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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 2, 115 - 126

SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Abstract. In the present paper we introduce some new sequence spaces defined by a sequence of modulus functions $F = (f_k)$. We study some topological properties and inclusion relations between these spaces.

1. Introduction and Preliminaries

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$, respectively. A linear functional φ on l_∞ is said to be Banach limit if it has the properties, $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and $\varphi(x_{k+1}) = \varphi(x_k)$ for all $x \in l_\infty$ see [5]. For more detail on the Banach limit, we may refer to Çolak and Çakar [7], Das [9] and references therein. The concept of almost convergence was defined by Lorentz in [13], using the idea of Banach limits. Lorentz proved that

$$\hat{c} = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

Maddox ([16], [17]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

Keywords and phrases: paranorm space, difference sequence space, modulus function .
(2010) Mathematics Subject Classification: 40C05, 46A45.

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [19] has defined the following sequence spaces :

$$[\hat{c}, p] = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^{p_k} = 0, \ \text{uniformly in } s\Big\},$$

$$[\hat{c}, p]_0 = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \ \text{uniformly in } s\Big\}$$

and

$$[\hat{c}, p]_\infty = \Big\{x = (x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty\Big\}.$$

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [10] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let u, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^u) = \{x = (x_k) \in w : (\Delta_v^u x_k) \in Z\},$$

where $\Delta_v^u x = (\Delta_v^u x_k) = (\Delta_v^{u-1} x_k - \Delta_v^{u-1} x_{k+v})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^u x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} u \\ i \end{pmatrix} x_{k+vi}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ *as* $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [25], Theorem 10.4.2, P-183).
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
(3) f is increasing
(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subseqentially, modulus function has been discussed in ([1], [2], [3], [4], [18], [20], [21], [22]) and many others.

Let $A = (a_{mk})$ be an infinite matrix of complex numbers. We write $Ax = (A_m(x))_{m=1}^{\infty}$ if $A_m(x) = \sum_{k=1}^{\infty} a_{mk}x_k$ converges for each $m \in \mathbb{N}$. A sequence $x = (x_k)$ is said to be summable $(C, 1)$ if and only if $\lim_n \sum_{i=1}^{n} x_i$ exists. Spaces of strongly Cesaro sequences were discussed by Kuttner [12] and this concept was generalized by Maddox [15] and some others. The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [14] as an extension of the definition of strongly Cesaro summable. Connor [8] extended his definition by replacing the Cesaro matrix with an arbitrary non-negative regular matrix summability method A. In [23] Savaş, following Connor [8] and defined the concept of strongly almost A-summability with respect to a modulus, but the definition introduced there is not very satisfactory and seems to be unnatural. By specializing the infinite matrix in the definition introduced in [23], we don't get strongly almost convergent sequences with respect to a modulus. In [24] Savas introduced an alternative definition of strongly almost A-summability with respect to a modulus. This definition seems to be more natural and contains the definition of strongly almost convergence with respect to a modulus as a special case. The sets $w_0(\hat{A}, f, p)$, $w(\hat{A}, f, p)$ and $w_\infty(\hat{A}, f, p)$ will called the spaces of strongly almost summable to zero, strongly almost summable and strongly almost bounded with respect to the modulus f respectively [24]. The argument s, that is, the factor k^{-s} was used by Bulut and Çakar [6], to generalize the Maddox sequence space $l(p)$ where $p = (p_k)$ be a bounded sequence of positive real numbers and $s \geq 0$. It performs an extension mission. For example, the space $l(p, s) = \{x \in w : \sum_{k=1}^{\infty} k^{-s}|x_k|^{p_k} < \infty$ as a subspace for $s > 0$ and it coincides with $l(p)$ only for $s = 0$.

Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $x_k \in X$ and usual coordinatewise operations. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$. Taking $X = \mathbb{C}$ we get w, the space of all complex-valued sequences.

Let $A = (a_{mk})$ be a non-negative matrix and $p = (p_k)$ be a bounded sequence of positive real numbers and $F = (f_k)$ be a sequence of modulus functions. Now, we define the following sequence spaces in this paper:

$$w_0(\hat{A}, p, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n}))\big]^{p_k} = 0, \text{uniformly in } n, s \geq 0\Big\},$$

$$w(\hat{A}, p, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \textstyle\sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n} - l))\big]^{p_k} = 0, \text{uniformly in } n, \text{ for some } l, s \geq 0\Big\};$$

and

$$w_\infty(\hat{A}, p, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \sup_{m,n} \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n}))\big]^{p_k} < \infty, s \geq 0\Big\}.$$

If we take $F(x) = x$, we get the spaces

$$w_0(\hat{A}, p, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \sum_k a_{mk} k^{-s} (q(\Delta_v^u x_{k+n}))^{p_k} = 0, \text{uniformly in } n, s \geq 0\Big\},$$

$$w(\hat{A}, p, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \sum_k a_{mk} k^{-s} (q(\Delta_v^u x_{k+n} - l))^{p_k} = 0, \text{uniformly in } n, \text{ for some } l, s \geq 0\Big\}$$

and

$$w_\infty(\hat{A}, p, \Delta_v^u, q, s) = \Big\{x \in S(X) : \sup_{m,n} \sum_k a_{mk} k^{-s} (q(\Delta_v^u x_{k+n}))^{p_k} < \infty, s \geq 0\Big\}.$$

If we take $p = (p_k) = 1$, we have

$$w_0(\hat{A}, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n}))\big] = 0, \text{uniformly in } n, s \geq 0\Big\},$$

$$w(\hat{A}, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \lim_{m\to\infty} \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n} - l))\big] = 0, \text{uniformly in } n, \text{ for some } l, s \geq 0\Big\}$$

and

$$w_\infty(\hat{A}, F, \Delta_v^u, q, s) = \Big\{x \in S(X) : \sup_{m,n} \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n}))\big] < \infty, s \geq 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of the present paper is to introduce and examine some new sequence spaces by using a sequence of modulus functions.

2. Main Results

Theorem 2.1 *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (a_{mk})$ be a non-negative regular matrix. Then $w_0(\hat{A}, p, F, \Delta_v^u, q, s) \subset w(\hat{A}, p, F, \Delta_v^u, q, s) \subset w_\infty(\hat{A}, p, F, \Delta_v^u, q, s)$.*

Proof. It is obvious that $w_0(\hat{A}, p, F, \Delta_v^u, q, s) \subset w(\hat{A}, p, F, \Delta_v^u, q, s)$. Suppose that $x \in w(\hat{A}, p, F, \Delta_v^u, q, s)$. Since q is a seminorm, there exists N integer such that $q(l) \leq N$. Since $F = (f_k)$ be a sequence of modulus functions, $A = (a_{mk})$ is a non-negative regular matrix and from (1.1), we can write

$$\sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n})) \big]^{p_k} \leq K \Big\{ \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n} - l)) \big]^{p_k} + \big[NF(1) \big]^H \sum_k a_{mk} k^{-s} \Big\}.$$

Therefore $x \in w_\infty(\hat{A}, p, F, \Delta_v^u, q, s)$ and this completes the proof.

Theorem 2.2 *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $w_0(\hat{A}, p, F, \Delta_v^u, q, s), w(\hat{A}, p, F, \Delta_v^u, q, s)$ and $w_\infty(\hat{A}, p, F, \Delta_v^u, q, s)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in w(\hat{A}, p, F, \Delta_v^u, q, s)$ and $\alpha, \beta \in \mathbb{C}$, suppose that $x \to l_1[w(\hat{A}, p, F, \Delta_v^u, q, s)]$ and $y \to l_2[w(\hat{A}, p, F, \Delta_v^u, q, s)]$. For α, β there exists the integers M_α and N_β such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq N_\beta$. From (1.1) and definitions of F and q, we have
$a_{mk} k^{-s} \big[f_k(q(\alpha \Delta_v^u x_{k+n} + \beta \Delta_v^u y_{k+n} - (\alpha l_1 + \beta l_2))) \big]^{p_k} \leq$

$K a_{mk} k^{-s} M_\alpha^H \big[f_k(q(\Delta_v^u x_{k+n} - l_1)) \big]^{p_k} + K a_{mk} k^{-s} N_\beta^H \big[f_k(q(\Delta_v^u y_{k+n} - l_2)) \big]^{p_k}$
when adding the above inequality from $k = 1$ to ∞, we get $\alpha x + \beta y \in w(\hat{A}, p, F, \Delta_v^u, q, s)$. This proves that $w(\hat{A}, p, F, \Delta_v^u, q, s)$ is a linear space. Similarly, we can prove that $w_0(\hat{A}, p, F, \Delta_v^u, q, s)$ and $w_\infty(\hat{A}, p, F, \Delta_v^u, q, s)$ are linear spaces.

Theorem 2.3 *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the spaces $w_0(\hat{A}, p, F, \Delta_v^u, q, s)$ and $w(\hat{A}, p, F, \Delta_v^u, q, s)$ are topological linear spaces paranormed by*

$$g(x) = \sup_m \Big\{ \sum_k a_{mk} k^{-s} \big[f_k(q(\Delta_v^u x_{k+n})) \big]^{p_k} \Big\}^{\frac{1}{M}}$$

where $M = \max(1, H = \sup_k p_k)$.

Proof. From Theorem 2.1, for each $x \in w_0(\hat{A}, p, F, \Delta_v^u, q, s), g(x)$ exists. Clearly $g(0) = 0, g(x) = g(-x)$ and by Minkowski's inequality $g(x+y) \le g(x) + g(y)$. For the continuity of scalar multiplication suppose that (μ^t) is a sequence of scalars such that $|\mu^t - \mu| \to 0$ and $g(x^t - x) \to 0$ for arbitrary sequence $(x^t) \in w_0(\hat{A}, p, F, \Delta_v^u, q, s)$. We shall show that $g(\mu^t x^t - \mu x) \to 0 (t \to \infty)$. Say $\tau_t = |\mu^t - \mu|$, then $\Big\{\sum_k a_{mk} k^{-s} [f_k(q(\mu^t \Delta_v^u x_{k+n}^t - \mu \Delta_v^u x_{k+n}))]^{p_k}\Big\}^{\frac{1}{M}} \le$ $\Big\{\sum_k \{a_{mk}^{\frac{1}{M}} k^{-\frac{s}{M}} [A(t,k,n)]^{\frac{p_k}{M}} + a_{mk}^{\frac{1}{M}} k^{-\frac{s}{M}} [B(t,k,n)]^{\frac{p_k}{M}}\}^M\Big\}^{\frac{1}{M}}$, where $A(t,k,n) = N f_k(q\Delta_v^u(x_{k+n}^t - x_{k+n})), B(t,k,n) = f_k(\tau_t q \Delta_v^u(x_{k+n}))$ and $N = 1 + \max\{1, \sup|\mu^t|\}$.
$g(\mu^t x^t - \mu x) \le N^{\frac{H}{M}} \sup_{m,n} \Big\{\sum_k a_{mk} k^{-s} \Big[\frac{A(t,k,n)}{N}\Big]^{p_k}\Big\}^{\frac{1}{M}} + \sup_{m,n} \Big\{\sum_k a_{mk} k^{-s} \big[B(t,k,n)\big]^{p_k}\Big\}^{\frac{1}{M}} = N^{\frac{H}{M}} g(x^t - x) + \sup_{m,n} \Big\{\sum_k a_{mk} k^{-s} \big[B(t,k,n)\big]^{p_k}\Big\}^{\frac{1}{M}}$. Because of $g(x^t - x) \to 0$ we must only show that $\sup_{m,n} \Big\{\sum_k a_{mk} k^{-s} \big[B(t,k,n)\big]^{p_k}\Big\}^{\frac{1}{M}} \to 0 (t \to \infty)$. There exist a positive integer t_0 such that $0 \le \tau_t \le 1$ for $t > t_0$. Write

$$\sup_{m,n} \Big\{\sum_{k=m+1}^{\infty} a_{mk} k^{-s} \big[f_k(q\Delta_v^u(x_{k+n}^t))\big]^{p_k}\Big\}^{\frac{1}{M}} \to 0 (m \to \infty).$$

Hence, for every $\epsilon > 0$, there exist a positive integer m_0 such that

$$\sup_{m,n} \Big\{\sum_{k=m_0+1}^{\infty} a_{mk} k^{-s} \big[f_k(q\Delta_v^u(x_{k+n}^t))\big]^{p_k}\Big\}^{\frac{1}{M}} < \frac{\epsilon}{2}.$$

For $t > t_0$, since $\tau_t q \Delta_v^u(x_{k+n}) \le q\Delta_v^u(x_k)$, we get

$$a_{mk} k^{-s} \big[f_k(\tau_t(q\Delta_v^u(x_{k+n})))\big]^{p_k} \le a_{mk} k^{-s} \big[f_k(q\Delta_v^u(x_{k+n}))\big]^{p_k}$$

for each n and k. This implies that $\sup_{m,n} \Big\{\sum_{k=m_0+1}^{\infty} a_{mk} k^{-s} \big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}}$

$\leq \sup_{m,n}\Big\{\sum_{k=m_0+1}^{\infty} a_{mk}k^{-s}\big[f_k(q\Delta_v^u(x_{k+n}^t))\big]^{p_k}\Big\}^{\frac{1}{M}} < \frac{\epsilon}{2}$. Hence, there exist a $\delta (0 < \delta < 1)$ such that

$\sup_{m,n}\Big\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\} \leq (\frac{\epsilon}{2})^M$ for $0 < \tau_t < \delta$. Also we can find a number h such that $\tau_t < \delta$ for $t > h$. So, for $t > h$, we have $\sup_{m,n}\Big\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}} < \frac{\epsilon}{2}$, so eventually,

$\sup_{m,n}\Big\{\sum_{k} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}}$

$\leq \sup_{m,n}\Big\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}}$

$+ \sup_{m,n}\Big\{\sum_{k=m_0+1}^{\infty} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}}$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $\sup_{m,n}\Big\{\sum_{k} a_{mk}k^{-s}\big[f_k(\tau_t(q\Delta_v^u(x_{k+n}^t)))\big]^{p_k}\Big\}^{\frac{1}{M}} \to 0 (t \to \infty)$. Thus $w_0(\hat{A}, p, F, q, \Delta_v^u, s)$ is paranormed space by g.

Theorem 2.4 *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the space $w_0(\hat{A}, p, F, q, \Delta_v^u, s)$ is complete with respect to its paranorm whenever (X, q) is complete.*

Proof. Suppose (x^i) is a Cauchy sequence in $w_0(\hat{A}, p, F, q, \Delta_v^u, s)$. Therefore

$$(2.1)\quad g(x^i - x^j) = \sup_{m,n}\Big\{\sum_{k} a_{mk}k^{-s}\big[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}^j))\big]^{p_k}\Big\}^{\frac{1}{M}} \to 0$$

as $i, j \to \infty$. Also, for each n and k

$$k^{-s}\big[f_k((q\Delta_v^u(x_{k+n}^i - x_{k+n}^j))\big]^{p_k} \to 0$$

as $i, j \to \infty$ and so $q\Delta_v^u(x_{k+n}^i - x_{k+n}^j) \to 0 (i, j \to \infty)$ from the continuity of f. It follows that the sequence (x_{k+n}^i) is a Cauchy in (X, q) for each fixed n and k. Then by the completeness of (X, q) we get the sequences $(x_{k+n}) \in X$ such that

$$(2.2)\qquad q\Delta_v^u(x_{k+n}^i - x_{k+n}) \to 0 (j \to \infty).$$

It is easy to see the validity of the inequality

$$\left|q\Delta_v^u\left(x_{k+n}^i - x_{k+n}^j\right) - q\Delta_v^u\left(x_{k+n}^i - x_{k+n}\right)\right| \le q\Delta_v^u\left(x_{k+n}^i - x_{k+n}\right).$$

We have

$$q\Delta_v^u\left(x_{k+n}^i - x_{k+n}^j\right) \to q\Delta_v^u\left(x_{k+n}^i - x_{k+n}\right)(j \to \infty).$$

Thus from (2.2), for each $\epsilon > 0$ there exist $i_0(\epsilon)$ such that $\left[g(x^i - x^j)\right]^M < \epsilon^M$ for $i, j > i_0$. Also $\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}^j))\right]^{p_k}\right\}$
$\le \sup_{m,n}\left\{\sum_k a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}^j))\right]^{p_k}\right\}$
$= \left[g(x^i - x^j)\right]^M$. Letting $j \to \infty$ we have

$$\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}^j))\right]^{p_k}\right\} \to \sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\right]^{p_k}\right\} < \epsilon^M$$

for $i > i_0$. Since m_0 is arbitrary, by taking $m_0 \to \infty$, we obtain

$$\sup_{m,n}\left\{\sum_k a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\right]^{p_k}\right\}^{\frac{1}{M}} < \epsilon$$

for all m and n that is

$$g(x^i - x) \to 0 \text{ as } i \to \infty.$$

We first need to show $x \in w_0(\hat{A}, p, f_k, q, \Delta_v^u, s)$. We know that $g(x^i)$ is bounded, say, $g(x^i) \le K$. Furthermore we have

$$a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\right]^{p_k} \to 0(i \to \infty).$$

Now we can determine a sequence $\eta_k \in w_0(0 < \eta_k^i \le 1)$ for each k, such that

$$a_{mk}k^{-s}\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\right]^{p_k} \le \eta_k^i a_{mk}k^{-s}[f_k(q\Delta_v^u(x_{k+n}^i))]^{p_k}.$$

On the other hand,

$$[f_k(q\Delta_v^u(x_{k+n}))]^{p_k} \le K\left\{\left[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\right]^{p_k} + \left[f_k(q\Delta_v^u(x_{k+n}^i))\right]^{p_k}\right\}$$

where $K = \max(1, 2^{H-1}); H = \sup p_k$. Also we have $a_{mk}k^{-s}[f_k(q\Delta_v^u(x_{k+n}))]^{p_k}$

$\leq Ka_{mk}k^{-s}\Big\{\big[f_k(q\Delta_v^u(x_{k+n}^i - x_{k+n}))\big]^{p_k} + \big[f_k(q\Delta_v^u(x_{k+n}^i))\big]^{p_k}\Big\}$
$\leq K(\eta_k^i + 1)a_{mk}k^{-s}\big[f_k(q\Delta_v^u(x_{k+n}^i))\big]^{p_k}$ from the last inequality above, we obtain $x \in w_0(\hat{A}, p, F, q, s)$ and this completes the proof of the theorem.

Lemma 2.1. *Let $F' = (f'_k)$, $F'' = (f''_k)$ are sequences of modulus functions and $0 < \delta < 1$. If $F'(t) > \delta$ for $t \in [0, \infty)$ then*

$$(F'' \circ F')(t) \leq \frac{2F''(1)}{\delta}F'(t).$$

Theorem 2.5. *Let $F' = (f'_k)$, $F'' = (f''_k)$ are sequences of modulus functions and $s, s_1, s_2 > 0$. Then*
(i) $\limsup \frac{F'(t)}{F''(t)} < \infty$ implies $w_0(\hat{A}, p, F'', q, s) \subset w_0(\hat{A}, p, F', q, s)$,
(ii) $w_0(\hat{A}, p, F', q, s) \cap w_0(\hat{A}, p, F'', q, s) \subseteq w_0(\hat{A}, p, F' + F'', q, s)$,
(iii) If the matrix $A = (a_{mk})$ is a regular matrix and $s > 1$, then $w_0(\hat{A}, p, F', q, s) \subseteq w_0(\hat{A}, p, F' \circ F'', q, s)$,
(iv) $s_1 \leq s_2$ implies $w_0(\hat{A}, p, F, q, s_1) \subset w_0(\hat{A}, p, F, q, s_2)$.

Proof. (i) Since there exists a $C > 0$ such that $F'(t) \leq F''(t)$ by the hypothesis, therefore we can write that

$$a_{mk}k^{-s}\big[f'_k(q\Delta_v^u(x_{k+n}))\big]^{p_k} \leq C^H a_{mk}k^{-s}\big[f''_k(q\Delta_v^u(x_{k+n}))\big]^{p_k}.$$

Let $x \in w_0(\hat{A}, p, F'', q, s)$. When adding the above inequality from $k = 1$ to ∞, we have $x \in w_0(\hat{A}, p, F', q, s)$.
(ii) The relation follows from the inequality $a_{mk}k^{-s}\big[(f'_k + f''_k)(q\Delta_v^u(x_{k+n}))\big]^{p_k}$
$= a_{mk}k^{-s}\big[f'_k(q\Delta_v^u(x_{k+n})) + f''_k(q\Delta_v^u(x_{k+n}))\big]^{p_k}$
$\leq Ka_{mk}k^{-s}\Big\{\big[f'_k(q\Delta_v^u(x_{k+n}))\big]^{p_k} + \big[f''_k(q\Delta_v^u(x_{k+n}))\big]^{p_k}\Big\}$ where $K = \max(1, 2^{H-1})$.
(iii) Let $0 < \delta < 1$, and define the sets $N_1 = \{k \in \mathbb{N} : f'_k(q\Delta_v^u(x_{k+n})) \leq \delta\}$ and $N_2 = \{k \in \mathbb{N} : f'_k(q\Delta_v^u(x_{k+n})) > \delta\}$. It follows from Lemma 2.1 that

$$(f''_k \circ f'_k)(q\Delta_v^u(x_{k+n})) \leq \frac{2f''_k(1)}{\delta}f'_k(q\Delta_v^u(x_{k+n})),$$

when $k \in N_2$. If $k \in N_1$ then

$$(f''_k \circ f'_k)(q\Delta_v^u(x_{k+n})) \leq f''_k(\delta),$$

and so

$$k^{-s}[(f''_k o f'_k)(q\Delta_v^u(x_{k+n}))]^{p_k} \leq \epsilon_1 k^{-s}$$

for $x \in w_0(\hat{A}, p, F', q, s)$, where $\epsilon_1 = \max\Big\{[f''_k(\delta)]^{\inf p_k}, [f''_k(\delta)]^{\sup p_k}\Big\}$. On the other hand

$$\begin{aligned} a_{mk}k^{-s}[(f''_k \circ f'_k)(q\Delta^u_v(x_{k+n}))]^{p_k} &\leq a_{mk}k^{-s}\Big[\frac{2f''_k(1)}{\delta}f'_k(q\Delta^u_v(x_{k+n}))\Big]^{p_k} \\ &\leq \epsilon_2 a_{mk}k^{-s}[f'_k(q\Delta^u_v(x_{k+n}))]^{p_k} \end{aligned}$$

for $k \in N_2$, where $\epsilon_2 = \max\Big\{\big[\frac{2f''_k(1)}{\delta}\big]^{\inf p_k}, \big[\frac{2f''_k(1)}{\delta}\big]^{\sup p_k}\Big\}$. Now, say $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and we get $\sum_k a_{mk}k^{-s}\big[(f'_k \circ f''_k)(q\Delta^u_v(x_{k+n}))\big]^{p_k}$ $\leq \Big\{\sum_k a_{mk}k^{-s} + \sum_k a_{mk}k^{-s}\big[f_k(q\Delta^u_v(x_{k+n}))\big]^{p_k}\Big\}$ for $k \in N_1 \cup N_2 = \mathbb{N}$. This implies $x \in w_0(\hat{A}, p, F' \circ F'', q, s)$.

Theorem 2.6. *Let $s > 1$, F be bounded and A be a non negative regular matrix. When $x \in w_0(\hat{A}, p, F, q, s)$ $\sum_k a_k x_k$ is convergent iff $(a_k) \in \phi$, where ϕ is a finite sequences.*

Proof. The sufficiency is trivial.

For the necessity, suppose that $a \notin \phi$. Then there is a sequence of positive integers $m_1 < m_2 < \cdots$ such that $|a_{mk}| > 0$. let us define

$$(y_k) = \begin{cases} \frac{u}{q(u)a_{m_k}}, & k = m_k\ \theta, \quad k \neq m_k \\ \\ \theta \quad , & k \neq m_k \end{cases}$$

where $u \in X$ such that $q(u) > 0$. Since F is bounded and $s > 1$,

$$\sum_k a_{mk}k^{-s}\big[f_k(q\Delta^u_v(y_{k+n}))\big]^{p_k} < \infty.$$

Hence $(y_k) \in w_\infty(\hat{A}, p, F, q, s)$ but $\displaystyle\sum_k a_k y_k = \sum_m 1 = \infty$. This is a contradiction to $\displaystyle\sum_k a_k y_k$ convergent. This completes the proof of the theorem.

Acknowledgement: The authors thank the referee for his valuable suggestion that improved the presentation of the paper.

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