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POINCARÉ INEQUALITIES BASED ON BANACH FUNCTION SPACES ON METRIC MEASURE SPACES

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Abstract. We introduce a new type of first order Poincaré inequality for functions defined on a metric measure space, that is an useful tool in the study of Newtonian spaces based on Banach function spaces. This Poincaré inequality extends the Orlicz-Poincaré inequality introduced by Aïssaoui (2004) and the Poincaré inequality based on Lorentz spaces, introduced by Costea and Miranda (2011), that in turn generalize the well-known weak $(1,p)$ −Poincaré inequality. Using very recent results of Durand-Cartagena, Jaramillo and Shanmugalingam (2012, 2013), it turns out that every complete metric space X, endowed with a doubling measure and supporting a weak Poincaré inequality based on a Banach function space is (thick) quasiconvex. We prove that the validity of the Poincaré inequality based on a Banach function space, on a doubling metric measure space, implies a pointwise estimate involving an appropriate maximal operator.

1. Introduction and preliminaries

Poincaré inequalities from the theory of Sobolev spaces are very useful in several fields, such as harmonic analysis, the calculus of variations, the theory of partial differential equations and nonlinear potential theory.

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From a historical viewpoint, the applicability of Poincaré inequalities is not surprising, since Poincaré arrived at this type of inequalities while studying a variational characterization of the eigenvalues of Fourier's problem for heat equation [16]. The first order calculus on metric measure spaces is based on the notion of upper gradient. An *upper gradient* of a function $u : X \to \mathbb{R}$ is a Borel function $g : X \to [0, \infty]$ satisfying

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_{\gamma} g ds \tag{1.1}$$

for every rectifiable curve $\gamma : [a, b] \to \mathbb{R}$.

The validity of a weak $(1, p)-$Poincaré inequality on a metric measure space provides a control on the mean oscillation of a locally integrable function on a arbitrary ball, in terms of the mean value of the p-th power of a function's upper gradient, on a related ball. The use of $(1, p)-$Poincaré inequalities on metric measure spaces can be traced back to [11].

Let (X, d, μ) be a metric measure space, that is, a metric space (X, d) endowed with a Borel regular measure μ which is finite and positive on balls. The measure μ is said to be *doubling* if there exists a constant $C_d \geq 1$ such that such that for every ball $B(x, r) \subset X$ the following inequality holds

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)). \tag{1.2}$$

In the following we assume that $u : X \to \mathbb{R}$ is a locally integrable function. We denote the mean value of u on A by $u_A = \frac{1}{\mu(A)} \int_A u \, d\mu$, whenever $A \subset X$ is a measurable set with $0 < \mu(A) < \infty$.

For a Borel measurable function $g : X \to [0, \infty]$ and $1 \leq p < \infty$, the pair (u, g) is said to satisfy a weak $(1, p)-$Poincaré inequality if there exist constants $C > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq Cr \left(\frac{1}{\mu(\tau B)} \int_{\tau B} g^p \, d\mu \right)^{1/p} \tag{1.3}$$

for every ball $B = B(x, r) \subset X$.

The space (X, d, μ) is said to support a weak $(1, p)-$Poincaré inequality if inequality (1.3) holds for every $u \in L^1_{loc}(X)$ and every upper gradient g of u, with fixed constants C and τ.

In this paper we give an unifying framework for two generalizations of the weak $(1,p)-$Poincaré inequality, the Orlicz-Poincaré inequality introduced by Aïssaoui [1] and the Poincaré inequality based on Lorentz spaces, introduced by Costea and Miranda [5]. In our case the role of $L^p(X)$ is played by an abstract Banach function space, defined as in [3].

Let (X,μ) be a complete and $\sigma-$finite measure space and let $M^+(X)$ be the collection of all $\mu-$measurable functions $f : X \to [0,\infty]$.

Definition 1.1. [3] A function $N : M^+(X) \to [0,\infty]$ is called a Banach function norm if, for all f, g, f_n $(n \geq 1)$ in $M^+(X)$, for all constants $a \geq 0$ and for all measurable sets $E \subset X$, the following properties hold:

(P1) i) $N(f) = 0$ if and only if $f = 0$ $\mu-$a.e.; ii) $N(af) = aN(f)$; iii) $N(f+g) \leq N(f) + N(g)$.

(P2) If $0 \leq g \leq f$ $\mu-$a.e., then $N(g) \leq N(f)$.

(P3) If $0 \leq f_n \uparrow f$ $\mu-$a.e., then $N(f_n) \uparrow N(f)$.

(P4) If $\mu(A) < \infty$, then $N(\chi_A) < \infty$.

(P5) If $\mu(A) < \infty$, then $\int\limits_A f \ d\mu \leq C_A N(f)$, for some constant $C_A \in (0,\infty)$ depending only on A and N.

Let $\mathbf{E}$ be the collection of the $\mu-$measurable functions $f : X \to [-\infty,\infty]$ for which $N(|f|) < \infty$.For $f \in \mathbf{E}$ define

$$\|f\|_{\mathbf{E}} = N(|f|).$$

Then $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is a seminormed space, that induces a normed space via the equivalence of functions that coincide $\mu-$a.e. The corresponding normed space, that will be still denoted by $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$, is complete [3, Theorem I.1.6].

A real-extended valued function on a metric measure space is said to be locally integrable if it is integrable on each ball. Note that property (P5) implies $\mathbf{E} \subset L^1_{loc}(X)$.

The $\mathbf{E}-$modulus of a family Γ of curves in X is defined by $Mod_{\mathbf{E}}(\Gamma) = \inf \|\rho\|_{\mathbf{E}}$, where the infimum is taken over all Borel functions $\rho : X \to [0,\infty]$ with $\int\limits_\gamma \rho \ ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$ [17]. Note that, in the case $\mathbf{E} = L^p(X)$ we have $Mod_{\mathbf{E}}(\Gamma) = (Mod_p(\Gamma))^{1/p}$ for $1 \leq p < \infty$ and $Mod_{\mathbf{E}}(\Gamma) = Mod_\infty(\Gamma)$ for $p = \infty$. Here $Mod_p(\Gamma)$ is the $p-$modulus of Γ [13], [7, Theorem 4.7]. A non-negative Borel function g on X is said to be a $\mathbf{E}-$weak upper gradient of $u : X \to \mathbb{R}$ if inequality (1.1) holds for all rectifiable curves γ except for a family Γ with $Mod_{\mathbf{E}}(\Gamma) = 0$.

Definition 1.2. [8, Definition 2.4] Fix $1 \leq p \leq \infty$. We say that X is $p-$thick quasiconvex if there is a constant $C \geq 1$ such that for every $x, y \in X$ with $x \neq y$ and each $0 < \varepsilon < d(x, y)/4$, and whenever $E \subset B(x, \varepsilon)$ and $F \subset B(y, \varepsilon)$ are measurable sets with $\mu(E)\mu(F) > 0$, we have

$$Mod_p(\Gamma(E, F, C)) > 0,$$

where $\Gamma(E, F, C)$ denotes the collection of all curves γ in X connecting E to F, with the length satisfying $l(\gamma) \leq Cd(x, y)$.

One shows that every $p-$thick quasiconvex is also quasiconvex, that is, there is a constant $C \geq 1$ such that whenever $x, y \in X$ there exists a curve γ connecting x to y with length $l(\gamma) \leq Cd(x, y)$.

The Newtonian space $N^{1,\mathbf{E}}(X)$ based of a Banach function space $\mathbf{E}$ was introduced in [17]. Let $\widetilde{N}^{1,\mathbf{E}}(X)$ be the class of all functions $u \in \mathbf{E}$ that have a $\mathbf{E}-$weak upper gradient in $\mathbf{E}$. For $u \in \widetilde{N}^{1,\mathbf{E}}(X)$ we define $\|u\|_{\widetilde{N}^{1,\mathbf{E}}(X)} = \|u\|_{\mathbf{E}} + \inf \|g\|_{\mathbf{E}}$, where the infimum is taken over all $\mathbf{E}-$weak upper gradients $g \in \mathbf{E}$ of u. The quotient space $N^{1,\mathbf{E}}(X) = \widetilde{N}^{1,\mathbf{E}}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{\widetilde{N}^{1,\mathbf{E}}(X)} = 0$, is a vector space, equipped with the norm $\|u\|_{N^{1,\mathbf{E}}(X)} := \|u\|_{\widetilde{N}^{1,\mathbf{E}}(X)}$. Note that for $\mathbf{E} = L^p(X)$ the space $N^{1,\mathbf{E}}(X)$ is the Newtonian space $N^{1,p}(X)$ introduced in [19].

The $(1, p)-$Poincaré inequality has several important implications and is instrumental in the extensions of nonlinear potential theory and quasiconformal theory to metric measure spaces. This inequality plays an important role in the study of the Newtonian space $N^{1,p}(X)$. Assume that (X, d, μ) is doubling and supports a $(1, p)-$Poincaré inequality, $1 \leq p < \infty$. Then Lipschitz functions are dense in $N^{1,p}(X)$ [19] and X admits a measurable differentiable structure with which Lipschitz functions can be differentiated a.e. [4]. Moreover, if $1 < p < \infty$, then $N^{1,p}(X)$ is reflexive [4], the quasiminimizers of $p-$Dirichlet integral satisfy Harnack's inequality, the strong maximum principle and are locally Hölder continuous [15] and $N^{1,p}(X) = M^{1,p}(X)$ isomorphically as Banach spaces [19]. Here $M^{1,p}(X)$ is the Hajłasz-Sobolev space introduced in [10].

The aim of this paper is introduce the notion of weak $(1, \mathbf{E})-$Poincaré inequality on a metric measure space (X, d, μ), where $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is a Banach function space over (X, μ). This Poincaré inequality extends the Orlicz-Poincaré inequality introduced by Aïssaoui [1] and the Poincaré inequality based on Lorentz spaces, introduced by Costea and Miranda [5], that in turn generalize the well-known weak $(1,p)$ −Poincaré inequality [12], [13]. We check that the weak $(1,\mathbf{E})$ −Poincaré inequality is a special case of first order Poincaré inequality for $\mathcal{F} = N^{1,\infty}(X)$, in the sense from [8].

It is easy to see that every metric measure space supporting a weak $(1,\mathbf{E})$−Poincaré inequality, for some Banach function space $\mathbf{E}$, also supports a weak ∞−Poincaré inequality. This remark enable us to use a very recent result of Durand-Cartagena, Jaramillo and Shanmugalingam [7, Theorem 4.7], showing that the validity of a weak ∞−Poincaré inequality to a complete metric space X, endowed with a doubling measure is equivalent to the geometric property of ∞−thick quasiconvexity of X. It follows that a complete doubling metric measure space supporting a weak Poincaré inequality based on a Banach function space is ∞−thick quasiconvex, in particular it is quasiconvex.

We also prove that the validity of the Poincaré inequality based on a Banach function space, on a doubling metric measure space, implies a pointwise estimate involving an appropriate maximal operator.

2. Old and new Poincaré inequalities

For $g \in L^p_{loc}(X)$ the weak $(1,p)$−Poincaré inequality (1.3) can be written as

$$\frac{1}{\mu(B)}\int\limits_B |u-u_B|\,d\mu \le Cr\frac{\|g\chi_{\tau B}\|_{L^p(X)}}{\|\chi_{\tau B}\|_{L^p(X)}}.$$

With an abuse of notation, we still write the above inequality if $\int\limits_{\tau B} g^p\,d\mu = \infty$, by taking $\|g\chi_{\tau B}\|_{L^p(X)} = \infty$.

By Hölder's inequality, if an open set supports a weak $(1,p)$−Poincaré inequality, then it supports a weak $(1,q)$−Poincaré inequality for each $p \le q < \infty$. So, the strongest $(1,p)$−Poincaré inequality with $1 \le p < \infty$ is that with $p = 1$. Looking for the weakest version of $(1,p)$−Poincaré inequality that still gives enough information on the geometry of the metric space, Durand-Cartagena, Jaramillo and Shanmugalingam [7] introduced the following ∞−Poincaré inequality.

Definition 2.1. A metric measure space (X,d,μ) is said to support a weak ∞−Poincaré inequality if there exist constants $C > 0$ and $\tau \ge 1$ such that, for every Borel measurable function $u : X \to \mathbb{R}\cup\{\infty\}$ and every upper gradient $g : X \to [0,\infty]$ of u, the pair (u,g) satisfies the inequality

$$\frac{1}{\mu(B)}\int\limits_B |u-u_B|\,d\mu \le Cr\,\|g\chi_{\tau B}\|_{L^\infty(X)} \tag{2.1}$$

for every ball $B=B(x,r)$.

If (X,d,μ) supports weak $(1,p)-$Poincaré inequality for some $1\le p<\infty$, then it supports the weak $\infty-$Poincaré inequality.

The following two versions of Orlicz-Poincaré inequality on metric measure spaces have been introduced by Tuominen in [20], respectively by Aïssaoui in [1]. Let $\Psi:[0,\infty)\to[0,\infty)$ be a strictly increasing Young function. The space (X,d,μ) is said to support a weak $(1,\Psi)-$Poincaré inequality if there exist some constants $C>0$ and $\tau\ge 1$ such that for every function $u\in L^1_{loc}(X)$ and every upper gradient g of u, the pair (u,g) satisfies the inequality

$$\frac{1}{\mu(B)}\int\limits_B |u-u_B|\,d\mu \le C\,r\Psi^{-1}\left(\frac{1}{\mu(\tau B)}\int\limits_{\tau B}\Psi(g)d\mu\right), \tag{2.2}$$

respectively

$$\frac{1}{\mu(B)}\int\limits_B |u-u_B|\,d\mu \le C\,r\,\|g\chi_{\tau B}\|_{L^\Psi(X)}\,\Psi^{-1}\left(\frac{1}{\mu\,(\tau B)}\right), \tag{2.3}$$

for every ball $B=B\,(x,r)\subset X$.

Note that $\Psi^{-1}\left(\frac{1}{\mu(\tau B)}\right)=\frac{1}{\|\chi_{\tau B}\|_{L^\Psi(X)}}$. For $\Psi\,(t)=t^p$ with $1\le p<\infty$ inequalities (2.2) and (2.3) become inequality (1.3).

On the other hand, Costea and Miranda [5] defined the following weak Poincaré inequality based on a Lorentz space $L^{p,q}\,(X)$, where $1<p<\infty$ and $1\le q\le\infty$. The space (X,d,μ) is said to support a weak $(1,L^{p,q})-$Poincaré inequality if there exist some constants $C>0$ and $\tau\ge 1$ such that for all balls $B=B\,(x,r)\subset X$, for all $\mu-$measurable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)}\int\limits_B |u-u_B|\,d\mu \le C\,r\frac{1}{\mu(\tau B)^{1/p}}\,\|g\chi_{\tau B}\|_{L^{p,q}(X,\mu)}\,. \tag{2.4}$$

Note that, assuming $1\le q\le p$, the quasinorm $\|\cdot\|_{L^{p,q}(X,\mu)}$ is a norm and $\frac{1}{\mu(\tau B)^{1/p}}=c\,(p,q)\,\frac{1}{\|\chi_{\tau B}\|_{L^{p,q}(X)}}$. For $p=q$ the inequality (2.4) becomes inequality (1.3).

Remark 2.2. *Let Ψ be a strictly increasing Young function. Assume that there exist some positive constants C_1 and C_2 such that $\Psi(C_1 st) \geq C_2 \Psi(s) \Psi(t)$ for all $s, t \in [0, \infty)$. Then every space supporting a $(1, \Psi)-$Poincaré inequality in the sense of Aïssaoui also supports a $(1, \Psi)-$Poincaré inequality in the sense of Tuominen. The converse holds true if we assume that Ψ satisfies, besides the above condition, the so-called $\Delta'-$ condition, i.e. there is some positive constant C such that $\Psi(st) \leq C\Psi(s)\Psi(t)$ for all $s, t \in [0, \infty)$.*

We introduce an extension of Poincaré-type inequalities (2.3) and (2.4), based on a Banach function space $\mathbf{E}$ over (X, μ).

Definition 2.3. The space (X, d, μ) is said to support a weak $(1, \mathbf{E})-$Poincaré inequality if there exist some constants $C > 0$ and $\tau \geq 1$ such that for all balls $B = B(x, r) \subset X$, for all locally integrable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq Cr \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}. \tag{2.5}$$

Here $\|g\chi_{\tau B}\|_{\mathbf{E}}$ stands for $N(g\chi_{\tau B})$, even in the case $N(g\chi_{\tau B}) = \infty$.

If (X, d, μ) supports a weak $(1, \mathbf{E})-$Poincaré inequality, then (2.5) holds whenever g is a $\mathbf{E}-$weak upper gradient of u, since for every $\mathbf{E}-$weak upper gradient g of a function u on X there is a sequence $(g_i)_{i \geq 1}$ of upper gradients of u, such that $\lim_{i \to \infty} \|g_i - g\|_{\mathbf{E}} = 0$ [17, Proposition 2].

As in the case of $(1, p)-$Poincaré inequality, we can easily obtain some topological implications of the validity of a weak $(1, \mathbf{E})-$Poincaré inequality.

Lemma 2.4. *Assume that (X, d, μ) supports a weak $(1, \mathbf{E})-$Poincaré inequality for some Banach function space $\mathbf{E}$. Then X is connected. Moreover, every sphere $S(x, r) := \{y \in X : d(x, y) = r\}$ with $r > 0$ is non-empty whenever $B(x, r) \neq X$.*

Proof. If X is not connected, there exist two disjoint non-empty open sets U and V such that $X = U \cup V$. Then U is pathwise disconnected from V, hence χ_U has the zero function as an upper gradient. Applying the weak $(1, \mathbf{E})-$Poincaré inequality for $u = \chi_U$ we see that u is constant $\mu-$a.e. on every ball $B \subset X$, which means that either $\mu(B \cap U) = 0$ or $\mu(B \cap V) = 0$ for all balls $B \subset X$. Let $x_0 \in U$ and let $\rho > 0$ such that $B(x_0, \rho) \subset X$. Writing

$X = \bigcup_{n=1}^{\infty} B(x_0, n)$ we see that there is an integer $n_0 \geq 1$ such that $B(x_0, n) \cap V$ is non-empty for all $n \geq n_0$. For all $n \geq \lfloor \rho \rfloor + 1$ we have $\mu(B(x_0, n) \cap U) \geq \mu(B(x_0, \rho)) > 0$, therefore $\mu(B(x_0, n) \cap V) = 0$. Consequently, $\mu(B(x_0, n) \cap V) = 0$ for all $n \geq 1$, whence we get $\mu(V) = 0$, which is false, since V is an non-empty open set and balls have positive measure.

Assume now that there is an empty sphere $S(x_1, r)$ with $r > 0$, such that $B(x_1, r) \neq X$. Then $B = B(x_1, r)$ is pathwise disconnected from $X \setminus B$, hence χ_B has the zero f unction as an upper gradient. Reasoning as above, we deduce that $\mu(X \setminus B) = 0$. This conclusion is false, since for every $y \in X \setminus B$ we have $B(y, d(y, x_1) - r) \subset X \setminus B$. $\square$

Let $B = B(x, r) \subset X$ be a fixed ball. Let $u \in L^1_{loc}(X)$ and let g be an upper gradient of u. If $g\chi_{\tau B}$ is not essentially bounded, we take $\|g\chi_{\tau B}\|_{L^\infty(X)} = \infty$ and (2.1) holds. Assuming that $\|g\chi_{\tau B}\|_{L^\infty(X)} < \infty$, it follows that

$$N(g\chi_{\tau B}) \leq \|g\chi_{\tau B}\|_{L^\infty(X)} N(\chi_{\tau B}) < \infty,$$

by (P2), (P1) and (P4) from Definition 1.1. Then $g\chi_{\tau B} \in \mathbf{E}$ and $\frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}} \leq \|g\chi_{\tau B}\|_{L^\infty(X)}$, therefore inequality (2.5) implies inequality (2.1). We obtain the following

Lemma 2.5. *If a metric measure space supports a weak* $(1, \mathbf{E})-$*Poincaré inequality, for some Banach function space* $\mathbf{E}$*, then it supports a weak* $\infty-$*Poincaré inequality.*

The validity of a weak $\infty-$Poincaré inequality has some remarkable geometric implications. Durand-Cartagena, Jaramillo and Shanmugalingam showed in [7, Proposition 3.4] that every complete doubling metric space supporting a weak $\infty-$Poincaré inequality is quasiconvex. Moreover, they proved in [7, Theorem 4.7] that a complete doubling metric space supports a weak $\infty-$Poincaré inequality if and only if it is $\infty-$thick quasiconvex. By [7, Theorem 4.7] and Lemma 2.5, we get

Corollary 2.6. *If a complete doubling metric measure space supports a weak* $(1, \mathbf{E})-$*Poincaré inequality, for some Banach function space* $\mathbf{E}$*, then it is* $\infty-$*thick quasiconvex.*

Very recently, Durand-Cartagena, Jaramillo and Shanmugalingam [8] studied first order Poincaré inequalities in metric measure spaces, following an approach that was considered for the first time in [9]. Let $\mathcal{F}$ be a family of locally integrable functions on X.

Definition 2.7. [8, Definition 3.1] Let $\mathcal{B}$ be the collection of all balls in X. It is said that (X,d,μ) supports a first order Poincaré inequality for $\mathcal{F}$ if for each function $f\in\mathcal{F}$ there exists $a_f:\mathcal{B}\to[0,\infty]$ such that

$$\frac{1}{\mu(B)}\int\limits_B |f-f_B|\,d\mu\le a_f(B) \tag{2.6}$$

for each ball $B\in\mathcal{B}$.

In [8, Definition 3.2], some geometric properties of the functional $f\mapsto a_f(B)$ are stated. If this functional satisfies the conditions from the above definition for $\mathcal{F}=N^{1,\infty}(X)$, it is said that $f\mapsto a_f(B)$ has a modulus of continuity if there exists a constant $C>0$ such that whenever $f\in\mathcal{F}$ and g_f is an upper gradient of f such that $\|g_f\|_{L^\infty(X)}\le 1$, then $a_f(B)\le C rad(B)$ for all $B\in\mathcal{B}$.

We shall see that if (X,d,μ) supports a weak $(1,\mathbf{E})-$Poincaré inequality for some Banach function space $\mathbf{E}$, then it supports a first order Poincaré inequality for $\mathcal{F}=N^{1,\infty}(X)$, where the functional $f\mapsto a_f(B)$ has a modulus of continuity. Assume that the conditions from Definition 2.3 hold. Given $f\in N^{1,\infty}(X)$, consider a_f defined by

$$a_f(B)=C rad(B)\inf_g\frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}},$$

for each ball $B\in\mathcal{B}$, where the infimum is taken over all upper gradients g of f. Here C and τ are the constants from Definition 2.3. Let $B\in\mathcal{B}$. Clearly, $a_f(B)\in[0,\infty]$ is well-defined. If $g_0\in L^\infty(X)$ is an upper gradient of f, then $0\le\frac{\|g_0\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}\le\|g_0\chi_{\tau B}\|_{L^\infty(X)}<\infty$, therefore $a_f(B)<\infty$. Moreover, if g_f is an upper gradient of f such that $\|g_f\|_{L^\infty(X)}\le 1$, then $\frac{\|g_f\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}\le\|g_f\chi_{\tau B}\|_{L^\infty(X)}\le 1$, hence $a_f(B)\le C rad(B)$.

Note that Corollary 2.6 can be regarded as a consequence of the generalization of [7, Proposition 3.4], namely Theorem 3.6 from [8], that guarantees the quasiconvexity of every complete doubling metric measure space (X,d,μ) supporting a first order Poincaré inequality for $\mathcal{F}=N^{1,\infty}(X)$ or for $\mathcal{F}=Lip^\infty(X)$, such that the functional $f\mapsto a_f(B)$ has a modulus of continuity.

3. A pointwise estimate implied by a Poincaré inequality based on a Banach function space

The purpose of this section is to find a bound for $\frac{|u(x)-u(y)|}{d(x,y)}$, where x and y are distinct points of X, in terms of a maximal function of g, provided that the pair (u, g) satisfies a weak Poincaré inequality on a doubling metric measure space.

The classical Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f| \, d\mu.$$

Some variants of the Hardy-Littlewood maximal operator are the restricted maximal operator $\mathcal{M}_R f(x) = \sup\limits_{0<r<R} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f| \, d\mu$, where $R > 0$ and the noncentered maximal operator $\mathcal{M}^* f(x) = \sup\limits_{B} \frac{1}{\mu(B)} \int\limits_{B} |f| \, d\mu$, where the supremum is taken over all balls $B \subset X$ containing the point x. Here f is any $\mu-$measurable function.

Let $\mathbf{E}$ be a Banach function space over (X, μ). We will consider an analogue of the maximal operator from [2], which was defined in the case $X = \mathbb{R}^n$. Assume that f is a $\mu-$measurable function. If $f\chi_B \notin \mathbf{E}$ for some ball B, then $N(f\chi_B) = \infty$ and we write $\|f\chi_B\|_{\mathbf{E}} = \infty$.

Definition 3.1. The noncentered maximal operator associated with the Banach function space $\mathbf{E}$ is defined by

$$\mathcal{M}_{\mathbf{E}} f(x) = \sup_{B} \frac{\|f\chi_B\|_{\mathbf{E}}}{\|\chi_B\|_{\mathbf{E}}},$$

where the supremum is taken over all balls $B \subset X$ containing the point x. Here f is any $\mu-$measurable function.

Note that in the case when $\mathbf{E} = L^p(X)$, $1 \le p < \infty$, we have $\mathcal{M}_{\mathbf{E}} f = (\mathcal{M}^*(|f|^p))^{1/p}$.

Hajłasz and Koskela proved in [12, Theorem 3.2] that a pointwise estimate is implied by the validity of the weak $(1, p)-$Poincaré inequality with $p > 0$ on a doubling metric measure space. If a pair (u, g) satisfies (1.3), then

$$|u(x) - u(y)| \le C' d(x, y) \left((\mathcal{M}_R g^p(x))^{1/p} + (\mathcal{M}_R g^p(y))^{1/p} \right)$$

for almost every $x, y \in X$, where $R = 2\tau d(x, y)$. Here C' is some constant depending only on the constant C associated with the weak $(1, p)-$Poincaré inequality (1.3) and on the constant C_d from the doubling condition (1.2) on μ. This result was used as a tool in proving the quasiconvexity of a complete doubling metric measure space supporting a weak $(1, p)-$Poincaré inequality with $1 \le p < \infty$ [12, Proposition 4.4].

Tuominen has shown that [12, Theorem 3.2] admits a generalization to the case when the $(1,p)$ −Poincaré inequality is replaced by a $(1,\Psi)$ −Poincaré inequality [20, Lemma 5.15]. If a pair (u,g) satisfies (2.2), then

$$|u(x)-u(y)| \leq C' d(x,y) \left(\Psi^{-1}\left(\mathcal{M}_R\left(\Psi\circ g\right)(x)\right) + \left(\mathcal{M}_R\left(\Psi\circ g\right)(y)\right)^{1/p}\right)$$

for almost every $x,y\in X$, where $R=2\tau d(x,y)$.

We extend Theorem 3.2 from [12] to show that the validity of a $(1,\mathbf{E})$ −Poincaré inequality implies a pointwise estimate.

Proposition 3.2. *Let (X,d,μ) be a doubling metric measure space and let $\mathbf{E}$ be a Banach function space over (X,μ). Assume that the pair (u,g) satisfies a weak $(1,\mathbf{E})$ −Poincaré inequality with constants C and τ. Then*

$$|u(x)-u(y)| \leq C' d(x,y)\left(\mathcal{M}_{\mathbf{E}} g(x) + \mathcal{M}_{\mathbf{E}} g(y)\right) \tag{3.1}$$

for almost every $x,y\in X$. Here C' is some constant depending only on C and C_d.

Proof. By Lebesgue's differentiation theorem [13] almost every point in X is a Lebesgue point for the locally integrable function u. Let $F\subset X$ be a set with $\mu(F)=0$ such that all points in $X\setminus F$ are Lebesgue points for u. We will show that (3.1) holds for all $x,y\in X\setminus F$, with some constant $C'>0$ depending only on C and C_d. We use a ball chaining argument. Consider $B_i(x) := B(x, 2^{-i}d(x,y))$ and $B_i(y) := B(y,2^{-i}d(x,y))$ for each $i\in\mathbb{N}$. Since $u(x)=\lim\limits_{i\to\infty} u_{B_i(x)}$, we have $u(x)-u_{B_0(x)} = \sum\limits_{i=0}^{\infty}\left(u_{B_{i+1}}(x) - u_{B_i(x)}\right)$. Then $\left|u(x)-u_{B_0(x)}\right| \leq \sum\limits_{i=0}^{\infty}\left|u_{B_{i+1}}(x)-u_{B_i(x)}\right|$.

By the inequality $\left|u_{B(x,s)}-k\right| \leq \frac{1}{\mu(B(x,s))}\int\limits_{B(x,s)}|u-k|\,d\mu$ and the doubling property of μ, it follows that

$$\left|u_{B(x,s)}-u_{B(x,r)}\right| \leq C_d\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)}\left|u-u_{B(x,r)}\right|d\mu$$

whenever $0<\frac{r}{2}\leq s\leq r$.

Let $i\in\mathbb{N}$. Since $\left|u_{B_{i+1}}(x)-u_{B_i(x)}\right| \leq C_d\frac{1}{\mu(B_i(x))}\int\limits_{B_i(x)}\left|u-u_{B_i(x)}\right|d\mu$, using the weak $(1,\mathbf{E})$ −Poincaré

Poincaré inequality, it follows that

$$\left|u_{B_{i+1}}(x)-u_{B_i(x)}\right|\le 2^{-i}C_dCd(x,y)\frac{\|g\chi_{B_i}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}}.$$

By the definition of the maximal operator $\mathcal{M}_{\mathbf{E}}$, the above inequality implies

$$\left|u_{B_{i+1}}(x)-u_{B_i(x)}\right|\le 2^{-i}C_dCd(x,y)\,\mathcal{M}_{\mathbf{E}}g(x).$$

We obtain

$$\left|u(x)-u_{B_0(x)}\right|\le 2C_dCd(x,y)\,\mathcal{M}_{\mathbf{E}}g(x). \tag{3.2}$$

Similarly, $\left|u(y)-u_{B_0(y)}\right|\le 2C_dCd(x,y)\,\mathcal{M}_{\mathbf{E}}g(y)$.

On the other hand, $\left|u_{B_0(x)}-u_{B_0(y)}\right|\le\left|u_{B_0(x)}-u_{2B_0(x)}\right|+\left|u_{2B_0(x)}-u_{B_0(y)}\right|$. As above,

$$\left|u_{B_0(x)}-u_{2B_0(x)}\right|\le 2C_dCd(x,y)\frac{\|g\chi_{2\tau B_0}\|_{\mathbf{E}}}{\|\chi_{2\tau B_0}\|_{\mathbf{E}}}\le 2C_dCd(x,y)\,\mathcal{M}_{\mathbf{E}}g(x). \tag{3.3}$$

Note that $B_0(y)\subset 2B_0(x)$, hence

$$\begin{aligned}\left|u_{2B_0(x)}-u_{B_0(y)}\right| &\le \frac{1}{\mu(B_0(y))}\int_{B_0(y)}\left|u-u_{2B_0(x)}\right|d\mu\\ &\le \frac{1}{\mu(B_0(y))}\int_{2B_0(x)}\left|u-u_{2B_0(x)}\right|d\mu.\end{aligned}$$

We also have $B_0(x)\subset 2B_0(y)$, therefore $\mu(2B_0(x))\le (C_d)^2\mu(B_0(y))$. It follows that

$$\left|u_{2B_0(x)}-u_{B_0(y)}\right|\le (C_d)^2\,Cd(x,y)\,\mathcal{M}_{\mathbf{E}}g(x). \tag{3.4}$$

Using (3.2) and its analogue, as well as (3.3) and (3.4), we get (3.1) with $C'=C_dC(4+C_d)$, q.e.d. □

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