"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 1, 77 - 84

SOME APPROXIMATION PROPERTIES OF q-DURRMEYER-SCHURER OPERATORS

CARMEN VIOLETA MURARU AND ANA MARIA ACU

Abstract. In recent years, there are many preoccupations in construction and study of generalized version in q-calculus of well-known linear and positive operators. In [5] is introduced a q-type of Schurer Bernstein operators. We will propose a Durrmeyer variant of q-Schurer operators of the form studied in [5]. Also, a Bohman-Korovkin type approximation theorem of these operators is considered. The rate of convergence by using the first modulus of smoothness is computed.

1. Introduction

It is well known the classical Bernstein operators $B_m : C[0,1] \to C[0,1]$ are defined for any $f \in C[0,1]$ by

$$B_m(f;x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$.

In [2], J.L. Durrmeyer introduced the integral modification of the Bernstein operators:

$$D_m(f;x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t)dt.$$

Keywords and phrases: q-Durrmeyer-Schurer operators, rate of convergence, Korovkin theorem, modulus of continuity.
(2010) Mathematics Subject Classification: 41A10.

Let $p \in N$ be fixed. In [7], Schurer introduced and studied the operators

$$\tilde{B}_{m,p} : C[0, 1 + p] \to C[0, 1], \text{ defined by}$$

$$\tilde{B}_{m,p}(f; x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \text{ where} \tag{1.1}$$

$$\tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}.$$

In [1], D. Barbosu modified the operators (1.1) in Durrmeyer sense as follows

$$\tilde{D}_{m,p}(f, x) = (m+p+1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_0^1 \tilde{p}_{m,k}(t) f(t) dt, \text{ where } f \in L_1[0, 1].$$

In recent years, there are many preoccupations in construction and study of generalized version in q-calculus of well-known linear and positive operators.

A. Lupaş ([4]) introduced in 1987 a q-type of the Bernstein operators and in 1996 another generalization of these operators based on q-integers was introduced by Philips ([6]).

We mention below certain basic definitions, which would be used in the main results of this paper. Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and q-factorial $[k]!$ are respectively defined by

$$[k] := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases}$$

$$[k]! := \begin{cases} [k][k-1] \cdots [1], & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For the integers n, k satisfying $n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

We denote

$$(a+b)_q^k = \prod_{j=0}^{k-1} (a + bq^j).$$

For $m, n > 0$ the q-Beta function is defined as $\beta_q(m, n) := \int_0^1 t^{m-1}(1-qt)_q^{n-1} d_q t$.

The q-analogue of integration (see [3]) in the interval $[0, a]$ is defined by

$$\int_0^a f(t)d_qt := a(1-q)\sum_{n=1}^{\infty} f(aq^n)q^n,\ 0<q<1.$$

Let $p \in N$ be fixed. For any $m \in N$, $f \in C[0, p+1]$, C. Muraru ([5]) constructs the class of generalized q-Schurer Bernstein operators as follows

$$B^*_{m,p}(f;q;x) = \sum_{k=0}^{m+p} p^*_{m,k}(q;x) f\left(\frac{[k]}{[m]}\right),\ \text{where}$$

$$p^*_{m,k}(q;x) = \begin{bmatrix} m+p \\ k \end{bmatrix} x^k(1-x)_q^{m+p-k},\ x \in [0,1].$$

Theorem 1.1. [5] *The operators defined above satisfy the following properties:*

(1) $B^*_{m,p}(e_0;q;x) = 1,$

(2) $B^*_{m,p}(e_1;q;x) = \frac{x[m+p]}{[m]},$

(3) $B^*_{m,p}(e_2;q;x) = \frac{[m+p]}{[m]^2}\left([m+p]x^2 + x(1-x)\right),$

where $e_j(x) = x^j$, $j = 0,1,2$ *are the test functions.*

2. q-Durrmeyer-Schurer operators

In this section we propose a Durrmeyer variant of q-Schurer operators of the form studied in [5]. Let $p \in N$ be fixed. For any $m \in N$, $f \in C[0, p+1]$ we introduce the following q-Durrmeyer-Schurer operators

(2.1)
$$D^*_{m,p}(f;q;x) = \frac{[m+p+1]}{(p+1)}\sum_{k=0}^{m+p} q^{-k}\tilde{p}^*_{m,k}(q;x)\int_0^{p+1} f(t)\tilde{p}^*_{m,k}(q,qt)d_qt,$$

where

$$\tilde{p}^*_{m,k}(q;x) = \frac{1}{(p+1)^{m+p}}\begin{bmatrix} m+p \\ k \end{bmatrix} x^k(p+1-x)_q^{m+p-k}. \tag{2.2}$$

Theorem 2.1. *The polynomials $\tilde{p}^*_{m,k}$ defined in (2.2) satisfy the following properties*

(1) $\sum_{k=0}^{m+p} \tilde{p}^*_{m,k}(q;x) = 1;$

(2) $\displaystyle\sum_{k=0}^{m+p}[k]\tilde{p}^*_{m,k}(q;x)=\frac{x[m+p]}{p+1}$;

(3) $\displaystyle\sum_{k=0}^{m+p}[k]^2\tilde{p}^*_{m,k}(q;x)=\frac{[m+p]}{(p+1)^2}\left\{q[m+p-1]x^2+x(p+1)\right\}$.

Proof. Using the known identity

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^k(1-x)_q^{n-k}=1,$$

we have

$$\sum_{k=0}^{m+p}\tilde{p}^*_{m,k}(q;x)=\sum_{k=0}^{m+p}\begin{bmatrix} m+p \\ k \end{bmatrix}\left(\tfrac{x}{p+1}\right)^k\left(1-\tfrac{x}{p+1}\right)^{m+p-k}=1,$$

and

$$\begin{aligned}\sum_{k=1}^{m+p}[k]\tilde{p}^*_{m,k}(q;x) &= \sum_{k=1}^{m+p}\tfrac{[m+p]!}{[k-1]![m+p-k]!}\left(\tfrac{x}{p+1}\right)^k\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k}\\ &= \sum_{k=0}^{m+p-1}\tfrac{[m+p]!}{[k]![m+p-k-1]!}\left(\tfrac{x}{p+1}\right)^{k+1}\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k-1}\\ &= [m+p]\tfrac{x}{p+1}\sum_{k=0}^{m+p-1}\begin{bmatrix} m+p-1 \\ k \end{bmatrix}\left(\tfrac{x}{p+1}\right)^k\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k-1}\\ &= [m+p]\tfrac{x}{p+1}.\end{aligned}$$

Applying the following well known property $[k+1]=1+q[k]$, we get

$$\begin{aligned}\sum_{k=1}^{m+p}[k]^2\tilde{p}^*_{m,k}(q;x) &= [m+p]\sum_{k=1}^{m+p}[k]\tfrac{[m+p-1]!}{[k-1]![m+p-k]!}\left(\tfrac{x}{p+1}\right)^k\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k}\\ &= [m+p]\sum_{k=0}^{m+p-1}[k+1]\tfrac{[m+p-1]!}{[k]![m+p-k-1]!}\left(\tfrac{x}{p+1}\right)^{k+1}\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k-1}\\ &= [m+p]\left\{\tfrac{x}{p+1}+q\sum_{k=0}^{m+p-2}\tfrac{[m+p-1]!}{[k]![m+p-k-2]!}\left(\tfrac{x}{p+1}\right)^{k+2}\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k-2}\right\}\\ &= [m+p]\left\{\tfrac{x}{p+1}+q[m+p-1]\sum_{k=0}^{m+p-2}\begin{bmatrix} m+p-2 \\ k \end{bmatrix}\left(\tfrac{x}{p+1}\right)^{k+2}\left(1-\tfrac{x}{p+1}\right)_q^{m+p-k-2}\right\}\\ &= [m+p]\left\{\tfrac{x}{p+1}+q[m+p-1]\tfrac{x^2}{(p+1)^2}\right\}\\ &= \tfrac{[m+p]}{(p+1)^2}\left\{q[m+p-1]x^2+x(p+1)\right\}\end{aligned}$$

Theorem 2.2. *The q-Durrmeyer-Schurer operators defined in (2.1) satisfy the following properties:*

(1) $D^*_{m,p}(e_0;q;x)=1$,

(2) $D^*_{m,p}(e_1;q;x)=\frac{qx[m+p]+p+1}{[m+p+2]}$,

(3) $D^*_{m,p}(e_2;q;x)=\frac{q^4[m+p][m+p-1]x^2+(p+1)q(q+1)^2[m+p]x+(p+1)^2(q+1)}{[m+p+2][m+p+3]}$,

where $e_j(x)=x^j$, $j=0,1,2$ *are the test functions.*

Proof. For $s=0,1,\cdots$ and by definition of q-Beta function, we have

$$\begin{aligned}
&\int_0^{p+1} e_s(t)\tilde{p}^*_{m,k}(q;qt)d_qt\\
&= (p+1)\int_0^1 e_s\left((p+1)t\right)\tilde{p}^*_{m,k}\left(q,q(p+1)t\right)d_qt\\
&= (p+1)^{s+1}q^k\begin{bmatrix} m+p\\ k\end{bmatrix}\int_0^1 t^{k+s}(1-qt)_q^{m+p-k}d_qt\\
&= (p+1)^{s+1}q^k\begin{bmatrix} m+p\\ k\end{bmatrix}\beta_q\left(k+s+1,m+p-k+1\right)\\
&= (p+1)^{s+1}q^k\frac{[m+p]![k+s]!}{[k]![m+p+s+1]!}.
\end{aligned}\tag{2.3}$$

Using the above result we obtain

$$\begin{aligned}
D^*_{m,p}(e_0;q;x) &= \frac{[m+p+1]}{(p+1)}\sum_{k=0}^{m+p} q^{-k}\tilde{p}^*_{m,k}(q;x)\int_0^{p+1}\tilde{p}^*_{m,k}(q,qt)d_qt\\
&= \sum_{k=0}^{m+p}\tilde{p}^*_{m,k}(q;x)=1.
\end{aligned}$$

Using definition (2.1) of $D^*_{m,p}$ and (2.3), for $s=1$ one has

$$\begin{aligned}
D^*_{m,p}(t;q;x) &= \sum_{k=0}^{m+p}\frac{(p+1)[k+1]}{[m+p+2]}\tilde{p}^*_{m,k}(q;x)\\
&= \frac{p+1}{[m+p+2]}\sum_{k=0}^{m+p}(1+q[k])\tilde{p}^*_{m,k}(q;x)=\frac{qx[m+p]+p+1}{[m+p+2]}
\end{aligned}$$

and for $s=2$, we have

$$\begin{aligned}
D^*_{m,p}(t^2;q;x) &= \frac{(p+1)^2}{[m+p+2][m+p+3]}\sum_{k=0}^{m+p}[k+1][k+2]\tilde{p}^*_{m,k}(q,x)\\
&= \frac{(p+1)^2}{[m+p+2][m+p+3]}\sum_{k=0}^{m+p}\left\{(1+q)[k+1]+q^2[k]+q^3[k]^2\right\}\tilde{p}^*_{m,k}(q,x)\\
&= \frac{q^4[m+p][m+p-1]x^2+(p+1)q(q+1)^2[m+p]x+(p+1)^2(q+1)}{[m+p+2][m+p+3]}.
\end{aligned}\tag{2.4}$$

Remark 2.3. *We have the following central moments*

$$\begin{aligned}
D^*_{m,p}(t-x;q;x) &= D^*_{m,p}(e_1;q;x)-xD^*_{m,p}(e_0;q;x)\\
&= \left(q\frac{[m+p]}{[m+p+2]}-1\right)x+\frac{p+1}{[m+p+2]}
\end{aligned}$$

$$\begin{aligned}
D^*_{m,p}((t-x)^2;q;x) &= D^*_{m,p}(e_2;q;x) - 2xD^*_{m,p}(e_1;q;x) + \\
&+ x^2 D^*_{m,p}(e_0;q;x) \\
&= \left(q^4 \tfrac{[m+p][m+p-1]}{[m+p+2][m+p+3]} - 2q\tfrac{[m+p]}{[m+p+2]} + 1\right) x^2 \\
&\quad +(p+1)\left(q(q+1)^2 \tfrac{[m+p]}{[m+p+2][m+p+3]} - \tfrac{2}{[m+p+2]}\right) x \\
&\quad + \tfrac{(p+1)^2(q+1)}{[m+p+2][m+p+3]}.
\end{aligned}$$

Theorem 2.4. *Let $q_m \in (0,1]$. Then the sequence $\{D^*_{m,p}(f;q_m;x)\}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,p+1]$ if and only if $\lim_{m\to\infty} q_m = 1$.*

Proof. The proof is based on the well known Korovkin theorem regarding the convergence of a sequence of linear and positive operators. So, it is enough to prove the conditions

$$\lim_{m\to\infty} D^*_{m,p}(e_i;q_m;x) = x^i,\ i = 0,1,2, \tag{2.5}$$

uniformly on $[0,1]$.

If $q_m \to 1$, then $\lim_{m\to\infty}[m] = \infty$ and for p a fixed natural number $\lim_{m\to\infty} \frac{[m+p]}{[m+p+2]} = 1$, $\lim_{m\to\infty} \frac{[m+p][m+p-1]}{[m+p+2][m+p+3]} = 1$, hence (2.5) follows from Theorem 2.2.

On the other hand, if we assume that for any $f \in C[0,p+1]$, $\{D^*_{m,p}(f;q_m;x)\}$ converges to f uniformly on $[0,1]$, then $q_m \to 1$. In fact, if the sequence (q_m) does not tend to 1, then it must contain a subsequence (q_{m_k}) such that $q_{m_k} \in (0,1)$, $q_{m_k} \to q_0 \in [0,1)$ as $k \to \infty$. We have

$$\lim_{k\to\infty} D^*_{m_k,p}(e_1;q_{m_k};x) = (1-q_0)(p+1) + q_0 x \neq x.$$

This leads to a contradiction. Hence, $q_m \to 1$.

3. The rate of convergence

We will estimate the rate of convergence in terms of modulus of continuity. Let $f \in C[0,b]$. The modulus of continuity of f denoted by $\omega_f(\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by next relation $\omega_f(\delta) = \sup_{|y-x|\le\delta} |f(y)-f(x)|,\ x,y \in [0,b]$ and posses the following property

$$\omega_f(\lambda\delta) \le (1+\lambda)\omega_f(\delta).$$

Theorem 3.1. *Let $(q_m)_m$ be a sequence satisfying $q_m \to 1$ as $m \to \infty$. Then*

$$\left|D^*_{m,p}(f;q_m;x) - f(x)\right| \le 2\omega_f(\sqrt{\delta_m})$$

for all $f \in C[0,p+1]$, where

$$\delta_m = D^*_{m,p}\left((t-x)^2;q_m;x\right)$$

Proof. Since $D^*_{m,p}(e_0;q_m;\cdot) = e_0$, we have

$$\begin{aligned}
&\left|D^*_{m,p}(f;q_m;x) - f(x)\right| \\
= &\ \tfrac{[m+p+1]}{p+1} \sum_{k=0}^{m+p} q_m^{-k} \tilde{p}^*_{m,k}(q_m,x) \int_0^{p+1} |f(t)-f(x)|\, \tilde{p}_{m,k}(q_m,q_m t) d_{q_m} t.
\end{aligned}$$

Using the following well known property of modulus of continuity

$$|f(t)-f(x)| \le \omega_f(\delta)\left(\tfrac{(t-x)^2}{\delta^2} + 1\right)$$

we get

$$\begin{aligned}
&\left|D^*_{m,p}(f;q_m;x) - f(x)\right| \\
\le &\ \tfrac{[m+p+1]}{p+1}\omega_f(\delta) \sum_{k=0}^{m+p} q_m^{-k} \tilde{p}^*_{m,k}(q_m,x) \int_0^{p+1} \left(\tfrac{(t-x)^2}{\delta^2} + 1\right) \tilde{p}_{m,k}(q_m,q_m t) d_{q_m} t \\
= &\ \left(\tfrac{1}{\delta^2} D^*_{m,p}\left((t-x)^2;q_m;x\right) + D^*_{m,p}(e_0;q_m;x)\right) \cdot \omega_f(\delta).
\end{aligned}$$

But, by Remark 2.3, $\lim_{m\to\infty} D^*_{m,p}\left((t-x)^2;q_m;x\right) = 0$ because $[m] \to \infty$ as $q_m \to 1$. So, letting $\delta_m = D^*_{m,p}\left((t-x)^2;x\right)$ and taking $\delta = \sqrt{\delta_m}$, we get the result.

References

[1] D. Barbosu, **Durrmeyer-Schurer type operators**, Facta Univ. (Niš), Ser Math. Inform 19(2004), 65-72.

[2] J.L. Durrmeyer, **Une formule d'inversion de la transformée de Laplace: Application à la theorie des moments. Thèse de 3e cycle**, Faculté des Sciences de l'Université de Paris, 1967.

[3] V. Kac, P. Cheung, **Quantum calculus**, Springer, New York, 2002.

[4] A. Lupaş, **A q-analogue of the Bernstein operator**, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, Preprint 9, 1987, 85-92.

[5] C. Muraru, **Note on q-Bernstein-Schurer operators**, Stud. Univ. Babes-Bolyai, Math. LVI, 2 (2011), 1-11.

[6] G. M. Philips, **On generalized Bernstein polynomials**, Numerical Analysis, D. F. Griffits, G. A. Watson (Eds.), 1996, 263-269.

[7] F. Schurer, **Linear positive operators in approximation theory**, Math. Inst. Tech., Univ. Delft Report, 1962.

Carmen Violeta Muraru
"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Department of Mathematics, Informatics and Education Sciences
157 Calea Mărăşeşti, Bacău, 600115, ROMANIA
e-mail: carmen_7419@yahoo.com

Ana Maria Acu
"Lucian Blaga" University of Sibiu
Department of Mathematics and Informatics
5 - 7 Dr. Ion Ratiu Street, Sibiu, 550012, ROMANIA
e-mail: acuana77@yahoo.com