, for all $x, y \in X$,

(G3) $G(x, x, y) \leq G(x, y, z)$, whenever $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables), and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.

Then G is called a generalized cone metric on X and X is called a generalized cone metric space or more specifically a G-cone metric space. We use the following Proposition 1.1 in G-cone metric space same as in G-metric space.

Definition 1.3 ([9]). A G-cone metric space X is symmetric if

$$G(x, y, y) = G(y, x, x) \quad \text{for all } x, yX.$$

Definition 1.4 ([9])**.** Let X be a G-cone metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is:

a) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$;

b) Convergent sequence if for every c in E with $0 \ll c$, there is N such that for all $n, m > N$, $G(x, x_n, x_m) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Remark 1.5 ([18])**.** *Let E be an ordered Banach (normed) space. Then c is an interior point of P, if and only if $[-c, c]$ is a neighborhood of* 0.

Remark 1.6. *i) If $a \le b$ and $b \ll c$, then $a \ll c$.*
Indeed, $c - a = (c - b) + (b - a) \ge c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

ii) If $0 \le u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.

Remark 1.7 ([7])**.** *If $c \in \operatorname{int} P$, $0 \le a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.*

Remark 1.8. *Let $0 \ll c$. If $0 \le G(x, x_n, x_m) \le b_n$ and $b_n \to 0$, then eventually $G(x, x_n, x_m) \ll c$, where x_n is sequence and x is given point in X.*

Remark 1.9 ([7])**.** *If $0 \le a_n \le b_n$ and $a_n \to a$, $b_n \to b$, then $a \le b$, for each cone P.*

Remark 1.10 ([7])**.** *If E is a real Banach space with cone P and if $a \le \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.*

Remark 1.11. *Let (X, G) be a G-cone metric space. Let us remark that the family $\{N(x, e) : x \in X, 0 \ll e\}$, where $N(x, e) = \{y \in X : G(x, y, y) \ll e\}$, is a subbasis for topology on X. We denote this cone topology by τ_0, and note that is τ_0 a Hausdorff topology (see, e.g.,* [15] *without proof).*

For the proof of the last statement, we suppose that for each c, $0 \ll c$ we have $N(x, c) \cap N(y, c) \ne \phi$. Thus, there exists $y \in X$ such that $G(x, a, a) \ll c/2$ and $G(a, y, y) \ll c/2$. Hence, $G(x, y, y) \le G(x, a, a) + G(a, y, y) \ll c/2 + c/2 = c$. Clearly, for each n, we have $c/n \in \operatorname{int} P$, so $c/n - G(x, y, y) \in \operatorname{int} P \subset P$. Now, $0 - G(x, y, y) \in P$, that is, $G(x, y, y) \in -P \cap P$, and we have $G(x, y, y) = 0$.

We use following proposition in G-cone metric space.

Proposition 1.12 ([8]). *Let (X, G) be a G-metric space. Then for any x, y, z and $a \in X$ it follows that:*
i) If $G(x, y, z) = 0$, then $x = y = z$,
ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
iii) $G(x, y, y) \leq G(y, x, x)$,
iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
v) $G(x, y, z) \leq 2/3(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$

Proposition 1.13 ([9]). *Let X be a G-cone metric space, $\{x_m\}$, $\{y_n\}$, $\{z_l\}$ be sequences in X such that $x_n \to x$, $y_n \to y$, $z_n \to z$ then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.*

Example 1.14. *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $G : X \times X \times X \to E$ defined by $G(x, y, z) = (|x-y| + |y-z| + |z-x|, \alpha[|x-y| + |y-z| + |z-x|])$ where $\alpha \geq 0$ is constant. Then (X, G) is a G-cone metric space.*

2. Fixed point theorems for generalized T-Kannan type mappings in G-cone metric spaces

J. Moralesa and E. Rojas [10] analyzed the existence (and uniqueness) of fixed points of T-Kannan type contractive mappings S defined on a complete cone metric space (M, d), as well as, T-Chaterjea mappings.

In this section, we obtain sufficient condition for the existence of a unique fixed point of generalized T-Kannan type mappings on complete G-cone metric spaces.

Definition 2.1. Let (X, G) be a G-cone metric space, P a normal cone with normal constant K and $T : X \times X$. Then

i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $T(x_n) = T(x)$, for all $\{x_n\}$ in X;

ii) T is said to be subsequentially convergent, if for every sequence $\{x_n\}$ in X, the transformed sequence $\{T(x_n)\}$ contains a convergent subsequence in X;

iii) T is said to be sequentially convergent if for every sequence $\{x_n\}$ in X, the transformed sequence $\{T(x_n)\}$ contains a convergent sequence in X.

Definition 2.2. Let (X, G) be a G-cone metric space and $T, S : M \to M$ two functions.

(K1) A mapping S is said to be a generalized T-Kannan contraction, (TK_1-generalized contraction) if there is $b \in [0, 1/6)$ constant such that

$$G(TSx, TSy, TSz) \leq b[G(Tx, TSx, TSx) + G(Ty, TSy, TSy) \\ + G(Tz, TSz, TSz)]$$

for $x, y, z \in X$.

(K2) A mapping S is said to be a T-Chatterjea contraction, (TK_2-generalized contraction) if there is $c \in [0, 1/6)$ constant such that

$$G(TSx, TSy, TSz) \leq c[G(Tx, TSy, TSy) + G(Ty, TSz, TSz) \\ + G(Tz, TSx, TSx)]$$

for all $x, y, z \in X$.

Example 2.3. *Let $E = (C_{[0,1]}, R)$, $P = \{\varphi \in E/\varphi \geq 0\} \subseteq E$, $X = R$ and $G : X \times X \times X \to E$ defined by $G(x, y, z) = (|x-y|+|y-z|+|z-x|)$ $f(x)$, where $f(x)$ is a positive function on $I[0, 1]$. Then (X, G) is a G-cone metric space.*

Here, S is a TK_1- generalized contraction. Moreover, it is not difficult to show that S is besides a TK_2- generalized contraction.

Theorem 2.4. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K. Suppose T is a one to one and continuous mapping from X into itself and $S : X \to X$ a TK_1-generalized contraction. Then,*
i) For every $x_0 \in X$

$$\lim_{n\to\infty} G(TS^n x_0, TS^{n+1} x_0, TS^{n+1} x_0) = 0;$$

ii) There is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v;$$

iii) If T is subsequentially convergent; then $(S^n x_0)$ has a convergent subsequence; iv) There is a unique $u \in X$ such that

$$Su = u;$$

v) If T is sequentially convergent; then for each $x_0 \in X$ the iterate sequence $(S^n x_0)$ converges to u.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence (x_n) by $x_{n+1} = Sx_n = S^n x_0$. We have

$$\begin{aligned} G(Tx_n, Tx_{n+1}, Tx_{n+1}) &= G(TSx_{n-1}, TSx_n, TSx_n) \\ &\leq b[G(Tx_{n-1}, TSx_{n-1}, TSx_{n-1}) \\ &\quad + G(Tx_n, TSx_n, TSx_n) \\ &\quad + G(Tx_n, TSx_n, TSx_n)] \end{aligned}$$

so,

$$\begin{aligned} G(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leq bG(Tx_{n-1}, TSx_{n-1}, TSx_{n-1}) \\ &\quad + 2G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ G(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leq \frac{b}{1-2b} G(Tx_{n-1}, Tx_n, Tx_n) \end{aligned}$$

and we can conclude, by repeating the same argument, we get that

$$\begin{aligned} &G(TS^n x_0, TS^{n+1} x_0, TS^{n+1} x_0) \\ &\leq \left(\frac{b}{1-2b}\right)^n G(Tx_0, TSx_0, TSx_0) \end{aligned} \tag{2.1}$$

From (2.1) we have

$$\|G(TS^n x_0, TS^{n+1} x_0, TS^{n+1} x_0)\| \leq \left(\frac{b}{1-2b}\right)^n K \|G(Tx_0, TSx_0, TSx_0)\|$$

where K is the normal constant of E. By inequality above we get

$$\lim_{n\to\infty} \|G(TS^n x_0, TS^{n+1} x_0, TS^{n+1} x_0)\| = 0$$

hence,

$$\lim_{n\to\infty} G(TS^n x_0, TS^{n+1} x_0, TS^{n+1} x_0) = 0 \tag{2.2}$$

By inequality (2.1), for every $m, n \in N$ with $m > n$ we have

$$\begin{aligned} &G(Tx_n, Tx_m, Tx_m) \\ &\quad \leq G(Tx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}) + \cdots \\ &\qquad + G(Tx_{m-1}, Tx_m, Tx_m) \\ &\quad \leq \left[\left(\frac{b}{1-2b}\right)^n + \left(\frac{b}{1-2b}\right)^{n+1} + \cdots + \left(\frac{b}{1-2b}\right)^{m-1}\right] \\ &\qquad \times G(Tx_0, TSx_0, TSx_0) \\ &\quad \leq \left(\frac{b}{1-2b}\right)^n \frac{1}{(1-\frac{b}{1-2b})} G(Tx_0, TSx_0, TSx_0) \end{aligned}$$

So, for all positive integer $m > n$, we have

$$G(TS^n x_0, TS^m x_0, TS^m x_0) \leq \left(\frac{b}{1-2b}\right)^n \frac{1}{\left(1-\frac{b}{1-2b}\right)} G(Tx_0, TSx_0, TSx_0) \tag{2.3}$$

from (2.3) we have,

$$\|G(TS^n x_0, TS^m x_0, TS^m x_0)\| \leq \left(\frac{b}{1-2b}\right)^n \frac{1}{\left(1-\frac{b}{1-2b}\right)} \|G(Tx_0, TSx_0, TSx_0)\|$$

where K is the normal constant of X. Taking limit and keeping in mind that $\left(\frac{b}{1-2b}\right) < 1$, we obtain

$$\lim_{n\to\infty} \|G(TS^n x_0, TS^m x_0, TS^m x_0)\| = 0\,.$$

In this way we have, $\lim_{n\to\infty} G(TS^n x_0, TS^m x_0, TS^m x_0) = 0$, which implies that $(TS^n x_0)$ is a Cauchy sequence in M. Since X is a complete G-cone metric space, then there is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v \tag{2.4}$$

Now, if T is subsequentially convergent, $(S^n x_0)$ has a convergent subsequence. So, there are $u \in X$ and (x_{n_i}) such that

$$\lim_{i\to\infty} S^{n_i} x_0 = u \tag{2.5}$$

Since T is continuous and by (2.5) we obtain

$$\lim_{i\to\infty} TS^{n_i} x_0 = Tu \tag{2.6}$$

By (2.4) and (2.6) we conclude that

$$Tu = v \tag{2.7}$$

On the other hand,

$$\begin{aligned}
G(TSu,Tu,Tu) &\le G(TSu,TS^{n_i}x_0,TS^{n_i}x_0)\\
&\quad + G(TS^{n_i}x_0,TS^{n_i+1}x_0,TS^{n_i+1}x_0)\\
&\quad + G(TS^{n_i+1}x_0,Tu,Tu)\\
&\le b[G(Tu,TSu,TSu)+2G(TS^{n_i-1}x_0,TS^{n_i}x_0,TS^{n_i}x_0)]\\
&\quad + \left(\frac{b}{1-2b}\right)^{n_i} G(Tx_0,TSx_0,TSx_0) + G(TS^{n_i+1}x_0,Tu,Tu)\\
G(TSu,Tu,Tu) &\le b[2G(TSu,Tu,Tu)+2G(TS^{n_i-1}x_0,TS^{n_i}x_0,TS^{n_i}x_0)]\\
&\quad + \left(\frac{b}{1-2b}\right)^{n_i} G(Tx_0,TSx_0,TSx_0)\\
&\quad + G(TS^{n_{i+1}}x_0,Tu,Tu)
\end{aligned}$$

hence,

$$\begin{aligned}
G(TSu,Tu,Tu) \le &\left(\frac{2b}{1-2b}\right) 2G(TS^{n_i-1}x_0,TS^{n_i}x_0,TS^{n_i}x_0)]\\
&+ \left(\frac{2b}{1-2b}\right)\left(\frac{b}{1-2b}\right)^{n_i} G(Tx_0,TSx_0,TSx_0)\\
&+ \left(\frac{2b}{1-2b}\right) G(TS^{n_{i+1}}x_0,Tu,Tu)
\end{aligned}$$

$$\begin{aligned}
\|G(TSu,Tu,Tu)\| \le &\left(\frac{bK}{1-2b}\right)\|G(TS^{n_i-1}x_0,TS^{n_i}x_0,TS^{n_i}x_0)\|\\
&\frac{K}{1-2b}\left(\frac{b}{1-2b}\right)^{n_i}\|G(TSx_0,Tx_0,Tx_0)\|\\
&+\frac{K}{1-2b}\|G(TS^{n_i+1}x_0,Tu,Tu)\| \to 0 \quad (i\to\infty),
\end{aligned}$$

where K is the normal constant of X. By convergence above we get that $G(TSu,Tu,Tu) = 0$, which implies the equality $TSu = Tu$. Since T is one to one, then $Su = u$, consequently S has a fixed point. Because S is a TK_1-generalized contraction we have

$$\begin{aligned}
&G(TSu,TSv,TSv)\\
&\quad \le b[G(Tu,TSu,TSu)+G(Tv,TSv,TSv)+G(Tv,TSv,TSv)].
\end{aligned}$$

If v is another fixed point of S, then from the injectivity of T we get $Su = Sv$, or is the same, the fixed point is unique. Finally, if T is sequentially convergent, by replacing (S^{n_i}) for (S^n) we conclude that

$$\lim_{n\to\infty} S^n x_0 = u.$$

This shows that $(S^n x_0)$ converges to the fixed point of S. $\square$

Theorem 2.5. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K. Suppose T is a one to one and continuous mapping from X into itself and $S : X \to X$ a TK$_2$-generalized contraction. Then,*

i) For every $x_0 \in X$

$$\lim_{n\to\infty} G(TS^n x_0, TS^{n+1}x_0, TS^{n+1}x_0) = 0.$$

ii) There is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v.$$

iii) If T is subsequentially convergent; then $(S^n x_0)$ has a convergent subsequence.

iv) There is a unique $u \in X$ such that

$$Su = u.$$

If T is sequentially convergent; then for each $x_0 \in X$, $(S^n(x_0))$ converges to u.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence (x_n) by $x_{n+1} = Sx_n = S^n x_0$. Since S is a TK$_2$-generalized contraction, we have

$$\begin{aligned}
G(TSx_n, TSx_{n+1}, TSx_{n+1}) &\le c[G(Tx_n, TSx_{n+1}, TSx_{n+1}) \\
&\quad + G(Tx_{n+1}, TSx_{n+1}, TSx_{n+1}) \\
&\quad + G(Tx_{n+1}, TSx_n, TSx_n)] \\
&\le c[G(TSx_{n-1}, TSx_{n+1}, TSx_{n+1}) \\
&\quad + G(TSx_n, TSx_{n+1}, TSx_{n+1}) \\
&\le c[G(TSx_{n-1}, TSx_n, TSx_n) \\
&\quad + G(TSx_n, TSx_{n+1}, TSx_{n+1}) \\
&\quad + G(TSx_n, TSx_{n+1}, TSx_{n+1})].
\end{aligned}$$

Thus,

$$\begin{aligned}
G(TSx_n, TSx_{n+1}, TSx_{n+1}) &\le \frac{c}{1-2c} G(TSx_{n-1}, TSx_n, TSx_n) \\
&= hG(TSx_{n-1}, TSx_n, TSx_n)
\end{aligned}$$

where $h = \dfrac{c}{1-2c}$, Recursively, we obtain

$$(2.8) \quad G(TSx_n, TSx_{n+1}, TSx_{n+1}) \le h^n G(TSx_0, TSx_1, TSx_1)$$
$$\|G(TSx_n, TSx_{n+1}, TSx_{n+1})\| \le h^n \cdot K\|G(TSx_0, TSx_1, TSx_1)\|$$

where K is the normal constant of X. Hence

$$\lim_{n\to\infty} \|G(TSx_n, TSx_{n+1}, TSx_{n+1})\| = 0,$$

this implies that

$$\lim_{n\to\infty} G(TSx_n, TSx_{n+1}, TSx_{n+1}) = 0,$$

By (2.8), for every $m, n \in N$ with $n > m$ we have,

$$\begin{aligned} G(TSx_n, TSx_m, TSx_m) &\le G(TSx_n, TSx_{n+1}, TSx_{n+1}) + \cdots \\ &\quad + G(TSx_{m-1}, TSx_m, TSx_m) \\ &\le [h^{n-1} + h^{n-2} + \cdots + h^m] G(TSx_0, TSx_1, TSx_1) \\ &\le \frac{h^m}{1-h} G(TSx_0, TSx_1, TSx_1), \end{aligned}$$

taking norm we get

$$\|G(TSx_n, TSx_m, TSx_m)\| \le \frac{h^m}{1-h} K \|G(TSx_0, TSx_1, TSx_1)\|$$

we have

$$\lim_{n\to\infty} \|G(TSx_n, TSx_m, TSx_m)\| = 0,$$

hence $(TS^n x_0)$ is a Cauchy sequence in X and since X is a complete G-cone metric space, there is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 2.6. *Our results in Theorem 2.1 extends Theorem 3.1 from Morales and Rojas* [10] *and Theorem 2.2 extends Theorem 3.5 from Morales and Rojas* [10].

3. Fixed point theorems in G-cone metric spaces

K. Jha [11] and G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic [7] proved Common Fixed Point Theorems for Weakly Compatible Pairs on Cone Metric Spaces. Also many authors [3, 4, 5, 14] obtained common fixed point theorems involving cone metric spaces.

In this section, we obtain necessary and sufficient conditions for the existence of common fixed points for two and three self mappings of a

G-cone metric space. Our results extend, generalize a number of fixed point theorems of Abbas and Jungck [13], Jungck and Rhoades [6] and K. Jha [11].

Definition 3.1. Let (X, G) be a G-cone metric space and P a cone with nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that the range of g contains the range of f, and $f(X)$ or $g(X)$ is a complete subspace of X. We will call the pair (f, g) as Abbas and Jungck's pair, or AJ's pair.

Definition 3.2 ([13])**.** Let f and g be self-maps of a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Proposition 3.3 ([13])**.** *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Now we generalize the results of K. Jha [11] in G-cone metric space

Theorem 3.4. *Let (X, G) be a G-cone metric space, and P be a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$\begin{aligned} G(fx, fy, fz) \leq r[&G(gy, fx, fx) + G(gx, fy, fy) + G(gx, fx, fx) \\ &+ G(gy, fy, fy) + G(gx, gy, gz)] \end{aligned}$$

where $r \in [0, 1/4)$ is a constant. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have an unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then, since $fX \subseteq gX$, we can choose a point x_1 in X such that $f(x_0) = g(x_1)$. Continuing like this, having chosen x_n in X, we get x_{n+1} in X such that $f(x_n) =$

$g(x_{n+1})$. Then,

$$\begin{aligned}
G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n)\\
&\leq r[G(gx_n, fx_{n-1}, fx_{n-1}) + G(gx_{n-1}, fx_n, fx_n)\\
&\quad + G(gx_{n-1}, fx_{n-1}, fx_{n-1}) + G(gx_n, fx_n, fx_n)\\
&\quad + G(gx_{n-1}, gx_n, gx_n)]\\
&\leq r[G(gx_n, gx_n, gx_n) + G(gx_{n-1}, gx_{n-1}, gx_{n-1})\\
&\quad + G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1})\\
&\quad + G(gx_{n-1}, gx_n, gx_n)]\\
&\leq r[2G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1})]
\end{aligned}$$

So, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq hG(gx_n, gx_{n-1}, gx_n), \quad \text{with } h = \frac{2r}{1-r}\,.$$

Now, for $n > m$, we get

$$\begin{aligned}
G(gx_n, gx_m, gx_m) \leq{}& G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2})\\
&+ G(gx_{n+2}, gx_{n+3}, gx_{n+3}) + \cdots + G(gx_{m-1}, gx_m, gx_m)
\end{aligned}$$

$$\begin{aligned}
&\leq (h^{n-1} + h^{n-2} + \cdots + h^m)G(gx_0, gx_1, gx_1)\\
&\leq \frac{h^m}{1-h}G(gx_0, gx_1, gx_1)
\end{aligned}$$

which, using the normality of cone P, implies that

$$\|G(gx_n, gx_m, gx_m)\| \leq \frac{h^m}{1-h}\|G(gx_0, gx_1, gx_1)\|$$

Then, $G(gx_n, gx_m, gx_m) \to 0$ as $n, m \to \infty$, and so $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is a complete subspace of X, so there exists q in $g(X)$ such that $gx_n \to q$, as $n \to \infty$. Consequently, we can find p

in X such that $g(p) = q$. Thus,

$$\begin{aligned}
G(gx_n, fp, fp) &= G(fx_{n-1}, fp, fp) \\
&\leq r[G(gp, fx_{n-1}, fx_{n-1}) + G(gx_{n-1}, fp, fp) \\
&\quad + G(gx_{n-1}, fx_{n-1}, fx_{n-1}) + G(gp, fp, fp) \\
&\quad + G(gx_{n-1}, gp, gp)] \\
&\leq r[G(gp, gx_n, gx_n) + G(gx_{n-1}, fp, fp) \\
&\quad + G(gx_{n-1}, gx_n, gx_n) + G(gp, fp, fp)] \\
G(q, fp, fp) &\leq r[G(q, fp, fp) + G(q, fp, fp)] \text{ as } n \to \infty. \\
G(q, fp, fp) &\leq 2r[G(q, fp, fp)]
\end{aligned}$$

Then by Remark 1.6, we have

$$G(q, fp, fp) = 0.$$

This implies $fp = q$.

The uniqueness of a limit in a cone metric space implies that $f(p) = g(p)$. Again, we show that f and g have a unique point of coincidence.

For this, if possible, assume that there exists another point t in X such that $f(t) = g(t)$.

Then, we have

$$\begin{aligned}
G(gt, gp, gp) &= G(ft, fp, fp) \\
&\leq r[G(gp, ft, ft) + G(gt, fp, fp) + G(gt, ft, ft) \\
&\quad + G(gp, fp, fp) + G(gt, gp, gp)] \\
G(gt, gp, gp) &\leq r[G(gp, gt, gt) + G(gt, gp, gp) \\
&\quad + G(gt, gt, gt) + G(gp, gp, gp) + G(gt, gp, gp)]
\end{aligned}$$

$$\begin{aligned}
G(gt, gp, gp) &\leq r[2G(gt, gp, gp) + G(gt, gp, gp) \\
&\quad + G(gt, gp, gp)] \\
G(gt, gp, gp) &\leq 4r[G(gt, gp, gp)]
\end{aligned}$$

Then by Remark 1.6, we have

$$G(gt, gp, gp) = 0.$$

This implies $gt = gp$.

Then by using the Proposition 3.1, we get that f and g have a unique common fixed point.

This completes the proof of the theorem. □

Corollary 3.5 ([11]). *Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$G(fx, fy) \leq r[G(fx, gy) + G(fy, gx) + G(fx, gx) + G(fy, gy)]$$

where $r \in [0, 1/4)$ is a constant. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Example 3.6. *Let $E = I^2$, $I = [0, 1]$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq I^2$, $G : I \times I \times I \to E$ such that $G(x, y, z) = (|x - y| + |y - z| + |z - x|, \alpha[|x - y| + |y - z| + |z - x|])$ where $\alpha > 0$ is constant. Define $f(x) = \dfrac{\alpha x}{1 + \alpha x}$, for all $x \in I$ and $gx = \alpha x$ for all $x \in I$. Then, for $\alpha = 1$, both the mappings f and g are weakly compatible and satisfy all the conditions of the above theorem with $x = 0$ as unique common fixed point.*

Now we will prove some fixed point theorems of contractive mappings for G-cone metric space. We generalize some results of [8, 13] by omitting the assumption of normality in the results.

Theorem 3.7. *Let (X, G) be a G-cone metric space. Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists*

$$u = u(x, y, z) \in \left\{ G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), \frac{G(gx, fy, fy) + G(gz, fx, fx)}{2} \right\} \tag{3.1}$$

such that

$$G(fx, fy, fz) \leq \lambda u. \tag{3.2}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and let $x_1 \in X$ be such that $gx_1 = fx_0 = y_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $gx_{n+1} = fx_n = y_n$.

We first claim that

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_n, y_n) \quad \text{for } n \geq 1 \tag{3.3}$$

We have

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(fx_n, fx_{n+1}, fx_{n+1}) \tag{3.4}$$

where

$$\begin{aligned} u \in \Big\{ & G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, fx_n, fx_n), \\ & G(gx_{n+1}, fx_{n+1}, fx_{n+1}), \\ & \frac{G(gx_n, fx_{n+1}, fx_{n+1}) + G(gx_{n+1}, fx_n, fx_n)}{2} \Big\} \\ = \Big\{ & G(y_{n-1}, y_n, y_n), G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), \\ & \frac{G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_n, y_n)}{2} \Big\} \\ = \Big\{ & G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), \frac{G(y_{n-1}, y_{n+1}, y_{n+1})}{2} \Big\} \end{aligned} \tag{3.5}$$

Now we have the following three cases.

If $u = G(y_{n-1}, y_n, y_n)$ then clearly (3.3) holds. If $u = G(y_n, y_{n+1}, y_{n+1})$ then according to Remark 1.6 $G(gx_n, gx_{n+1}, gx_{n+1}) = 0$ and (3.3) is immediate. Finally, suppose that $u = (1/2)G(y_{n-1}, y_{n+1}, y_{n+1})$. Now,

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\leq \lambda(1/2)G(y_{n-1}, y_{n+1}, y_{n+1}) \\ &\leq \frac{\lambda}{2}G(y_{n-1}, y_n, y_n) + \frac{1}{2}G(y_n, y_{n+1}, y_{n+1}) \end{aligned} \tag{3.6}$$

Hence, $G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_n, y_n)$ and we proved (3.3).

Now, we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda^n G(y_0, y_1, y_1) \tag{3.7}$$

We will show that $\{y_n\}$ is a Cauchy sequence. For $n > m$, we have

$$\begin{aligned} G(y_n, y_m, y_m) \leq\ & G(y_n, y_{n-1}, y_{n-1}) + G(y_{n-1}, y_{n-2}, y_{n-2}) + \cdots \\ & + G(y_{m+1}, y_m, y_m) \end{aligned} \tag{3.8}$$

and we obtain

$$\begin{aligned} G(y_n, y_m, y_m) &\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m)G(y_0, y_1, y_1) \\ &\leq \frac{\lambda^m}{1-\lambda}G(y_0, y_1, y_1) \to 0 \text{ as } m \to \infty \end{aligned} \tag{3.9}$$

From Remark 1.6 it follows that for $0 \ll c$ and large m : $\lambda^m(1-\lambda)^{-1}G(y_0, y_1, y_1) \ll c$; thus, according to Remark 1.2(i), $G(y_n, y_m, y_m)$

$\ll c$. Hence, by Definition, $\{y_n\}$ is a Cauchy sequence. Since $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is complete, there exists a $q \in g(X)$ such that $gx_n \to q \in g(X)$ as $n \to \infty$. Consequently, we can find $p \in X$ such that $gp = q$.

Let us show that $fp = q$. For this we have

$$\begin{aligned} G(fp, q, q) &\leq G(fp, fx_n, fx_n) + G(fx_n, q, q) \\ &\leq \lambda \cdot u_n + G(fx_n, q, q) \end{aligned} \tag{3.10}$$

where

$$u_n \in \Big\{ G(gp, gx_n, gx_n), G(gp, fp, fp), G(gx_n, fx_n, fx_n), \frac{G(gp, fx_n, fx_n) + G(gx_n, fp, fp)}{2} \Big\} \tag{3.11}$$

Let $0 \ll c$. Clearly there are following four cases hold for infinitely many n.

Case 1.

$$\begin{aligned} G(fp, q, q) &\leq \lambda G(gp, gx_n, gx_n) + G(fx_n, q, q) \\ &\ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c \end{aligned} \tag{3.12}$$

Case 2.

$$\begin{aligned} G(fp, q, q) &\leq \lambda G(gx_n, fx_n, fx_n) + G(fx_n, q, q) \\ &\leq 2G(fx_n, gx_n, gx_n) + G(fx_n, q, q) \\ &\leq 2\lambda[G(fx_n, q, q) + G(q, gx_n, gx_n)] + G(fx_n, q, q) \\ &\leq (2\lambda + 1)G(fx_n, q, q) + G(q, gx_n, gx_n) \\ &\ll (2\lambda + 1) \cdot \frac{c}{2(2\lambda + 1)} + \lambda \frac{c}{2\lambda} = c \end{aligned} \tag{3.13}$$

Case 3.

$$\begin{aligned} G(fp, q, q) &\leq \lambda G(gp, gp, fp,) + G(fx_n, q, q) \\ &\leq 2\lambda G(fp, gp, gp,) + G(fx_n, q, q) \\ G(fp, q, q) &\leq \frac{1}{1 - 2\lambda} G(fx_n, q, q) \\ &\ll \frac{1}{1 - 2\lambda} \cdot \frac{c}{\frac{1}{1-2\lambda}} = c \end{aligned} \tag{3.14}$$

Case 4.

(3.15)
$$
\begin{aligned}
G(fp,q,q) &\leq \lambda\left(\frac{G(gp,fx_n,fx_n)+G(gx_n,fp,fp)}{2}\right)+G(fx_n,q,q)\\
&\leq \lambda G(fx_n,gp,gp)+\lambda G(fp,q,q)+G(fx_n,q,q)\\
G(fp,q,q) &\leq \frac{\lambda+1}{1-\lambda}G(fx_n,q,q)\\
&\ll \frac{\lambda+1}{1-\lambda}\cdot\frac{c(1-\lambda)}{\lambda+1}=c
\end{aligned}
$$

In all cases, we obtain $G(fp,q,q)\ll c$ for each $c\in \operatorname{int}P$. Using Remark 1.2(ii), it follows that $G(fp,q,q)=0$, or $fp=q$.

Hence, we proved that f and g have a coincidence point $p\in X$ and a point of coincidence $q\in X$ such that $q=f(p)=g(p)$. If q_1 is another point of coincidence, then there is $p_1\in X$ with $q_1=fp_1=gp_1$. Now,

$$
G(q,q_1,q_1)=G(fp,fp_1,fp_1)\leq \lambda u \tag{3.16}
$$

where

$$
\begin{aligned}
u \in &\left\{G(gp,fp_1,fp_1),G(gp,fp,fp),G(gp_1,fp_1,fp_1),\right.\\
&\left.\frac{G(gp,fp_1,fp_1)+G(gp_1,fp,fp)}{2}\right\}\\
&=\left\{G(q,q_1,q_1),G(q,q,q),0,\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}\\
&=\{G(q,q_1,q_1),0\}
\end{aligned} \tag{3.17}
$$

This implies

$$
\begin{aligned}
G(q,q_1,q_1)&=G(fp,fp_1,fp_1)\\
&\leq \lambda\left\{G(q,q_1,q_1),0,\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}.
\end{aligned}
$$

By Remark 1.6 we get, $G(q, q_1, q_1) = 0$, that is, $q = q_1$.

$$\begin{aligned} G(q, q_1, q_1) &= G(fp, fp_1, fp_1) \\ &\leq \lambda \left\{ \frac{G(q, q_1, q_1) + G(q_1, q, q)}{2} \right\} \end{aligned}$$

$$G(q, q_1, q_1) \leq \frac{\lambda}{2}[G(q, q_1 q_1) + G(q_1, q, q)]$$

$$G(q, q_1, q_1) \leq \left(\frac{\frac{\lambda}{2}}{1 - \frac{\lambda}{2}} = K \right) [G(q_1, q, q)]$$

Again by same argument,

$$G(q_1, q, q) \leq (K)[G(q, q_1, q_1)]$$

So

$$G(q, q_1, q_1) \leq K^2[G(q_1, q, q)], \text{ since } K < 1$$

Then this implies that $q = q_1$.

Since $q = f(p) = g(p)$ is the unique point of coincidence of f and g, and f and g are weakly compatible, q is the unique common fixed point of f and g by Proposition 3.1 [13]. □

Theorem 3.8. *Let (X, G) be a G-cone metric space. Suppose that (f, g) is AJ's pair, and that for some constant $(0, 1)$ and for every $x, y \in X$, there exists*

(3.18)
$$u = u(x, y, z) \in \left\{ G(gx, gy, gz), \frac{G(gx, fx, fx) + G(gy, fy, fy)}{2}, \frac{G(gx, fy, fy) + G(gy, fx, fx)}{2} \right\},$$

such that

$$G(fx, fy, fz) \leq \lambda u. \tag{3.19}$$

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and let $x_1 \in X$ be such that $gx_1 = fx_0 = y_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $gx_{n+1} = fx_n = y_n$.

We first show that

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_n, y_n) \text{ for } n \geq 1 \tag{3.20}$$

Notice that

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(fx_n, fx_{n+1}, fx_{n+1}) \leq \lambda u_n \tag{3.21}$$

where

$$u_n \in \Big\{G(gx_n, gx_{n+1}, gx_{n+1}), \tag{3.22}$$
$$\frac{G(gx_n, fx_n, fx_n) + G(gx_{n+1}, fx_{n+1}, fx_{n+1})}{2},$$
$$\frac{G(gx_n, fx_{n+1}, fx_{n+1}) + G(gx_{n+1}, fx_n, fx_n)}{2}\Big\}$$
$$u \in \Big\{G(y_{n-1}, y_n, y_n), \frac{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})}{2},$$
$$\frac{G(y_{n-1}, y_{n+1}, y_{n+1})}{2}\Big\}$$

As in Theorem 3.1, we have to consider three cases.

If $u = G(y_{n-1}, y_n, y_n)$, then clearly (3.20) holds.

If $u = \left\{\frac{G(y_{n-1},y_n,y_n)+G(y_n,y_{n+1},y_{n+1})}{2}\right\}$, then
from (3.19) with $x = x_n$, $y = x_{n+1}$ and $z = x_{n+1}$, we have

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\leq \lambda \left\{\frac{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})}{2}\right\} \\ &\leq \lambda \frac{G(y_{n-1}, y_n, y_n)}{2} + \frac{G(y_n, y_{n+1}, y_{n+1})}{2} \end{aligned} \tag{3.23}$$

Hence, $G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_n, y_n)$, and in this case (3.20) holds. Finally, if $u = G(y_{n-1}, y_{n+1}, y_{n+1})/2$, then

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\leq \lambda \frac{G(y_{n-1}, y_{n+1}, y_{n+1})}{2} \\ &\leq \lambda \left\{\frac{G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})}{2}\right\} \\ &\leq \lambda \frac{G(y_{n-1}, y_n, y_n)}{2} + \lambda \frac{G(y_n, y_{n+1}, y_{n+1})}{2} \end{aligned} \tag{3.24}$$

and (3.20) holds. Thus, we proved that in all three cases (3.20) holds.

Now, from the proof of Theorem 3.1,we know that $\{gx_{n+1}\} = \{fx_n\} = \{y_n\}$ is a cauchy sequence. Hence, there exists q in $g(X)$ and $p \in X$ such that $gx_n \to q$, $n \to \infty$ and $g(p) = q$.

Now we have to show that $fp = q$. For this we have

$$G(fp,q,q) \leq G(fp,fx_n,fx_n) + G(fx_n,q,q) \\ \leq \lambda \cdot u_n + G(fx_n,q,q) \tag{3.25}$$

where

$$u_n \in \left\{ G(gp,gx_n,gx_n), \frac{G(gp,fp,fp)+G(gx_n,fx_n,fx_n)}{2}, \frac{G(gp,fx_n,fx_n)+G(gx_n,fp,fp)}{2} \right\} \tag{3.26}$$

Let $0 \ll c$. Clearly at least one of the following three cases holds for infinitely many n.

Case 1.

$$G(fp,q,q) \leq \lambda G(gp,gx_n,gx_n) + G(fx_n,q,q) \leq \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} \tag{3.27}$$

Case 2.

(3.28)

$$\begin{aligned} G(fp,q,q) &\leq \lambda \frac{G(gx_n,fx_n,fx_n)+G(gp,fp,fp)}{2} + G(fx_n,q,q) \\ &\leq 2\lambda \frac{G(fx_n,gx_n,gx_n)+G(fp,gp,gp)}{2} + G(fx_n,q,q) \\ &\leq \lambda G(fx_n,q,q) + \lambda \cdot G(fp,q,q) + G(fx_n,q,q), \end{aligned}$$

$$\begin{aligned} G(fp,q,q) - \lambda G(fp,q,q) &\leq (\lambda+1)G(fx_n,q,q) \\ G(fp,q,q) &\leq \frac{\lambda+1}{1-\lambda} G(fx_n,q,q) \\ &\ll \frac{\lambda+1}{1-\lambda} \frac{c(1-\lambda)}{\lambda+1} = c \end{aligned} \tag{3.29}$$

Case 3.

$$\begin{aligned} G(fp,q,q) &\leq \lambda \frac{G(gp,fx_n,fx_n)+G(gx_n,fp,fp)}{2} + G(fx_n,q,q) \\ &\leq 2\lambda \frac{G(fx_n,gp,gp)+G(fp,gx_n,gx_n)}{2} + G(fx_n,q,q) \\ &\leq \lambda G(fx_n,q,q) + \lambda \cdot G(fp,q,q) + G(fx_n,q,q) \end{aligned}$$

$$
\begin{aligned}
G(fp,q,q)-\lambda G(fp,q,q) &\leq (\lambda+1)G(fx_n,q,q)\\
G(fp,q,q) &\leq \frac{\lambda+1}{1-\lambda}G(fx_n,q,q)\\
&\ll \frac{\lambda+1}{1-\lambda}\,\frac{c(1-\lambda)}{\lambda+1}=c
\end{aligned}
$$

In all cases we obtain $G(fp,q,q)\ll c$ for each $c\in \operatorname{int}P$. Using Remark 1.2(ii), it follows that $G(fp,q,q)=0$ or $fp=q$.

Thus we showed that f and g have a coincidence point $p\in X$, that is, point of coincidence $q\in X$ such that $q=fp=gp$. If q_1 is another point of coincidence then there is $p_1\in X$ with $q_1=fp_1=gp_1$. Now from (3.19), it follows that

$$G(q,q_1,q_1)=G(fp,fp_1,fp_1)\leq \lambda u, \tag{3.30}$$

where

$$
\begin{aligned}
u\in &\left\{G(gp,gp_1,gp_1),\frac{G(gp,fp,fp)+G(gp_1,fp_1,fp_1)}{2},\right.\\
&\left.\frac{G(gp,fp_1,fp_1)+G(gp_1,fp,fp)}{2}\right\}\\
=&\left\{G(q,q_1,q_1),\frac{G(q,q,q)+G(q_1,q_1,q_1)}{2},\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}\\
=&\left\{G(q,q_1,q_1),0,\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}
\end{aligned}
$$

This implies

$$
\begin{aligned}
G(q,q_1,q_1)&=G(fp,fp_1,fp_1)\\
&\leq \lambda\left\{G(q,q_1,q_1),0,\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}.
\end{aligned}
$$

By Remark 1.6 we get, $G(q,q_1,q_1)=0$, that is, $q=q_1$.

$$
\begin{aligned}
G(q,q_1,q_1)&=G(fp,fp_1,fp_1)\\
&\leq \lambda\left\{\frac{G(q,q_1,q_1)+G(q_1,q,q)}{2}\right\}\\
G(q,q_1,q_1)&\leq \frac{\lambda}{2}[G(q,q_1q_1)+G(q_1,q,q)]\\
G(q,q_1,q_1)&\leq \left(\frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}=K\right)[G(q_1,q,q)]
\end{aligned}
$$

Again by same argument,

$$G(q_1, q, q) \leq (K)[G(q, q_1, q_1)]$$

So

$$G(q, q_1, q_1) \leq K^2[G(q_1, q, q)], \quad \text{since } K < 1$$

Then this implies that $q = q_1$.

If f and g are weakly compatible, then as in the proof of Theorem 3.1, we have that q is a unique common fixed point of f and g. Hence proved the theorem. □

Now we generalize results of G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic [7] in G-cone metric space by extending number of factors also.

Theorem 3.9. *Let (X, G) be a G-cone metric space. Suppose that (f, g) is AJ's pair, and that there exist nonnegative constants a_i satisfying $\sum_{i=1}^{7} a_i < 1$ such that, for each $x, y \in X$*

(3.31)

$$\begin{aligned} G(fx, fy, fz) \leq a_1 G(gx, gy, gz) &+ a_2 G(gx, fx, fx) + a_3 G(gy, fy, fy) \\ &+ a_4 G(gx, fy, fy) + a_5 G(gy, fz, fz) \\ &+ a_6 G(gz, fx, fx) + a_7 G(gz, fz, fz) \end{aligned}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let us define the sequences x_n and y_n as in the proof of Theorem 3.1. We have to show that

(3.32)

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_n, y_n) \text{ for some } \lambda \in (0, 1), \; n \geq 1.$$

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &= G(fx_n, fx_{n+1}, fx_{n+1}), \\ &\leq a_1 G(y_{n-1}, y_n, y_n) + a_2 G(y_{n-1}, y_n, y_n) \\ &\quad + a_3 G(y_n, y_{n+1}, y_{n+1}) + a_4 G(y_{n-1}, y_{n+1}, y_{n+1}) \\ &\quad + a_5 G(y_n, y_n, y_n) + a_6 G(y_n, y_n, y_n) \\ &\quad + a_7 G(y_n, y_{n+1}, y_{n+1}) \\ G(y_n, y_{n+1}, y_{n+1}) &\leq (a_1 + a_2 + a_4) G(y_{n-1}, y_n, y_n) \\ &\quad + (a_3 + a_7) G(y_n, y_{n+1}, y_{n+1}), \end{aligned}$$

where $\lambda = (a_1 + a_2 + a_4)(1 - a_3 - a_7)^{-1} \in (0,1)$ and we get (3.32). Now, from the proof of Theorem 3.1, we know that $\{gx_{n+1}\} = \{fx_n\} = \{y_n\}$ is a Cauchy sequence. Hence, there exist q in $g(X)$ and $p \in X$ such that $gx_n \to q$, $n \to \infty$ and $gp = q$.

We have to show that $fp = q$. For this we have

(3.33)

$$\begin{aligned}
G(fp,q,q) &\leq G(fp,fx_n,fx_n) + G(fx_n,q,q)\\
&\leq a_1G(gp,gx_n,gx_n) + a_2G(gp,fp,fp) + a_3G(gx_n,fx_n,fx_n)\\
&\quad + a_4G(gp,fx_n,fx_n) + a_5G(gx_n,fp,fp)\\
&\quad + a_6G(gx_n,fp,fp) + a_7G(gx_n,gx_n,gx_n) + G(fx_n,q,q)\\
&\leq a_1G(q,gx_n,gx_n) + a_2G(q,fp,fp) + a_3G(gx_n,fx_n,fx_n)\\
&\quad + a_4G(q,fx_n,fx_n) + (a_5+a_6)G(gx_n,fp,fp)\\
&\quad + G(fx_n,q,q)\\
&\leq a_1G(q,gx_n,gx_n) + 2a_2G(fp,q,q) + 2a_3G(fx_n,q,q)\\
&\quad + 2a_4G(fx_n,q,q) + 2(a_5+a_6)G(fp,q,q) + G(fx_n,q,q)\\
&\leq 2(a_2+a_5+a_6)G(fp,q,q) + (2a_3+2a_4+1)G(fx_n,q,q)\\
G(fp,q,q) &\leq \left(\frac{2a_3+2a_4+1}{1-2a_2-2a_5-2a_6}\right)G(fx_n,q,q)\\
&\leq \left(\frac{2a_3+2a_4+1}{1-2a_2-2a_5-2a_6}\right)\cdot\left(\frac{c\cdot(1-2a_2-2a_5-2a_6)}{2a_3+2a_4+1}\right)\\
&\ll c\,.
\end{aligned}$$

Then according to Remark 1.2(ii), $G(fp,q,q) = 0$, that is, $fp = q$.

Thus we showed that f and g have a coincidence point $p \in X$ that is, point of coincidence $q \in X$ such that $q = fp = gp$. If q_1 is another point of coincidence then there is $p_1 \in X$ with $q_1 = fp_1 = gp_1$. Now,

$$\begin{aligned}
G(q,q_1,q_1) &= G(fp,fp_1,fp_1)\\
&\leq a_1G(gp,gp_1,gp_1) + a_2G(gp,fp,fp) + a_3G(gp_1,fp_1,fp_1)\\
&\quad + a_4G(gp,fp_1,fp_1) + a_5G(gp_1,fp,fp) + a_6G(gp_1,fp,fp)\\
&\quad + a_7G(gp_1,fp_1,fp_1)\\
&\leq a_1G(q,q_1,q_1) + a_2G(q,q,q) + a_3G(q_1,q_1,q_1)\\
&\quad + a_4Gt(q,q_1,q_1) + a_5G(q_1,q,q) + a_6G(q_1,q,q)\\
&\quad + a_7G(q_1,q_1,q_1)
\end{aligned}$$

$$\leq (a_1 + a_4)G(q, q_1, q_1) + (a_5 + a_6)G(q_1, q, q)$$

$$G(q, q_1, q_1) \leq \left(\frac{a_5 + a_6}{1 - a_1 - a_4} = \lambda\right) G(q_1, q, q)$$

Again by same argument,

$$G(q_1, q, q) \leq (\lambda)[G(q, q_1, q_1)] \tag{3.34}$$

So

$$G(q, q_1, q_1) \leq \lambda^2[G(q_1, q, q)], \text{ since } \lambda < 1$$

Then this implies that $q = q_1$. f and g are weakly compatible, then as in the proof of Theorem 3.1, we have that q is a unique common fixed point of f and g. Hence proved the theorem. □

Corollary 3.10. *Let (X, d) be a cone metric space. Suppose that (f, g) is AJ's pair, and that there exists nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that, for each $x, y \in X$*

$$\begin{aligned} d(fx, fy) \leq\; & a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) \\ & + a_4 d(gx, fy) + a_5 d(gy, fx). \end{aligned} \tag{3.35}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Here, we prove the common fixed point theorem for three mappings:

Lemma 3.11. *Let X be a non-empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.*

Proof. Since v is point of coincidence S, T and f. Therefore, $v = fu = Su = Tu$ for some $u \in X$. By weakly compatibility of (S, f) and (T, f) we have

$$Sv = Sfu = fSu = fv \quad \text{and} \quad Tv = Tfu = fTu = fv.$$

It implies that $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T and f. Therefore, $v = w$ by uniqueness. Thus v is a unique common fixed point of S, T and f. □

Theorem 3.12. *Let (X, G) be a G-cone metric space and the mappings $S, T, f : X \to X$ satisfy:*

$$G(fx, fy, fz) \leq \lambda G(Sx, Ty, Tz) \tag{3.36}$$

for all $x, y, z \in X$ where $0 \leq \lambda < 1$. If $f(X) \subset S(X) \cap T(X)$ and if one of $S(X)$ and $T(X)$ is complete, then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then $fx_0 \in X$. Since $f(X)$ is contained in $S(X)$, there exists a point x_1 in X such that $fx_0 = Sx_1$. Since $f(X)$ is also contained in $T(X)$, we can choose a point x_2 in X such that $fx_1 = Tx_2$. Continuing this process and having chosen x_n in X. We obtain x_{n+1} in X such that

$$fx_{2k+1} = Tx_{2k+2} \qquad fx_{2k} = Sx_{2k+1}, \quad k = 0, 1, 2, \ldots .$$

Then,

$$\begin{aligned} G(fx_{2k}, fx_{2k+1}, fx_{2k+1}) &\leq \lambda G(Sx_{2k}, Tx_{2k+1}, Tx_{2k+1}) \\ &= \lambda G(fx_{2k-1}, fx_{2k}, fx_{2k}). \end{aligned}$$

Similarly,

$$\begin{aligned} G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) &\leq \lambda G(Sx_{2k+1}, Tx_{2k+2}, Tx_{2k+2}) \\ &= \lambda G(fx_{2k}, fx_{2k+1}, fx_{2k+1}) \end{aligned}$$

Now by induction, we obtain for each $k = 0, 1, 2, \ldots$,

$$G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) \leq \lambda^{2k+1} G(fx_0, fx_1, fx_1).$$

Let

$$y_n = fx_n, \quad n = 0, 1, 2, \ldots .$$

Now for all n, we have

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\leq \lambda G(y_{n-1}, y_n, y_n) \\ &\leq \cdots \leq \lambda^n G(y_0, y_1, y_1). \end{aligned}$$

Now for any $m > n$,

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + G(y_{n+2}, y_{n+3}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_m) \\ &\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) G(y_0, y_1, y_1) \\ &\leq \frac{\lambda^m}{1 - \lambda} G(y_0, y_1, y_1) \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that

$$c + \{x \in E : \|x\| < \delta\} \subseteq P.$$

Also, choose a natural number N_1, such that

$$\frac{\lambda^m}{1-\lambda}G(y_0, y_1, y_1) \in \{x \in E : \|x\| < \delta\}, \quad \text{for all } n \geq N_1.$$

Then

$$\frac{\lambda^m}{1-\lambda}G(y_0, y_1, y_1) \ll c, \quad \text{for all } n \in N_1.$$

Thus,

$$m > n \geq N_1 \Rightarrow G(y_n, y_m, y_m) \leq \frac{\lambda^m}{1-\lambda}G(y_0, y_1, y_1) \ll c,$$

it implies that $\{y_n\}$ is a Cauchy sequence. Since $S(X)$ and $T(X)$ both are complete, there exists $u, v \in X$ such that $y_n \to v = Su = Tu$. Choose a natural number N_2 such that

$$G(y_n, v, v) \ll \frac{c}{4}, \text{ for all } n \geq N_2.$$

Hence, for all $n \geq N_2$,

$$\begin{aligned} G(Su, fu, fu) &\leq G(Su, y_{2n+2}, y_{2n+2}) + G(y_{2n+2}, fu, fu) \\ &\leq G(v, y_{2n+2}, y_{2n+2}) + G(fx_{2n+2}, fu, fu) \\ &\leq G(v, y_{2n+2}, y_{2n+2}) + \lambda G(Sx_{2n+2}, Tu, Tu) \\ &\leq 2G(y_{2n+2}, v, v) + G(y_{2n+1}, Tu, Tu) \\ &\ll \frac{c}{2} + \frac{c}{4} = \frac{3c}{4} \end{aligned}$$

Thus

$$G(Su, fu, fu) \ll \frac{c}{m}, \text{ for all } \ m > 1.$$

So,

$$\frac{c}{m} - G(Su, fu, fu) \in P, \text{ for all } m > 1.$$

Since $\dfrac{c}{m} \to 0$ (as $m \to \infty$) and P is closed, $-G(Su, fu, fu) \in P$, but $P \cap (-P) = \{0\}$. Therefore, $G(Su, fu, fu) = 0$. Hence $fu = Su$.

Similarly, by using

$$G(Tu, fu, fu) \leq G(Tu, y_{2n+2}, y_{2n+2}) + G(y_{2n+2}, fu, fu),$$

we can show that $fu = Tu$, it implies that v is a common point of coincidence of S, T and f that is $v = fu = Su = Tu$.

Now we show that f, S and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that

$v^* = fu^* = Su^* = Tu^*$ for some u^* in X. Now,

$$\begin{aligned} G(v, v^*, v^*) &= G(fu, fu^*, fu^*) \leq \lambda G(Su, Tu^*, Tu^*) \\ &\leq \lambda G(v, v^*, v^*) \end{aligned}$$

This implies that $v^* = v$. If (S, f) and (T, f) are weakly compatible, by Lemma 3.1, S, T and f have a unique common fixed point. $\square$

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