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PSEUDO STRONGLY θ-CONTINUOUS FUNCTIONS

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Abstract. A new class of functions called 'pseudo strongly θ-continuous' functions is introduced. Their place in the hierarchy of variants of continuity which already exist in the literature is highlighted. The interplay between topological properties and pseudo strong θ-continuity is investigated.

1. Introduction

Strongly θ-continuous functions were introduced by Noiri [24] and almost strongly θ-continuous functions are due to Noiri and Kang [25]. A new class of functions called 'pseudo strongly θ-continuous' functions is introduced, which properly contains each of the classes of (i) quasi θ-continuous functions [26] (ii) (almost) strongly θ-continuous functions ([25] [24]) and (iii) pseudo D_δ-supercontinuous functions [15] and is contained in the class of slightly continuous functions [6]. The organization of the paper is as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3 we introduced the concept of pseudo strongly θ-continuous functions, wherein examples are included and observations are made to reflect upon the distinctiveness of the notion so introduced from the existing ones in the literature. Section 4 is devoted to study their basic properties and in Section 5 we discuss the interplay between topological properties and pseudo strongly θ-continuous functions.

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2. Preliminaries and basic definitions

A subset A of a topological space X is called a **regular G_δ-set** [22] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^\circ$, where each F_n is a closed subset of X (here F_n° denotes the interior of F_n). The complement of a regular G_δ-set is called a **regular F_σ-set**. Any union of regular F_σ-sets is called ***d_δ-open*** [10]. The complement of a d_δ-open set is referred to as a ***d_δ-closed set***. A point $x \in X$ is called a ***θ-adherent point*** [31] of $A \subset X$ if every closed neigbourhood of x intersects A. Let $cl_\theta\ A$ denote the set of all θ-adherent points of A. The set A is called ***θ-closed*** [31] if $A = cl_\theta A$. The complement of a θ-closed set is referred to as a ***θ-open set***. A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e., $A = \overline{A}^\circ$. The complement of a regular open set is referred to as a ***regular closed set***. A union of regular open sets is called **δ-open** [31]. The complement of a δ-open set is referred to as a ***δ-closed set***.

Definition 2.1. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*
(a) ***supercontinuous [23]*** (**D_δ-supercontinuous [10]**) *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular open set (regular F_σ-set) U containing x such that $f(U) \subset V$.*
(b) ***strongly θ-continuous*** ([20] [24]) *if for each $x \in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$.*
(c) ***D_δ-continuous [11] (z-continuous) [27])*** *if for each point $x \in X$ and each regular F_σ set (respectively cozero set) V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.*
(d) ***almost continuous [28] (faintly continuous [21])*** *if for each point $x \in X$ and each regular open set (respectively θ-open set) V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.*
(e) ***θ -continuous*** [3] *if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset \overline{V}$.*
(f) ***weakly continuous*** [19] *if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset \overline{V}$.*
(g) ***quasi θ-continuous function*** [26] *if for each $x \in X$ and each θ-open set V containing $f(x)$ there exists an θ-open set U containing*

x such that $f(U) \subset V$.
(h) ***slightly continuous*** [6][1] *if $f^{-1}(V)$ is open in X for every clopen set $V \subset Y$.*
(i) d_δ***-map***[12] *if for each regular F_σ-set U in Y, $f^{-1}(U)$ is a regular F_σ-set in X.*
(j) ***almost strongly θ-continuous*** [25] *if for each $x \in X$ and for each regular open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$.*
(k) ***pseudo z-supercontinuous*** [16] *if for each $x \in X$ and for each regular F_σ-set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.*
(l) ***δ-continuous*** [24] *if for each $x \in X$ and for each regular open set V containing $f(x)$,there exists a regular open set U containing x such that $f(U) \subset V$.*
(m) ***quasi (pseudo)supercontinuous*** [29]*if for each $x \in X$ and for each θ open set (regular F_σ-set) V containing $f(x)$,there exists a regular open set U containing x such that $f(U) \subset V$.*
(n) ***almost (respectively quasi, respectively pseudo) D_δ-supercontinuous***([17] [15]) *if for $x \in X$ and for each regular open set (respectively θ-open set, respectively regular F_σ-set) V containing $f(x)$ there exists a regular F_σ-set U containing x such that $f(U) \subset V$.*

Definition 2.2. *A space X is said to be*
(i) D_δ***-Hausdorff [11] (θ-Hausdorff*** [2] [30]) *if every pair of distinct points in X are contained in disjoint regular F_σ-sets (θ-sets).*
(ii) $D_\delta T_0$***-space*** [15] *if for each pair of distinct points x, y in X, there is a regular F_σ-set U containing one of the points x and y but not both.*
(iii) ***weakly δ-normal [13] (weakly θ-normal*** [7]) *if for every pair of disjoint regular G_δ-sets (θ-closed sets) A and B, there exist disjoint open sets U and V containing A and B, respectively.*
(iv) D_δ***-compact [12] (θ-compact*** [8] [5][2]) *if every cover of X by regular F_σ-sets (θ-open sets) has a finite subcover.*
(v) D_δ***-completely regular*** [14] *if it has a base of regular F_σ-sets.*
(vi) ***θ-completely regular*** [30] *if for each θ-closed set F and a point $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.*

[1]slightly continuous functions have been referred to as cl-continuous in ([11] [18])

[2]θ-sets have been called θ-compact by Jafari [5]. For example of a θ-set which is not θ-compact see [8, Remark 2.2]

Definition 2.3. *A subset S of a space X is said to be **regular** G_δ-**embedded** [1] in X if every regular G_δ-set in S is the intersection of a regular G_δ-set in X with S; or equivalently every regular F_σ-set in S is the intersection of a regular F_σ-set in X with S.*

Definition 2.4. *A filter $\mathcal{F}$ is said to $u\theta$-**converge** [8] (d_δ-**converge** [10])to a point x, written as $\mathcal{F} \xrightarrow{u\theta} x$($\mathcal{F} \xrightarrow{d_\delta} x$)if every θ open set (regular F_σ-set) of x contains a member of $\mathcal{F}$.*

Definition 2.5. *A net (x_α) in a topological space is said to $u\theta$-**converge** [8] to x, written as $(x_\alpha) \xrightarrow{u\theta} x$, if for each θ open set V containing x it is eventually in V.*

Definition 2.6. *A net (x_α) in a topological space is said to d_δ-**converge** [10] to x, written as$(x_\alpha) \xrightarrow{d_\delta} x$,if for each regular F_σ-set V containing x the net (x_α) is eventually in V.*

Definition 2.7. *Let (X, τ) be a topological space.*

i) Let B_{d_δ} denote the collection of all regular F_σ-sets. Since the intersection of two regular F_σ-sets is a regular F_σ-set, the collection B_{d_δ} is a base for a topology τ_{d_δ} on X such that $\tau_{d_\delta} \subset \tau$. The topology τ_{d_δ} has been used in [10], [11].

ii) Let B_θ denote the collection of θ-open sets of the space (X, τ). Since arbitrary unions and finite intersections of θ -open sets are θ-open, the collection B_θ is indeed a topology on X. We shall denote this topology by τ_θ. The topology τ_{d_θ} has been extensively referred to in the literature (see [21], [31]).

3. Pseudo strongly $\theta-$ continuous functions

A function $f : X \to Y$ from a topological space X into a topological space Y is said to be ***pseudo strongly*** $\theta-$ ***continuous*** at $x \in X$ if for each regular F_σ-set V containing $f(x)$ there exists an open set U containing x such that $f(\overline{U}) \subset V$. The function f is said to be pseudo strongly θ-continuous if it is pseudo strongly θ-continuous at each $x \in X$.

The following diagram supplements the diagrams already existing in the literature and well reflects the place of pseudo strong θ-continuity in the hierarchy of variants of continuity that already exist in the lore of mathematical literature. It also well depicts the relations and interrelations that exist among pseudo strong θ-continuity and other variants of continuity existing in the literature.

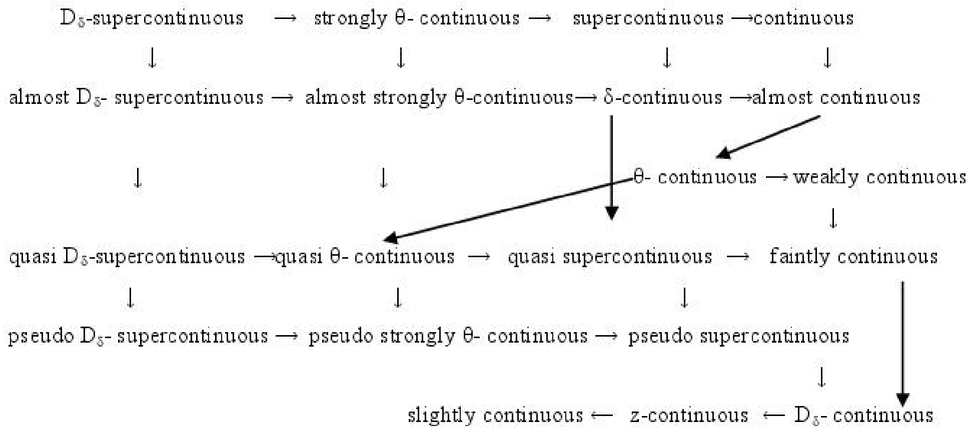

Figure 1

However, none of the above implications is reversible as shown by examples in [11], [23], [24], [29] and the following examples/observations.

3.1 **Example**: Let X denote the skyline space due to Heldermann [4] which is a T_1-regular space but not a D_δ-completely regular space. Then every D_δ-continuous function from $f : X \to Y$ is pseudo strongly θ-continuous but fails to be pseudo D_δ -supercontinuous.

3.2 **Example**: Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and let Y be the skyline space due to Heldermann [4] which is a T_1-regular space. Let $f : X \to Y$ be defined as $f(a) = f(b) = f(c) = p^-$, $f(d) = p^+$. Then f is pseudo strongly θ continuous, since Y is the only regular F_σ-set containing both p^- and p^+ but it is not quasi supercontinuous as $V(c) = \{(x, y) : c < x\} \cup \{p^+\}$is a θ-open set containing $p^+$and its inverse image is not even open.

3.3 **Example**: Let $\mathbb{N}$ be the set of positive integers. Define a topology τ on $\mathbb{N}$ by taking every singleton consisting of an odd integer to be open and a set $U \subset \mathbb{N}$is open if for every even integer $p \in U$, the predecessor and successor of p are also in U. Let $Y = \mathbb{N} \cup \{\infty\}$ denote the one point compactification of the space $(\mathbb{N}, \tau)$. Let $X = \{a, b, c, d\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then $f : X \to Y$ defined as $f(a) = f(b) = f(c) = 2$, $f(d) = \infty$ is quasi supercontinuous but not pseudo strongly θ continuous.

3.4 **Proposition**: *If X is a regular space and $f : X \to Y$ is D_δ-continuous, then f is a pseudo strongly θ-continuous function.*

Proof:In a regular space every open set is $\theta-$ open.

3.5 **Proposition**: *Every pseudo strongly θ-continuous function into a D_δ-completely regular space is strongly θ-continuous.*

Proof: Let V be an open set in Y. Since Y is D_δ-completely regular, $V = \cup_\alpha V_\alpha$, where each V_α is a regular $F_\sigma$set. Again, since f is pseudo strongly θ-continuous, $f^{-1}(V) = f^{-1}(\cup_\alpha V_\alpha) = \cup_\alpha f^{-1}(V_\alpha)$, where each $f^{-1}(V_\alpha)$ is $\theta-$open in X and since arbitrary unions of θ open sets is θ open so $f^{-1}(V)$ is θ open.

3.6 **Proposition**: *Every d_δ-map $f : X \to Y$ is pseudo D_δ-supercontinuous.*

3.7 **Proposition**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous function. If $g : Y \to Z$ is D_δ-supercontinuous (respectively almost D_δ-supercontinuous, respectively quasi D_δ-supercontinuous,respectively pseudo D_δ-supercontinuous, respectively z-continuous), then $g \circ f$ is strongly θ-continuous (respectively almost strongly θ-continuous, respectively quasi θ-continuous, respectively pseudo strongly θ-continuous, respectively z-strongly θ-continuous).*

4. Basic Properties of Pseudo strongly θ-continuous Functions

4.1. Theorem: *For a function $f : (X, \tau) \to (Y, \upsilon)$ the following statements are equivalent.*
(a) *f is pseudo strongly θ-continuous.*
(b) *For every $x \in X$ and for each regular F_σ-set V containing $f(x)$, there exists a θ-open set U containing x such that $f(U) \subset V$.*
(c) *$f^{-1}(V)$ is θ-open in X for every regular F_σ-set $V \subset Y$.*
(d) *$f^{-1}(V)$ is θ-open in X for every d_δ-open set $V \subset Y$.*
(e) *$f^{-1}(B)$ is θ-closed in X for every regular G_δ-set $B \subset Y$.*
(f) *$f^{-1}(B)$ is θ-closed in X for every d_δ-closed set $B \subset Y$.*
(g) *The function $f : (X, \tau_\theta) \to (Y, \upsilon_{d_\delta})$is continuous.*
(h) *The function $f : (X, \tau) \to (Y, \upsilon_{d_\delta})$ is strongly θ-continuous.*
(i) *The function $f : (X, \tau_\theta) \to (Y, \upsilon)$ is D_δ-continuous.*
(j) *For every filter $\mathcal{F}$ with $\mathcal{F} \xrightarrow{u\theta} x$, we have $f(\mathcal{F}) \xrightarrow{d_\delta} f(x)$.*
(k) *For every net (x_α) in X with $(x_\alpha) \xrightarrow{u\theta} x$, we have $f(x_\alpha) \xrightarrow{d_\delta} f(x)$.*

Proof: $(a) \Rightarrow (b)$: Let f be a pseudo strongly θ-continuous function. Let $x \in X$ and V be an open set containing $f(x)$. So there exists an open set W containing x such that $f(\overline{W}) \subset V$. Then $x \in W \subset (\overline{W}) \subset f^{-1}(V)$,where $f^{-1}(V)$ is a θ-open set. Let $U = f^{-1}(V)$. Then $f(U) = f(f^{-1}(V)) \subset V$.
$(b) \Rightarrow (c)$: Let $V \subset Y$ be a regular F_σ-set. Let $x \in f^{-1}(V)$. Since $f(x) \in V$, there exists *a* θ-open set U_x containing x such that $f(U_x) \subset V$. Then $x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)$. It follows that

$f^{-1}(V) = \bigcup\limits_{x \in f^{-1}(V)} U_x$ is a union of θ-open sets, hence it is a θ-open set.
$(c) \Rightarrow (d)$: Let V be a d_δ-open set in Y. Then $V = \cup_\alpha V_\alpha$, where each V_α is a regularF_σ-set in Y. Now $f^{-1}(V) = f^{-1}(\cup_\alpha V_\alpha) = \cup_\alpha f^{-1}(V_\alpha)$, where each $f^{-1}(V_\alpha)$ is $\theta-$open in X by (c) and since arbitrary unions of $\theta-$open sets are $\theta-$open, $f^{-1}(V)$ is $\theta-$open.
$(d) \Rightarrow (e)$: Let B be a regular G_δ-set in Y. Then $Y \setminus B$ is regular F_σ-set. Since every regular F_σ-set is d_δ-open, therefore by (d) $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is θ-open and hence $f^{-1}(B)$ is θ-closed.
$(e) \Rightarrow (f)$: Every d_δ-closed set $B \subset Y$ is an intersection of regular G_δ-set, hence $f^{-1}(B)$ is an intersection of θ-closed sets.
$(f) \Rightarrow (g)$: Let B be a closed set in (Y, υ_{d_δ}), hence d_δ-closed in (Y, υ). By (f), $f^{-1}(B)$ is θ-closed in (X, τ), hence closed in (X, τ_θ).
$(g) \Rightarrow (h)$: Let V be an open set in (Y, υ_{d_δ}). By (g), the set $f^{-1}(V)$ is open in (X, τ_θ) and so is θ-open in (X, τ).
$(h) \Rightarrow (i)$: This follows from definitions 2.1 and 2.7.
$(i) \Rightarrow (j)$: Let V be a regular F_σ-set in Y containing $f(x)$. By (i) $f^{-1}(V)$ is open in (X, τ_θ) which implies $f^{-1}(V)$ is θ-open in (X, τ). Since $\mathcal{F} \xrightarrow{u\theta} x$, there exists a member $G \in \mathcal{F}$ such that $G \in f^{-1}(V)$. Clearly $f(G) \subset V$ and $f(\mathcal{F}) \xrightarrow{d_\delta} f(x)$.
$(j) \Rightarrow (k)$: Let $(x_\alpha) \xrightarrow{u\theta} x$, and let $\mathcal{F}_{x_\alpha}$be the filter generated by (x_α). Then each θ-open set containing x contains a member of $\mathcal{F}_{x_\alpha}$and so $\mathcal{F}_{x_\alpha} \xrightarrow{u\theta} x$. By (j) we have $f(\mathcal{F}_{x_\alpha}) \xrightarrow{d_\delta} f(x)$, which in turn implies that $\mathcal{F}_{f(x_\alpha)} \xrightarrow{d_\delta} f(x)$. Thus every regular F_σ-set V containing $f(x)$ contains a member of $\mathcal{F}_{f(x_\alpha)}$and so $f(x_\alpha)$ is eventually in V and thus $f(x_\alpha) \xrightarrow{d_\delta} f(x)$.
$(k) \Rightarrow (a)$: Suppose f is not pseudo strongly θ-continuous at x. Then there exists a regular F_σ-set V containing x such that $f(U) \not\subset V$ for every θ-open set U containing x. Thus for every θ-open set U containing x, we can choose $x_U \in U$ such that $f(x_U) \notin V$. But then (x_U) is a net in X such that $(x_U) \xrightarrow{u\theta} x$, but $f(x_U) \overset{d_\delta}{\not\longrightarrow} f(x)$.
4.2 **Theorem**: *If $f : X \to Y$ is a pseudo strongly θ-continuous function and $g : Y \to Z$ is d_δ-map, then $g \circ f$ is pseudo strongly θ-continuous.*
4.3 **Corollary**: *If $f : X \to Y$ is pseudo strongly θ-continuous and $g : Y \to Z$ is continuous, then the composition $g \circ f$ is pseudo strongly θ-continuous.*

4.4 **Theorem**: *If $f : X \to Y$ is pseudo strongly θ-continuous and $A \subset X$. Then the restriction $f|A : A \to Y$ is pseudo strongly θ-continuous, further, if $f(A)$ is regular G_δ-embedded in Y, then $f|A : A \to f(A)$ is pseudo strongly θ-continuous.*

4.5 **Theorem**: *If $f : X \to Y$ is pseudo strongly θ-continuous and Y is a subspace of Z, then $g : X \to Z$ defined by $g(x) = f(x)$ for all $x \in X$ is pseudo strongly θ-continuous.*

4.6 **Theorem**: *If $f : X \to Y$ is pseudo strongly θ-continuous and $f(X)$ is regular G_δ-embedded in Y, then $f : X \to f(X)$ is pseudo strongly θ-continuous.*

4.7 **Theorem**: *Let $f : X \to Y$ be a function. Then the following statements are true.*

(a) *Let $\{U_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of X and suppose that $f_\alpha = f|U_\alpha$ is pseudo strongly θ-continuous for each α. Then f is pseudo strongly θ-continuous.*

(b) *Let $\{F_i : i = 1, ..., n\}$ be a θ-closed cover of X and suppose that for each $i = 1, ..., n$, $f_i = f|F_i$ is pseudo strongly θ-continuous. Then f is pseudo strongly θ-continuous.*

Proof: (a) Let V be a regular F_σ-set in Y. Then $f^{-1}(V) = \cup\{f_\alpha^{-1}(V) : \alpha \in \Lambda\}$. Since each f_α is pseudo strongly θ-continuous, each $f_\alpha^{-1}(V)$ is θ-open in U_α and hence in X. Since arbitrary unions of θ-open sets is θ-open, $f^{-1}(V)$ is θ-open and so f is pseudo strongly θ-continuous.

(b) Let F be any regular G_δ-set in Y. Then $f^{-1}(F) = \cup_{i=1}^{n} f_i^{-1}(F)$. Since each $f_i (i = 1, ..., n)$, is pseudo strongly θ-continuous, $f_i^{-1}(F)$ is a θ-closed set in F_i and hence in X. Since finite union of θ-closed sets is θ-closed, $f^{-1}(F)$ is θ-closed in X and so f is pseudo strongly θ-continuous.

4.8 **Lemma** ([29]): *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha) \in X$ and V is a regular F_σ-set in the product space containing x, then there exists a basic regular F_σ-set $\prod V_\alpha$ such that $x \in \prod V_\alpha \subset V$, where V_α is a regular F_σ-set in X_α for each $\alpha \in \Lambda$ and $V_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.*

4.9 **Theorem**: *Let $\{f_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$ be a family of functions and let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. Then f is pseudo strongly θ-continuous if and only if each f_α is pseudo strongly θ-continuous.*

Proof: Suppose f is pseudo strongly θ-continuous. Then for each α, we have $f_\alpha = \pi_\alpha \circ f$, where π_α denotes the projection map

$\pi_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$. So, in view of Corollary 4.3, each f_α is pseudo strongly θ continuous.

Conversely, suppose that each $f_\alpha : X \to X_\alpha$ is pseudo strongly θ-continuous. To prove that f is pseudo strongly θ-continuous, it suffices to prove that $f^{-1}(U)$ is θ-open for each regular F_σ-set U in the product space $\prod_{\alpha\in\Lambda} X_\alpha$. Since arbitrary unions and finite intersections of θ-open sets is θ-open, in view of Lemma 4.8 it is sufficient to prove that $f^{-1}(S)$ is θ-open for every subbasic regular F_σ-set S in the product space $\prod_{\alpha\in\Lambda} X_\alpha$. Let $U_\beta \times \prod_{\alpha\in\Lambda} X_\alpha$ be a subbasic regular F_σ-set in $\prod_{\alpha\in\Lambda} X_\alpha$. Then $f^{-1}(U_\beta \times \prod_{\alpha\neq\beta} X_\alpha) = f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$ is θ-open in X and so f is pseudo strongly θ-continuous.

4.10 **Corollary**: *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is pseudo strongly θ-continuous if and only if f is pseudo strongly θ-continuous.*

Proof: The graph function can be written as $g = 1_X \times f$, where 1_X denotes the identity mapping defined on X.

4.11 **Theorem**: *Let $\{f_\alpha : X_\alpha \to Y_\alpha : \alpha \in \Delta\}$ be a family of functions and let $f : \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$ be defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each $(x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha$. Then f is pseudo strongly θ-continuous if and only if each f_α is pseudo strongly θ-continuous.*

Proof: Suppose that f is pseudo strongly θ-continuous and let V_β be a regular F_σ-set in Y_β. Then $V_\beta \times (\prod_{\alpha\neq\beta} Y_\alpha)$ is a subbasic regular F_σ-set in the product space $\prod_{\alpha\in\Lambda} Y_\alpha$. In view of pseudo strongly θ-continuity of f, $f^{-1}(V_\beta \times (\prod_{\alpha\neq\beta} Y_\alpha)) = f^{-1}(V_\beta) \times (\prod_{\alpha\neq\beta} X_\alpha)$ is a θ-open set in $\prod X_\alpha$. Consequently, $f^{-1}(V_\beta)$ is a θ-open set in X_β and hence f_β is pseudo strongly θ-continuous for each $\beta \in \Delta$.

Conversely, suppose that each f_α is pseudo strongly θ-continuous. Let $V = V_\beta \times (\prod_{\alpha\neq\beta} Y_\alpha)$ be a subbasic regular F_σ-set in the product space $\prod Y_\alpha$. Again since each f_α is pseudo strongly θ-continuous and since $f^{-1}(V) = f^{-1}(V_\beta \times (\prod_{\alpha\neq\beta} Y_\alpha) = f_\beta^{-1}(V_\beta) \times (\prod_{\alpha\neq\beta} X_\alpha)$, $f^{-1}(V)$ is θ-open and so f is pseudo strongly θ-continuous.

4.12 **Theorem**: *Let f, $g : X \to Y$ be pseudo strongly θ-continuous functions from a space X into a D_δ-Hausdorff space Y. Then the equalizer $E = \{x : f(x) = g(x)\}$ of f and g is a θ-closed subset in X.*

Proof: Let $x \in (X - E)$. Then $f(x) \neq g(x)$, and so by hypothesis on Y, there are disjoint regular F_σ-sets U and V containing $f(x)$ and $g(x)$, respectively. Since f and g are pseudo strongly θ-continuous, the sets $f^{-1}(U)$ and $g^{-1}(V)$ are θ-open and contain the point x. Let $G = f^{-1}(U) \cap g^{-1}(V)$. Then G is a θ-open set containing x and

$G \cap E = \emptyset$. Thus E is θ-closed in X.
4.13 **Corollary**: *The set of fixed points of pseudo strongly θ-continuous function into a D_δ-Hausdorff space is θ-closed in X.*
4.14 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous function. If X is θ completely regular, then f is pseudo z-supercontinuous.*
Proof: Let $x \in X$ and let V be a regular F_σ-set containing f(x). Since f is pseudo strongly θ-continuous, there exists a θ open set U containing x such that $f(U) \subset V$. In view of θ complete regularity of X there exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 0$ and $h(X \setminus U) = 1$. Then $W = h^{-1}[0, 1)$ is a cozero set containing x and contained in U, and $f(W) \subset V$. Thus f is pseudo z-supercontinuous.
4.15 **Definition**: *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. The graph $G(f)$ of f is said to be* ***θ-D_δ-closed*** *with respect to $X \times Y$ if for each $(x, y) \notin G(f)$ there exists a θ-open set U containing x and a regular F_σ-set V containing y such that $(U \times V) \cap G(f) = \emptyset$.*
4.16 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous function from a space X into a D_δ-Hausdorff space Y. Then $G(f)$, the graph of f is θ-D_δ-closed with respect to $X \times Y$.*
Proof: Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is D_δ-Hausdorff, there exist disjoint regular F_σ-sets V and W containing $f(x)$ and y, respectively. Since f is pseudo strongly θ-continuous, there exists a θ open set U containing x such that $f(U) \subset V \subset (Y - \overline{W})$. Consequently, $(U \times W) \cap G(f) = \emptyset$ and so $G(f)$ is θ-D_δ-closed with respect to $X \times Y$.

5. Topological Properties and pseudo strongly θ continuous functions

The following lemma will be of use in the sequel.
5.1 **Lemma** ([7] [9]): *Let X be a topological space. A subset U of X is θ-open if and only if for each $x \in U$ there exists an open set V such that $x \in V \subset \overline{V} \subset U$.*
5.2 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous injection into a $D_\delta T_0$-space Y. Then X is Hausdorff. Moreover, if Y is D_δ-Hausdorff, then X is θ-Hausdorff.*
Proof: Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is an injection, $f(x_1) \neq f(x_2)$. Now, since Y is a $D_\delta T_0$-space, there exists a regular F_σ-set V containing one of the points $f(x_1)$ and $f(x_2)$ but not both. For definiteness, assume $f(x_1) \in V$. Since f is pseudo strongly θ-continuous, $f^{-1}(V))$

is a θ-open set containing x_1. By Lemma 5.1, there exists an open set U such that $(x_1) \in U \subset \overline{U} \subset f^{-1}(V))$. Then U and $(X \setminus \overline{U})$ are disjoint open sets containing x_1 and x_2, respectively and so X is Hausdorff.

Now, if Y is D_δ-Hausdorff, there exist disjoint regular F_σ-sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively. Then, in view of Theorem 4.1, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint θ-open sets containing x_1 and x_2, respectively and so X is a θ-Hausdorff space.

5.3 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous surjection from a θ-compact space X onto a space Y. Then Y is a D_δ-compact space.*

Proof: Let $\beta = \{B_\alpha : \alpha \in \Lambda\}$ be a cover of Y by regular F_σ-sets. Since f is pseudo strongly θ-continuous, the collection $G = \{f^{-1}(B_\alpha) : B_\alpha \in \beta\}$ is a cover of X by θ-open sets. Since X is θ-compact, there exists a finite subcollection $\{f^{-1}(B_{\alpha_i})|i = 1, ..., n\}$ of G which covers X. Now, since f is onto, $\{B_{\alpha_i}|i = 1, ..., n\}$ is a finite subcollection of β which covers Y. Hence Y is a D_δ-compact space.

5.4 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous, closed surjection defined on a weakly θ-normal space X. Then Y is a weakly D_δ-normal space.*

Proof: Let A and B be disjoint regular G_δ-subsets of Y. Since f is pseudo strongly θ-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint θ-closed sets in X. Again, since X is a weakly θ-normal space, there exist disjoint open sets U and V containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. Since f is a closed surjection, $f(X \setminus U)$ and $f(X \setminus V)$ are disjoint closed sets in Y. It is easily verified that $Y \setminus f(X \setminus U)$ and $Y \setminus f(X \setminus V)$ are disjoint open sets containing A and B, respectively. Hence, Y is a weakly D_δ-normal space.

5.5 **Theorem**: *Let $f : X \to Y$ be a pseudo strongly θ-continuous injection which maps open sets in X to d_δ-open sets in Y. Then X is a regular space.*

Proof: To prove that X is a regular space, we shall show that every open set in X is θ-open. To this end, let U be an open set in X containing x. Then by the hypothesis on f, $f(U)$ is a d_δ-open set containing $f(x)$. Let $f(U) = \cup_{\alpha\in\Lambda} V_\alpha$,where each V_α is a regular F_σ-set in Y. Since f is pseudo strongly θ-continuous, each $f^{-1}(V_\alpha)$ is open in X. Since f is an injection, $U = f^{-1}(f(U)) = \cup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$. Again, since an arbitrary unions of θ-open sets is θ-open, U is θ-open and so X is a regular space.

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