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GRÜSS AND OSTROWSKI TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. We establish new inequalities for the cumulative distribution function, the incomplete special function, the expectation, the variance and the k-th order moments.

1. Introduction

Let X be a random variable with the probability density function $f : [a, b] \to \mathbb{R}_+$ and the cumulative distribution function $F(x) = \int_a^x f(t)dt$, i.e. $F(x) = Pr\,(X \le x)$.

The expectation, $E(X)$, respectively variance $\sigma^2(X)$ of the random variable X are defined such that

$$E(X) := \int_a^b t f(t)dt, \text{ respectively}$$

$$\sigma^2(X) := \int_a^b [t - E(X)]^2 f(t)dt.$$

Many applications in mathematical statistics of the cumulative distribution function, expectation and variance motivates us to give new approximations of these statistical indicators. These results were obtained by using the classical mathematical inequalities.

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In the last years many authors, using Grüss and Ostrowski inequalities, obtained in this field interesting results and very useful in insurance, econometrics and actuarial mathematics ([3], [7], [10], [12], [19]). That particular case of the cumulative distribution function can be considered the incomplete special function which have many applications in physics and engineering mathematics. In section 2 we propose a very easy method to approximates these kind of functions. One can here point to some important samples such as the incomplete special function ([13], [16]):

i) incomplete beta function

$$B(x;p,q)=\int_0^x t^{p-1}(1-t)^{q-1}dt,\ p,q>0,\ 0\leq x\leq 1.$$

ii) incomplete gamma function

$$\gamma(x;p)=\int_0^x t^{p-1}e^{-t}dt,\ p\neq 0,-1,-2,\dots,\ x>0,$$

iii) first and second kind of incomplete elliptic functions

$$E_1(x;p)=\int_0^x \frac{d\theta}{\sqrt{1-p^2\sin^2\theta}};\ E_2(x;p)=\int_0^x \sqrt{1-p^2\sin^2\theta}d\theta;\ |p|<1,$$

iv) error function

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,$$

v) Fresnel functions

$$C(x)=\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt,\ S(x)=\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt.$$

We recall here a few from the celebrated classical inequalities of Grüss and Ostrowski type, which are used to give main results of this paper.

Denote by

$$T(f,g):=\tfrac{1}{b-a}\int_a^b f(t)g(t)dt-\tfrac{1}{b-a}\int_a^b f(t)dt\cdot\tfrac{1}{b-a}\int_a^b g(t)dt, \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable functions. The functional (1.1) is well known in the literature as the Čebyšev functional.

In 1934, Grüss [8] obtained the following results

Theorem 1.1. *Let f and g be two functions defined and integrable on $[a,b]$. If $m \leq f(x) \leq M$ and $p \leq g(x) \leq P$ for all $x \in [a,b]$, then*

$$|T(f,g)| \leq \frac{1}{4}(M-m)(P-p). \tag{1.2}$$

The constant 1/4 is the best possible.

In 1882 Čebyšev [4] obtained for $T(f,g)$ the following inequality

Theorem 1.2. *If $f, g \in C^1[a,b]$, then*

$$|T(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2,$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)|$.

The constant $\frac{1}{12}$ cannot be improved in the general case.

In 1970, Ostrowski [18] proved the following result

Theorem 1.3. *If f is Lebesgue integrable on $[a,b]$ and satisfying $m \leq f(x) \leq M$, $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$, then*

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty.$$

The constant $\frac{1}{8}$ is sharp.

These inequalities are known in literature as the Grüss type inequalities.

Another celebrated classical inequality was proved by A.M. Ostrowski [17] in 1938 which we cite below in the form given by Anastassiou in 1995 (see [2]).

Theorem 1.4. *Let f be in $C^1[a,b]$, $x \in [a,b]$. Then*

$$\left| f(x) - \tfrac{1}{b-a}\int_a^b f(t)dt \right| \leq \tfrac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty. \tag{1.3}$$

In [6], Cheng gave a sharp version of the above inequality but also generalized it as follows.

Theorem 1.5. *Let $I \subset \mathbb{R}$ be an open interval $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is a differentiable function such that there exists constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. Then for all $x \in [a,b]$, we have*

$$\left| \tfrac{1}{2} f(x) - \tfrac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \tfrac{1}{b-a}\int_a^b f(t)dt \right| \leq \tfrac{(x-a)^2+(b-x)^2}{8(b-a)}(\Gamma-\gamma). \tag{1.4}$$

Definition 1.6. [11] The function $f:[a,b]\to\mathbb{R}$ is said to be (l,L)-Lipschitzian on $[a,b]$ if $l(x_2-x_1)\le f(x_2)-f(x_1)\le L(x_2-x_1)$, for $a\le x_1\le x_2\le b$, where $l,L\in\mathbb{R}$, with $l<L$.

In [11], Liu proved the following Ostrowski inequality for functions (l,L)-Lipschitzian on $[a,b]$.

Theorem 1.7. *Let $f:[a,b]\to\mathbb{R}$ be (l,L)-Lipschitzian on $[a,b]$, $a<b$. Then we have*

$$\left|f(x)-\left(x-\tfrac{a+b}{2}\right)S-\tfrac{1}{b-a}\int_a^b f(t)dt\right|\le(b-a)M_0, \tag{1.5}$$

for all $x\in[a,b]$, where $S=\frac{f(b)-f(a)}{b-a}$ and $M_0=\min\left\{\frac{L-l}{8},\frac{S-l}{2},\frac{L-S}{2}\right\}$.

2. The applications of Grüss and Ostrowski inequalities

Since the incomplete special functions cannot be evaluate at any arbitrary point we will give a very easy model to approximate the values of these kind of functions. The inequalities from this section was obtained using inequalities of Grüss type.

Proposition 2.1. *If $x>0$ and $p>1$, then*

$$\left|\tfrac{x^{p+1}}{p(e^x-1)}-\gamma(x;p)\right|\le\begin{cases}\frac14 x^2(p-1)^{p-1}e^{-(p-1)}, & \text{for } x\ge p-1,\\[2mm] \frac14 x^{p+1}e^{-x}, & \text{for } x<p-1.\end{cases} \tag{2.1}$$

Proof. In Grüss inequality (1.2) we consider $f:[0,x]\to\mathbb{R}$, $f(t)=e^t$, $g:[0,x]\to\mathbb{R}$, $g(t)=t^{p-1}e^{-t}$. Since for all $t\in[0,x]$ the functions f and g verified $1\le f(t)\le e^x$, $0\le g(t)\le(p-1)^{p-1}e^{-(p-1)}$, for $x\ge p-1$ and $0\le g(t)\le x^{p-1}e^{-x}$, for $x<p-1$, then the inequality (2.1) holds.

Proposition 2.2. *If $x\in(0,1)$, $p>1$, $q>0$, then*

$$\left|B(x;p,q)+x^{p-1}\tfrac{(1-x)^q-1}{pq}\right|\le\begin{cases}\frac14 x^p\left[(1-x)^{q-1}-1\right], & 0<q<1,\\[2mm] \frac14 x^p\left[1-(1-x)^{q-1}\right], & q\ge1.\end{cases} \tag{2.2}$$

Proof. In Grüss inequality (1.2) we consider $f:[0,x]\to\mathbb{R}$, $f(t)=t^{p-1}$, $g:[0,x]\to\mathbb{R}$, $g(t)=(1-t)^{q-1}$.

Proposition 2.3. *If $x\in(0,1)$ and $p,q>1$, $q\ne2$, then*

$$\left|\tfrac{x^{p+1}(2-q)}{p[(1-x)^{2-q}-1]}+B(x;p,q)\right|\le \tag{2.3}$$

$$\begin{cases} \frac{x^2(2-q)}{4[1-(1-x)^{2-q}]} \cdot \frac{(1-p)^{p-1}(1-q)^{q-1}}{(2-p-q)^{p+q-2}} \left[(1-x)^{1-q}-1\right], \frac{1-p}{2-p-q} \leq x < 1, \\ \\ \frac{x^2(2-q)}{4[1-(1-x)^{2-q}]} \cdot x^{p-1}(1-x)^{q-1} \cdot \left[(1-x)^{1-q}-1\right], \ 0 < x < \frac{1-p}{2-p-q}. \end{cases}$$

Proof. In Grüss inequality (1.2) we consider $f : [0,x] \to \mathbb{R}$, $f(t) = t^{p-1}(1-t)^{q-1}$, $g : [0,x] \to \mathbb{R}$, $g(t) = (1-t)^{1-q}$.

Proposition 2.4. *If $x > 0$, then*

$$\left| \frac{2\left(1-e^{-x^2}\right)}{x\sqrt{\pi}} - erf(x) \right| \leq \frac{x}{\sqrt{\pi}} \left(1-e^{-x^2}\right). \tag{2.4}$$

To obtained inequalities for cumulative distribution functions very useful are inequalities of Ostrowski type. For example using classical inequality of Ostrowski type which was recalled by Theorem 1.4 can be obtained the following inequality.

Theorem 2.5. *Let X be a random variable with the probability density function $f : [a,b] \to \mathbb{R}_+$ and the cumulative distribution function F. Then*

$$\left| F(x) - \frac{b-E(X)}{b-a} \right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)} \|f\|_\infty, \tag{2.5}$$

where $E(X)$ is expectation of X.

Proof. This inequality can be easy obtained if in Ostrowski inequality to replace f with the cumulative distribution function F.

Let X be a random variable which has Beta distribution, therefore, the density probability function is

$$f(x) = \begin{cases} \frac{1}{B(p,q)} \cdot x^{p-1}(1-x)^{q-1}, \ x \in [0,1], \\ \\ 0, \ x \in \mathbb{R} \setminus [0,1], \end{cases}$$

where $B(p,q) = \displaystyle\int_0^1 t^{p-1}(1-t)^{q-1}dt$.

Since the mathematical expectation of X is $E(X) = \frac{p}{p+q}$ and the norm of density probability function is

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \frac{1}{B(p,q)} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}} \quad \text{for } p,q > 1,$$

the inequality (2.5) can be written

$$\left| Pr(X \leq x) - \frac{q}{p+q} \right| \leq \frac{1}{B(p,q)} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}} \cdot \frac{x^2+(1-x)^2}{2} \quad \text{for } p,q > 1.$$

In the last years many applications in statistics and econometrics utilize the truncated distributions ([14], [15]).

Suppose we have a continuous distributions with probability density function (pdf) g and cumulative distribution function (cdf) G. Let X be a random variable representing the truncated version of this distribution over the interval $[a,b]$ where $-\infty < a < b < \infty$.

Then the pdf and the cdf of the truncated distribution can be written

$$f(x)=\frac{g(x)}{G(b)-G(a)},\quad F(x)=\frac{G(x)-G(a)}{G(b)-G(a)},$$

respectively, for $-\infty < a < b < \infty$.

Since in the above results we consider the compact interval $[a,b]$, then all of these results can be applied in the case of truncated distribution.

Theorem 2.6. *Let X be a random variable with the expectation $E(X)$ and the variance $\sigma^2(X)$. Then*

$$0\le\left(E(X)-\frac{a+b}{2}\right)^2+\sigma^2(X)\le\frac{(b-a)^2}{2}. \tag{2.6}$$

Proof. Let $f:[a,b]\to\mathbb{R}_+$ be the probability density function and $F(x)=\displaystyle\int_a^x f(t)dt$ be the cumulative distribution function of random variable X. Define by $H(x)=\displaystyle\int_a^x F(t)dt$. Since $H(b)=b-E(X)$, $\displaystyle\int_a^b H(x)dx=b(b-E(X))-\frac{b^2}{2}+\frac{1}{2}E(X^2)$, if we replace in (1.4) function f with H, we obtain the relation (2.6).

Theorem 2.7. *Let X be a random variable with the expectation $E(X)$ and the variance $\sigma^2(X)$. Then*

$$\left|\left(E(X)-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2+\sigma^2(X)\right|\le 2(b-a)^2M_0, \tag{2.7}$$

where $M_0=\min\left\{\frac{1}{8},\frac{b-E(X)}{2(b-a)},\frac{E(X)-a}{2(b-a)}\right\}$.

Proof. Let $H(x)=\displaystyle\int_a^x F(t)dt$, where F is the cumulative distribution function of random variable X. Since $0\le H(x_2)-H(x_1)=\displaystyle\int_{x_1}^{x_2}F(x)dx\le(x_2-x_1)$, for all $a\le x_1<x_2\le b$, the function H is $(0,1)$-Lipschitzian. If we replace in (1.5) function f with H, we obtain the relation (2.7).

3. Moments inequalities of a random variable

The k-th order moment, the k-th central moment, respectively, of the random variable X are defined in the following way

$$M_k(X) = \int_a^b t^k f(t)dt, \ \nu_k(X) = \int_a^b (t-E(X))^k f(t)dt, \ k = 0, 1, 2, \ldots,$$

respectively, where $f : [a, b] \to \mathbb{R}_+$ is the probability density function.

It may be noted that $M_0(X) = 1$, $M_1(X) = E(X)$, $\nu_0(X) = 1$, $\nu_1(X) = 0$ and $\nu_2(X) = \sigma^2(X)$ is the variance of the random variable X.

If we know the probability density function, then the moments can be determined. Since there are situations when the exact forms of probability distribution functions are not know or the moments can not be calculated, in this section our goal is to find some estimations of the moments of the random variable using the classical inequalities of Grüss-Ostrowski type. Numerous statistical, actuarial applications of probability theory motivate in the last years many authors to develop some estimations of the moments, the covariance and the variance of random variables ([9], [19]).

Theorem 3.1. *For the random variable X with the probability density function $f : [a, b] \to \mathbb{R}_+$, $0 \le a < b$,*

$$|M_{k+1}(X) - aM_k(X)| \le M\tfrac{(b-a)^2}{2}b^k,$$

where $m \le f(t) \le M$ on $[a, b]$.

Proof. In Ostrowski inequality (1.3) we replace the function f with $F(x) = \int_a^x t^k f(t)dt.$

Theorem 3.2. *For the random variable X with the probability density function $f : [a, b] \to \mathbb{R}_+$, $0 \le a < b$,*

$$\left|M_{k+1}(X) - \tfrac{a+b}{2}M_k(X)\right| \le \tfrac{(b-a)^2}{8}(b^k M - a^k m),$$

where $m \le f(t) \le M$ on $[a, b]$.

Proof. To prove in similar way with the above result using inequality (1.4).

In [1] we proved the following Grüss inequality on a compact metric space for more than two functions.

Theorem 3.3. [1] *Let $A : C(X) \to \mathbb{R}$ be a positive, linear functional, $A(1) = 1$, defined on the metric space $C(X)$. Then*

(3.1)
$$|A(f_1 f_2 \dots f_n) - A(f_1)A(f_2)\dots A(f_n)| \leq \frac{1}{4} \sum_{i,j=1, i<j}^{n} \theta(f_i)\theta(f_j) \prod_{k=1, k\neq i,j}^{n} \|f_k\|_\infty,$$

where $\theta(f) := \max\limits_X f - \min\limits_X f$.

Let X be a continuous random variable having probability density function w defined on a finite interval $[a, b]$. By definition

$$E(f(X)) = \int_a^b f(x)w(x)dx, \;\; M_k(f(X)) = \int_a^b f(x)^k w(x)dx,$$

are the mean value, the k-th order moment of the random variable $f(X)$, respectively.

If in (3.1) we choose the functional $A(f) = \displaystyle\int_a^b f(x)w(x)dx$, then

$$|M_n(f(X)) - [E(f(X))]^n| \leq \frac{n(n-1)}{8}(M - m)^2 M^{n-2},$$

where $m \leq f(x) \leq M$, $x \in [a, b]$.

Theorem 3.4. *Let X be a continuous random variable such that $m \leq X \leq M$ for some finite $m, M \in \mathbb{R}$. Then*

$$|M_n(X) - [E(X)]^n| \leq \tfrac{n(n-1)}{8}(M - m)^2 M^{n-2}.$$

Theorem 3.5. *If $f : [a, b] \to \mathbb{R}_+$ is a probability density function and $m \leq f(x) \leq M$ for $x \in [a, b]$, then*

$$\left| \tfrac{1}{b-a} \int_a^b (f(t))^n \, dt - \left(\tfrac{1}{b-a}\right)^n \right| \leq \tfrac{1}{8} n(n-1)(M - m)^2 M^{n-2}.$$

Proof. In (3.1) we choose $A(f) = \frac{1}{b-a} \displaystyle\int_a^b f(t)dt$ and $f_i = f$, $i = \overline{1, n}$.

In [9], Kumar proved the following inequality.

Theorem 3.6. [9] *For the random variable X with distribution function $F : [a, b] \to [0, 1]$,*

$$\left| \int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s} \tfrac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \right| \leq \tfrac{1}{4}(b-a)(M-m)^2,$$

where $m \leq f \leq M$ a.e. on $[a, b]$ and $s, r \leq 0$.

In next part of this section we will give a new inequality for integral $I = \displaystyle\int_a^b (b-t)^r (t-a)^s f(t)dt$. This integral plays an important role in computed of the central moment (see [9]).

Theorem 3.7. *Let $f : [a,b] \to \mathbb{R}_+$ be a probability density function and $m \le f(x) \le M$ for $x \in [a,b]$. Then*

$$\left| \int_a^b (b-t)^r (t-a)^s f(t)dt - \tfrac{(b-a)^{r+s}}{(r+1)(s+1)} \right| \le \tfrac{1}{4}(b-a)^{r+s+1}(3M-2m),$$

where $s, r \ge 0$.

Proof. In (3.1) we consider $A(g) = \frac{1}{b-a}\displaystyle\int_a^b g(x)dx$ and $n = 3$ with $f_1(t) = (b-t)^r$, $f_2(t) = (t-a)^s$, $f_3(t) = f(t)$.

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