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R - MECHANICAL SYSTEMS

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Abstract. In the present paper we study a particular case of Finslerian mechanical system This is defined by a triple $\sum_R = (M, F^2, F_e)$, where M is the configuration space, $F(x,y) = \alpha(x,y) + \beta(x,y)$ is a Randers metric and $F_e = a^i_{jk}(x)\, y^j y^k \frac{\partial}{\partial y^i}$ are the external forces.

1. Introduction and Preliminaries

Let M be an n-dimensional, real C^∞ manifold. Denote by (TM, τ, M) the tangent bundle of M and let $F^n = (M, F(x,y))$ be a Finsler space, where $F : TM \to R_+$ is it fundamental function, i.e., F verifies the following axioms:
i) F is a differentiable function on $\widetilde{TM} = TM - \{0\}$ and it is continuous on the null section of the projection $\tau : TM \to M$;
ii) F is positively 1- homogeneous with respect to the variables y^i;
iii) For every $(x,y) \in \widetilde{TM}$ the Hessian of F^2 with respect y^i is positively defined.

Consequently, the d-tensor field $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positively defined. It is called the fundamental tensor or metric tensor of F^n.

For a non-zero vector $y \in T_pM$, define $h_y(u,v) = g_y(u,v) - F^{-2}(y)\, g_y(y,u)\, g_y(y,v)$, with $u, v \in T_pM.h = \{h_y\}$ is called the angular metric of F.The geodesics of F are characterized locally by $\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0$, where $G^i = \frac{1}{4} g^{ik}\left\{2\frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\right\} y^p y^q$.

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2. Randers spaces

Let $F^n = (M, F(x,y))$ be a Finsler space with the fundamental function $F(x,y) = \alpha(x,y) + \beta(x,y)$ where $\alpha(x,y) = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta(x,y) = b_i(x) y^i$. $a = a_{ij}(x) dx^i dx^j$ is a pseudo-Riemannian metric on M and $b_i(x)$ is a covector field on the manifold M .We denote by "$|$" the covariant differentiation with respect to the Levi-Civita connection $\gamma = \left\{\gamma^i_{jk}\right\}$ of the associated Riemannian space (M, α^2). We shall use the notation as follows [9]:

$$\tilde{l}_i = \frac{\partial \alpha}{\partial y^i} = \frac{a_{ij} y^j}{\alpha}, l_i = \frac{\partial F}{\partial y^i} = \tilde{l}_i + b_i, l^i = \frac{y^i}{F}, \tilde{l}^i = \frac{y^i}{\alpha},$$

$$b_{j|k} = \frac{\partial b_j}{\partial x^k} - b_s \gamma^s_{jk}, r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i_j = a^{ih} s_{hj}, s_j = b_i s^i_j, e_{ij} = r_{ij} + b_i s_j + b_j s_i,$$

$$e_{00} = e_{ij} y^i y^j, s_0 = s_i y^i, s^i_0 = s^i_j y^j.$$

The fundamental tensor g_{ij} of the Randers metric $F = \alpha + \beta$ is expressed by

$$g_{ij} = \frac{F}{\alpha}(a_{ij} - \tilde{l}_i \tilde{l}_j) + l_i l_j. \tag{2.1}$$

The functions $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x,y) y^j y^k$ are the components of the geodesic spray $S = y^i \dfrac{\partial}{\partial x^i} - 2G^i \dfrac{\partial}{\partial y^i}$ and $\Gamma^i_{jk}(x,y)$ are the Christoffel symbols of the metric tensor g_{ij}. By a direct calculation

$$G^i = (\gamma^i_{jk} + l^i b_{j|k}) y^j y^k + (a^{ij} - l^i b^j)(b_{j|k} - b_{k|j}) \alpha y^k, \tag{2.2}$$

or

$$G^i = \tilde{G}^i + \frac{1}{2F}(r_{kl} y^k y^l - 2\alpha b_r a^{rp} s_{pl} y^l) y^i + \alpha a^{ir} s_{rl} y^l, \tag{2.3}$$

or, equivalently,

$$G^i = \tilde{G}^i + \frac{e_{00}}{2F} y^i - s_0 y^i + \alpha s^i_0, \tag{2.4}$$

with $\tilde{G}^i$ the components of the geodesic spray of the Riemannian space. The Cartan nonlinear connection $\overset{C}{N}$ for the Finsler space $F^n = (M, F = \alpha + \beta)$ has the coefficients

$$\overset{C}{N}{}^i_j = \frac{\partial G^i}{\partial y^j}. \tag{2.5}$$

Definition 2.1.The Finsler space $F^n = (M, F = \alpha + \beta)$ equiped with Cartan nonlinear connection $\overset{C}{N}$ is called a *Randers space* and it is denoted by $RF^n = \left(M, \alpha + \beta, \overset{C}{N}\right)$.

The local basis adapted to the Cartan nonlinear connection is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=\overline{1,n}}$ with

$$\frac{\overset{C}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{C}{N}{}_i^k \frac{\partial}{\partial y^k}. \tag{2.6}$$

Theorem 2.1. *There exists an unique $\overset{C}{N}$-metrical connection $C\Gamma\left(\overset{C}{N}\right) = \left(F^i_{jk}, C^i_{jk}\right)$ of the Randers space RF^n which verifies the following axioms:*

i) $\overset{C}{\nabla}{}^H_k\, g_{ij} = 0$; $\overset{C}{\nabla}{}^V_k\, g_{ij} = 0$;

ii) $\overset{C}{T}{}^i_{jk} = 0$; $\overset{C}{S}{}^i_{jk} = 0$.

The connection $C\Gamma\left(\overset{C}{N}\right)$ has the coefficients expressed by the generalized Christoffel symbols:

$$\begin{cases} \overset{C}{F}{}^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\overset{C}{\delta}\, g_{sj}}{\delta x^k} + \frac{\overset{C}{\delta}\, g_{sk}}{\delta x^j} - \frac{\overset{C}{\delta}\, g_{jk}}{\delta x^s} \right) \\ \overset{C}{C}{}^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{cases} \tag{2.7}$$

The integrability tensor of $\overset{C}{N}$ is

$$\overset{C}{R}{}^i_{jk} = \frac{\overset{C}{\delta}\, \overset{C}{N}{}^i_j}{\delta x^k} - \frac{\overset{C}{\delta}\, \overset{C}{N}{}^i_k}{\delta x^j}. \tag{2.8}$$

3. R-Mechanical Systems

For a manifold M, that is the configuration space of Finslerian dynamical system, let us consider the tangent bundle TM to which we refer to as the velocity space. Suppose that there exists a Randers metric $F = \alpha + \beta$ on $\widetilde{TM}$ and $a^i_{jk}(x)$ a symmetric tensor on the configuration space M of type $(1, 2)$.

Definition 3.1. An *R-mechanical system* is a triple $\sum_R = (M, F^2, F_e)$,where $F_e = a^i_{jk}(x)\, y^j y^k \frac{\partial}{\partial y^i}$ are the external forces, and $F^n = (M, F = \alpha + \beta)$ is a Randers space endowed with Cartan nonlinear connection N.

We denote $F^i(x,y) = a^i_{jk}(x)\, y^j y^k$ and one considers $F_i(x,y) = g_{ij} F^j(x,y)$ the covariant components of the external forces F_e.

The energy of the Randers space $RF^n = (M, F = \alpha + \beta)$ is

$$\varepsilon_{F^2} = y^i \frac{\partial (\alpha + \beta)^2}{\partial y^i} - (\alpha + \beta)^2 = (\alpha + \beta)^2 = g_{ij}(x,y)\, y^i y^j. \tag{3.1}$$

If the external forces F_e are global defined on $\widetilde{TM}$ we obtain

Theorem 3.2. *[8] For the Finslerian mechanical system $\sum_R = \left(M, (\alpha+\beta)^2, F_e\right)$ the following properties hold true:*

i) The operator S defined by

$$S = y^i \frac{\partial}{\partial x^i} - (2\overset{R}{G^i} - \frac{1}{2} a^i_{jk}(x)\, y^j y^k) \frac{\partial}{\partial y^i} \tag{3.2}$$

is a vector field, global defined on the phase space $\widetilde{TM}$.
ii) S is a semispray which depends only on $\sum_R$and it is a spray if F_e are 2-homogeneous with respect to y^i.
iii) The integral curves of the vector field S are the evolution curves given by the Lagrange equations of $\sum_R$:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk}(x, \frac{dx}{dt}) \frac{dx^j}{dt} \frac{dx^k}{dt} = \frac{1}{2} F^i(x, \frac{dx}{dt}). \tag{3.3}$$

The semispray S (3.2) has the coefficients $\overset{R}{G^i}$ expressed by

$$2\overset{R}{G^i} = 2G^i - \frac{1}{2} a^i_{jk}(x)\, y^j y^k = \Gamma^i_{jk}(x,y)\, y^j y^k - \frac{1}{2} a^i_{jk}(x)\, y^j y^k. \tag{3.4}$$

Thus, the canonical nonlinear connection $\overset{R}{N}$ of the Finslerian mechanical system $\sum_R$ has the coefficients

$$\overset{R}{N^i_j} = \frac{\partial \overset{R}{G^i}}{\partial y^j} = \overset{C}{N^i_j} - a^i_{jk}(x)\, y^k \tag{3.5}$$

The nonlinear connection $\overset{R}{N}$ determines the horizontal distribution $\overset{R}{N}$ which is supplementary to the natural vertical distribution on the tangent manifold $\widetilde{TM}$. A local adapted basis to these distributions is $\left(\frac{\overset{R}{\delta}}{\delta x^j}, \frac{\partial}{\partial y^j}\right)_{j=\overline{1,n}}$ where

$$\frac{\overset{R}{\delta}}{\delta x^j} = \frac{\partial}{\partial x^i} - \overset{R}{N}{}^i_j \frac{\partial}{\partial y^j} = \frac{\partial}{\partial x^i} - \overset{C}{N}{}^i_j \frac{\partial}{\partial y^j} + a^i_{jk}(x)\, y^k \frac{\partial}{\partial y^j} = \frac{\overset{C}{\delta}}{\delta x^i} + a^i_{jk}(x)\, y^k \frac{\partial}{\partial y^j}. \tag{3.6}$$

The adapted cobasis is $\left(dx^i, \overset{R}{\delta} y^i\right)_{i=\overline{1,n}}$ with

$$\overset{R}{\delta} y^i = dy^i + \overset{R}{N}{}^i_j dx^j = dy^i + \overset{C}{N}{}^i_j dx^j - a^i_{jk}(x)\, y^k dx^j = \overset{C}{\delta} y^i - a^i_{jk}(x)\, y^k dx^j. \tag{3.7}$$

We determine the torsion $\overset{R}{T}{}^i_{jk}$ and the curvature $\overset{R}{R}{}^i_{jk}$ of the canonical connection $\overset{R}{N}$ by a direct calculation:

$$\overset{R}{T}{}^i_{jk} = \frac{\partial \overset{R}{N}{}^i_j}{\partial y^k} - \frac{\partial \overset{R}{N}{}^i_k}{\partial y^j} = 0. \tag{3.8}$$

$$\overset{R}{R}{}^i_{jk} = \frac{\overset{R}{\delta} \overset{R}{N}{}^i_j}{\delta x^k} - \frac{\overset{R}{\delta} \overset{R}{N}{}^i_k}{\delta x^j} = \overset{C}{R}{}^i_{jk} + (N^s_k a^i_{jk} - N^s_j a^i_{sk}). \tag{3.9}$$

Applying the theory from the book [9] the following theorem holds:

Theorem 3.3. *Let $\sum_R = \left(M, (\alpha+\beta)^2, F_e\right)$ be a R- mechanical system and $\overset{R}{N}$ the canonical nonlinear connection of $\sum_R$. There exists an unique d-connection $R\Gamma(\overset{R}{N}) = (\overset{R}{F}{}^i_{jk}, \overset{R}{C}{}^i_{jk})$ determined by the following axioms:*

i) $\overset{R}{\nabla}{}^H_k g_{ij} = 0$; $\overset{R}{\nabla}{}^V_k g_{ij} = 0$;

ii) $\overset{R}{T}{}^i_{jk} = 0$; $\overset{R}{S}{}^i_{jk} = 0$, *where*

$$\overset{R}{\nabla}{}^{H}_{k}\, g_{ij} = \frac{\overset{R}{\delta}\, g_{ij}}{\delta x^k} - \overset{R}{F}{}^{s}_{ik}\, g_{sj} - \overset{R}{F}{}^{s}_{jk}\, g_{is} \tag{3.10}$$
$$\overset{R}{\nabla}{}^{V}_{k}\, g_{ij} = \frac{\partial g_{ij}}{\partial y^k} - \overset{R}{C}{}^{s}_{ik}\, g_{sj} - \overset{R}{C}{}^{s}_{jk}\, g_{is}.$$

We call this connection the canonical metrical d-connection of $\sum_R$.

Theorem 3.4. *The local coefficients of the canonical metrical d-connection of* $\sum_R$ *are*

$$\begin{cases} \overset{R}{F}{}^{i}_{jk} = \frac{1}{2} g^{is} \left(\dfrac{\overset{R}{\delta}\, g_{sj}}{\delta x^k} + \dfrac{\overset{R}{\delta}\, g_{sk}}{\delta x^j} - \dfrac{\overset{R}{\delta}\, g_{jk}}{\delta x^s} \right) \\ \overset{R}{C}{}^{i}_{jk} = \frac{1}{2} g^{is} \left(\dfrac{\partial g_{sj}}{\partial y^k} + \dfrac{\partial g_{sk}}{\partial y^j} - \dfrac{\partial g_{jk}}{\partial y^s} \right) \end{cases} \tag{3.11}$$

In order to calculate $\overset{R}{F}{}^{i}_{jk}$ and $\overset{R}{C}{}^{i}_{jk}$ we have from (3.6):

$$\frac{\overset{R}{\delta}\, g_{sj}}{\delta x^k} = \frac{\overset{C}{\delta}\, g_{sj}}{\delta x^k} + a^{i}_{jk}\,(x)\, y^k \frac{\partial g_{sj}}{\partial y^h}. \tag{3.12}$$

Now, the developed expression of the coefficients $\overset{R}{F}{}^{i}_{jk}$ and $\overset{R}{C}{}^{i}_{jk}$ is given in the next theorem:

Theorem 3.5. *The canonical metrical d-connection of* $\sum_R$ *has the coefficients*

$$\begin{cases} \overset{R}{F}{}^{i}_{jk} = \overset{C}{F}{}^{i}_{jk} + \frac{1}{2} g^{is} \left(a^{h}_{kp} y^p \dfrac{\partial g_{sj}}{\partial y^h} + a^{h}_{jp} y^p \frac{\partial g_{sk}}{\partial y^h} - a^{h}_{sp} y^p \frac{\partial g_{jk}}{\partial y^h} \right) \\ \overset{R}{C}{}^{i}_{jk} = \overset{C}{C}{}^{i}_{jk}\,. \end{cases}$$

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