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THE USE OF THE CHROMATIC POLYNOMIAL OF A GRAPH IN ENUMERATIVE COMBINATORICS

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Abstract. The chromatic polynomial of a graph on the set $N = \{1, 2, ..., n\}$ is used as an instrument to find the number of possible partitions of N under some particular restraints and the number of functions on N with special properties.

1. Introduction

Let $n \geq 3$ be an integer, $N := \{1, 2, ..., n\}$ and $Part(N)$ is the family of all partitions of the set N.

We recall that $\pi \in Part(N)$ if and only if

1^o. $(\forall A)(A \in \pi \Rightarrow A \neq \emptyset)$;

2^o. $(\forall A)\,(\forall B)(A, B \in \pi$ and $A \neq B \Rightarrow A \cap B = \emptyset)$;

3^o. $\bigcup_{A \in \pi} A = N$.

It is known that $|Part(N)| = B_n$, the Bell number of rank n.

If $\pi \in Part(N)$, the sets in π will be called *classes*. When $i \in N$, we denote the class in π containing i by $C(i, \pi)$. Now we define a set P^* of restricted partitions by

$$P^* := \{\pi \in Part(N) : C(i, \pi) \neq C(i + 1, \pi) \text{ for all } 1 \leq i < n\}.$$

Problem 1. To find the cardinality of P^*.

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2. The chromatic polynomial of a graph

Troughout this paper $G = (N, E)$ is a simple graph with $V(G) = N$ the set of nodes in G and the edge set $E = E(G)$.

A stable set or *an independence set* in G is a set of pairwise non-adjacent vertices.

We define for a graph $G = (V, E)$:

$$\begin{aligned} Ind(G) &:= \{A \subset V, A \neq \emptyset,\ A \text{ is an independent set in } G\}, \\ Ind_0(G) &:= Ind(G) \cup \{\emptyset\}. \end{aligned}$$

Then $Ind_0(G)$ is a simplicial complex of the lattice $(\mathcal{P}(N), \subseteq)$:

$$A \in Ind_0(G) \text{ and } B \subseteq A \text{ imply } B \in Ind_0(G),$$

where $P(N) := \{S : S \subseteq N\}$.

A partition $\pi := \{V_1, V_2, ..., V_r\}$ of the set V is *a color partition of* the graph G if

$$V_i \in Ind(G) \text{ for } i = 1, 2, ..., r.$$

The sets $V_1, V_2, ..., V_r$ are called *color classes.*

By $\pi_r(G)$ we denote the number of color partitions of G into r color classes.

Definition 2.1 ([B, p. 57])**.** *The chromatic polynomial* of a graph $G = (V, E)$ is the polynomial $\sum_{r=1}^{n} \pi_r(G)\lambda^{(r)}$, where $n = |V|$ and λ is a complex variable and we denote

$$\lambda^{(r)} = \prod_{i=1}^{r} (\lambda + 1 - i).$$

A r- coloring of vertices of the graph G is a function $c : V(G) \to \{1, 2, ..., r\}$, where

$$c^{-1}\{1\} \in Ind(G) \text{ for } i = 1, 2, ..., r.$$

Theorem 2.2 ([B, p. 57])**.** *If r is a natural number, $P(G; r)$ is the number of vertex colorings of the graph G with at most r colors.*

3. Normal algebra

Let $\mathbb{R}[\lambda]$ be the algebra over $\mathbb{R}$ of all real polynomials in the variable λ with usual addition $(+)$, multiplication $(\cdot)$ and scalar multiplication.

Definition 3.1 ([A, p. 100])**.** A *polynomial sequence* is a family $(P_n(\lambda))_{\lambda \geq 0}$ of monic real polynomials such that

$$\deg P_n(\lambda) = n$$

for all $n \in \mathbb{N}_0 := \{z \in \mathbb{Z} : z \geq 0\}$.

Obviously, any polynomial sequence is a basis of the $\mathbb{R}$ - vector space $\mathbb{R}[\lambda]$.

Definition 3.2. A *polynomial sequence* $(P_n(\lambda))_{\lambda \geq 0}$ is called a normal basis if

$$P_n(0) = 0, n \geq 1.$$

The following polynomial sequences are normal bases.

i) *the standard basis*

$$\sigma := (\lambda^n)_{n \geq 0}$$

ii) *the falling factorial basis*

$$\varphi := (\lambda^{(n)})_{n \geq 0},$$

where

$$\lambda^{(n)} = \prod_{i=1}^{r} (\lambda + 1 - i) \text{ and } \lambda^{(0)} = 1$$

iii) *the trees-basis*: $\tau = (t_n(\lambda))_{n \geq 0}$, where

$$t_n(\lambda) = \lambda (\lambda - 1)^{n-1}.$$

Let $\beta := (P_n(\lambda))_{\lambda \geq 0}$ be a normal basis.

Definition 3.3. The β - convolution is a binary operation on $\mathbb{R}[\lambda]$ defined by

$$f(\lambda) *_\beta g(\lambda) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j \right) P_k(\lambda),$$

where

$$f(\lambda) = a_0 P_0(\lambda) + ... + a_m P_m(\lambda)$$

and

$$g(\lambda) = b_0 P_0(\lambda) + ... + b_n P_n(\lambda).$$

It is easy to see from Definition 2.5 that $(\mathbb{R}[\lambda], +, *_\beta)$ is an algebra over $\mathbb{R}$, called β - normal algebra.

Now we define *the* φ *- functional sum of coefficients*

$$L_\varphi : \mathbb{R}[\lambda] \to \mathbb{R}$$

which is uniquely determined by linearity and by correspondences $\lambda^{(n)} \mapsto 1$ $(n \geq 0)$. If $f(\lambda) = a_0 \lambda^{(0)} + ... + a_n \lambda^{(n)}$, then $L_\varphi f(\lambda) = a_0 + ... + a_n$.

If $G = (V, E)$ is a graph and $P(G; \lambda)$ is the chromatic polynomial of G, then $L_\varphi P(G; \lambda)$ is the number of color partitions of the graph G.

Since any two polynomial sequences $(p_n(\lambda))_{n\geq 0}$ and $(q_n(\lambda))_{n\geq 0}$ of $\mathbb{R}[\lambda]$, are basses, each $q_n(\lambda)$ can be uniquely expressed as a linear combination of polynomials $p_k(\lambda)$ $(0 \leq k \leq n)$:

$$q_n(\lambda) = \sum_{k=0}^{n} a(n,k) p_k(\lambda),$$

conversely

$$p_n(\lambda) = \sum_{k=0}^{n} b(n,k) q_k(\lambda) \ (n \geq 0).$$

The coefficients $a(n,k)$, $b(n,k)$ are called the *connecting coefficients.*

Example 3.4. *1) For the polynomials sequence $((\lambda-1)^n)_{n\geq 0}$ and for the normal basis τ we have that*

$$(\lambda-1)^k(\lambda-1)^k - (-1)^{k-1}(\lambda-1)^{k-1} = (-1)^k t_k(\lambda) \ (n \geq 1)$$

and so

$$(-1)^n(\lambda-1)^n - 1 = \sum_{k=1}^{n} t_k(\lambda) \ (n \geq 0).$$

Conversely,

$$t_n(\lambda) = (\lambda-1)^n + (\lambda-1)^{n-1} (n \geq 1)$$

and $t_0(\lambda) = (\lambda-1)^0$.

2) For $n \in \mathbb{N}_0$ we have

$$\lambda^{(n)} = \sum_{k=0}^{n} (-1)^{n-k} \left[\begin{array}{c} n \\ k \end{array} \right] \lambda^k$$

and

$$\lambda^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \lambda^{(k)}$$

the connecting coefficients are $\left[\begin{array}{c} n \\ k \end{array} \right]$ and $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$, the first and respectively, the second kind Stirling's numbers.

In the following, we write $* := *_\varphi$ for short.

Theorem 3.5. *The following properties hold whenever $t \in \mathbb{N}_0$ and $f(\lambda) \in \mathbb{R}[\lambda]$:*

1) $\lambda^{(t)} * f(\lambda) = \lambda^{(t)} \cdot f(\lambda - t)$

2) $\lambda^t * f(\lambda) = \sum\limits_{k=0}^{t} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \lambda^{(k)} \cdot f(\lambda - k)$

3) $L_\varphi(\lambda^{(t)} * f(\lambda)) = L_\varphi f(\lambda)$.

Proof. 1). Let $f(\lambda) = \sum\limits_{k=0}^{m} a_k \lambda^{(k)}$. Since $\lambda^{(t+k)} = \lambda^{(t)}(\lambda - t)^{(k)}$ for all natural numbers t, k, we obtain

$$\begin{aligned} \lambda^{(t)} * f(\lambda) &= \lambda^{(t)} * \sum_{k=0}^{m} a_k \lambda^{(k)} = \sum_{k=0}^{m} a_k \lambda^{(t+k)} \\ &= \lambda^{(t)} \sum_{k=0}^{m} a_k (\lambda - t)^{(k)} = \lambda^{(t)} \cdot f(\lambda - t). \end{aligned}$$

2) follows immediately from 1) and the formula $\lambda^t = \sum\limits_{k=0}^{t} \left\{ \begin{matrix} t \\ k \end{matrix} \right\} \lambda^{(k)}$.

3) follows from definition of the functional L_φ. □

4. Number of partitions in P^*

From the condition $C(i, \pi) \neq C(i+1, \pi)$ we define the graph G with $V(G) = N$ and $E(G) = \{\{i, i+1\} : i = 1, 2, ..., n-1\}$.

Now, $\pi \in P^*$ if and only if π is a color partition of the graph G.

Since G is a tree with n vertices, we have by [B, Corollary 9.4] and Theorem 3.5 1):

$$\begin{aligned} P(G; \lambda) &= \lambda(\lambda - 1)^{n-1} \\ &= \lambda * \lambda^{n-1} \end{aligned}$$

Now, by Theorem 3.5 3) we get

$$\begin{aligned} L_\varphi(P(G; \lambda)) &= L_\varphi \lambda^{n-1} \\ &= L_\varphi \sum_{k=0}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \lambda^{(k)} \\ &= \sum_{k=0}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} = B_{n-1}. \end{aligned}$$

In conclusion, the number of partitions in P^* is B_{n-1}.

5. The problem of $\mathcal{S}$ -restricted partitions

Let $\mathcal{S}$ be a simplicial complex of the lattice $(\mathcal{P}(N), \subseteq)$, i.e. $A \in \mathcal{S}$ and $B \subseteq A \Rightarrow B \in \mathcal{S}$.

A graph $G = (N, E)$ on N is called $\mathcal{S}$ - graph if $Ind_0(G) = \mathcal{S}$.

The simplicial complex $\mathcal{S}$ is *a simplicial complex graphic on* N if there exists a $\mathcal{S}$ - graphic on the set N.

Theorem 5.1. *Let $\mathcal{S}$ be a simplicial complex on the set N. Then there exists at most one graph G on N such that $Ind_0(G) = \mathcal{S}$.*

Proof. Let G and H be graphs on N with properties: $E(G) \neq E(H)$ and $Ind_0(G) = Ind_0(H) = \mathcal{S}$. Then there exists $a \neq b$ in N with
(i) $ab \in E(G)$ and $ab \notin E(H)$ or
(ii) $ab \notin E(G)$ and $ab \in E(H)$.
For example, in the case (i) we have: $\{a, b\} \notin Ind_0(G) = \mathcal{S}$ and $\{a, b\} \in Ind_0(H) = \mathcal{S}$, a contradiction. □

Let $\mathcal{S}$ be a simplicial complex on the set N. We define the family of partitions:

$$Part(N, \mathcal{S}) := \{\pi \in Part(N) : \pi \subseteq \mathcal{S}\}.$$

The *problem of the $\mathcal{S}$ - restricted partitions* is to find the cardinality of $Part(N, \mathcal{S})$.

In order to solve this problem we use *the method of chromatic polynomial.*

Step 1. We determine the $\mathcal{S}-$ graph G , where $\mathcal{S}$ is a simplicial complex graphic on N).

Step 2. We determine the chromatic polynomial $P(G; \lambda)$.

Step 3. We have $|Part(N, \mathcal{S})| = L_\varphi P(G; \lambda)$.

The following theorem describes two useful techniques for calculating chromatic polynomials. We denote by $G_1 + G_2$ the join of the graphs G_1 and G_2. If G_1, G_2 are two simple graphs. Then $G_1 + G_2$ is a simple graph [B, p. 59] with:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2)$$

and

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{ab \mid a \in V(G_1), b \in V(G_2)\}.$$

Theorem 5.2. *The following properties hold:*

i) Reduction Formula. Let a and b be two adjacent vertices of G. Then

$$P(G; \lambda) = P(G - ab; \lambda) - P(G/ab; \lambda).$$

ii) [B, Corollary 9.6, p. 60] The chromatic polynomial of $G_1 + G_2$ is given by

$$P(G_1 + G_2; \lambda) = P(G_1; \lambda) *_\varphi P(G_2; \lambda)$$

where G_1, G_2 are two simple graphs.

Application 1.

For the set $N = \{1, 2, ..., n\}$ ($n \geq 3$ is an integer) we define

$$\mathcal{S} = \{S \subseteq N : 1 < x - y \leq n - 1 \text{ for all } x \neq y \text{ in } S\}.$$

Then an $\mathcal{S}$ - graph is the circuit graph

$$C_n := (N, \{12, 23, ..., (n-1)\, n, n1\}).$$

Since

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1),$$

by [B, p. 59], we have:

$$|Part(N, \mathcal{S})| = L_\varphi P(C_n; \lambda).$$

From the Reduction Formula we have

$$P(C_n; \lambda) = \lambda(\lambda - 1)^{n-1} - P(C_{n-1}; \lambda) \quad (1).$$

Applying L_φ in both sides of (1) we obtain

$$s_n = B_{n-1} - s_{n-1},$$

where we denote $s_k = L_\varphi(C_k; \lambda)$ for each natural number $k \geq 1$.

Solving the recursion

$$s_n + s_{n-1} = B_{n-1},\ n \geq 2 \text{ and } s_1 = 1,$$

we obtain the number of partitions in $Part(N, \mathcal{S})$, which is

$$L_\varphi P(C_n; \lambda) = s_n = (-1)^{n-1} + \sum_{k=2}^{n} (-1)^{n-k} B_{k-1}.$$

Application 2.

Let $n > m \geq 2$ be integers.

Let $f : N \to \{0, 1, ..., m-1\} =: \{M\}$ be a function defined by

$$f(x) := x - m \left\lfloor \frac{x}{m} \right\rfloor.$$

We set

$$F := \{g : N \to N \mid \text{exists } h : N \to M \text{ with } h \circ g = f\}.$$

We want to compute $|F|$.

For $i = 0, 1, ..., m-1$ we set

$$N_i := f^{-1}\{i\}.$$

We define:

$$\mathcal{S} := \{S \subseteq N : f(x) \neq f(y) \text{ for all } x \neq y \text{ in S}\}.$$

Write $n = mq + r$, where $q = \left\lfloor \frac{n}{m} \right\rfloor$ and $r = f(n)$.

The Turán's graph is the graph

$$T_n^m = G_0^{q+1} + G_1^{q+1} + ... + G_r^{q+1} + G_{r+1}^q + ... + G_{m-1}^q,$$

where $G_i^t = (V, E)$ with $|V| = t, E = \emptyset, i = 0, 1, ..., m-1$ and $t \in \{q, q+1\}$.

We have

$$P(T_n^m; \lambda) = \underbrace{\lambda^{q+1} * \lambda^{q+1} * ... * \lambda^{q+1}}_{r \text{ times}} * \underbrace{\lambda^q * ... * \lambda^q}_{m-r-1 \text{ times}}.$$

The Turán's graph T_n^m is an $\mathcal{S}$ - graph. We have

$$g \in F \Leftrightarrow \ker(g) \in Part(N, \mathcal{S}),$$

where

$$\ker(g) = \{g^{-1}\{i\} : i \in g(N)\}.$$

Since g is an n - coloring of the graph T_n^m, we have:

$$|F| = P(T_n^m; n),$$

the value of the chromatic polynomial at $\lambda = n$.

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