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GRAVITATIONAL FIELD OF LAGRANGIAN NONHOLONOMIC MECHANICAL SYSTEM

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Abstract. One associates to a Lagrangian nonholonomic mechanical system Σ a canonical semispray S^* on the phase space TM, which has the integral curves given by the evolution equations of Σ. The Lagrange geometry of the system Σ is the geometry of semispray S^*. We study h- and v-electromagnetic tensors and then we apply the Ricci identities for the gravitational potentials.

1. INTRODUCTION

The geometrization of holonomic mechanical systems was done by Levi-Civita, while, in 1926, Gh. Vrănceanu, by introducing the notion of Riemannian honholonomic space realized a first geometric model for the nonholonomic mechanical system.

In this paper, we study the gravitational field of Lagrangian nonholonomic mechanical systems:

(1.1) $$\Sigma = \left(M, L(x,y), F_i(x,y), Q_\sigma(x,y)\right),$$

where $L^n = \left(M, L(x,y)\right)$ is a Lagrange space, $F_i(x,\dot{x})$ are external forces and the Pfaff equations $Q_\sigma(x,dx) = a_{\sigma_i}(x)dx^i = 0$, $(\sigma = m+1,...,n)$ are the kinematic constrains of the system.

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The equations of evolution of the system Σ are Lagrange equations:

(1.2)
$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^i}\right) - \dfrac{\partial L}{\partial x^i} = \sum\limits_{\sigma=m+1}^{n} \lambda^{\sigma}(x) a_{\sigma_i}(x) + F_i(x,y), & y^i = \dfrac{dx^i}{dt}, \\ Q_{\sigma}(x,dx) = a_{\sigma_i}(x) dx^i = 0, & \end{cases}$$

where $\lambda^{\sigma}(x)$ are Lagrange multipliers.

We study only scleronomic systems associating to them a canonical semispray S^*, whose integral curves are given by the evolution equations (1.2).

The vector field S^* is a dynamical system on *TM*.

We denote with $g_{ij}(x,y)$ the fundamental tensor of the Lagrange space $L^n = (M, L(x,y))$ and with $g^{ij}(x,y)$ its contravariant part. As it is known, $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$, rank$\|g_{ij}\| = n$ on $TM \setminus \{0\}$ and g_{ij} has constant signature.

External forces $F_i(x,y)$ determine a d-covariant vector field and

(1.3.)
$$F_{ij} = \frac{\partial F_j}{\partial y^i} - \frac{\partial F_i}{\partial y^j}$$

is an antisymmetric d-tensor field, named *elicoidal tensor of system* Σ.

The functions that determine the constrains of the system

$$Q_{\sigma}(x,y) = a_{\sigma_i}(x) y^i, \quad (\sigma = m+1, \ldots, n)$$

are scalars with respect to the changes of the coordinates on TM.

So, $a_{\sigma_i}(x)$ are n-covariant fields on M and

(1.4.)
$$\sum_{\sigma=m+1}^{n} \lambda^{\sigma}(x) Q_{\sigma}(x,y)$$

is also a scalar function on TM. The functions $\lambda^{\sigma}(x)$ are the Lagrange multipliers.

In [7] and [8] we studied the canonical semispray and nonlinear connection of system Σ.

The canonic semispray S^* of the system will be determined by a vector field S^* on the phase space

(1.5.)
$$S^* = y^i \frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i}.$$

The integral curves of the canonic semispray S^* are given by the evolution equations of the system Σ.

We defined Lagrange geometry of system Σ as being Lagrange geometry on the phases space TM of the canonic semispray S^*.

The nonlinear connection N^* of S^* is called *canonical nonlinear connection* of system Σ.

Theorem 1.1. *The canonical nonlinear connection N^* of system Σ, has the coefficients:*

$$N_j^{*i} = \frac{\partial G^{*i}}{\partial y^j} = N_j^i - \frac{1}{4}\left(\frac{\partial F^i}{\partial y^j} + \lambda^\sigma \frac{\partial a_\sigma^i}{\partial y^j}\right). \tag{1.6.}$$

Let "$_|$" and "$|$" h and v covariant derivates defined by N^* - linear connection.

Theorem 1.2. [2]. *There exists only one N^*- linear connection $C\Gamma\left(N^*\right) = \left(L_{jk}^{*i}, C_{jk}^i\right)$ without h and v-torsions, having the properties:*

$$\begin{gathered} g_{ij\overset{*}{|}k} = 0, g_{ij} \overset{*}{|}_k = 0 \\ T_{jk}^{*i} = L_{jk}^{*i} - L_{kj}^{*i} = 0, S_{jk}^{*i} = C_{jk}^{*i} - C_{kj}^{*i} = 0, \end{gathered} \tag{1.7.}$$

where $g_{ij\overset{*}{|}k} = \frac{\delta^* g_{ij}}{\delta x^k} - g_{sj} L_{ik}^{*s} - g_{is} L_{jk}^{*s}; g_{ij} \overset{*}{|}_k = \frac{\partial g_{ij}}{\partial y^k} - g_{sj} C_{ik}^{*s} - g_{is} C_{jk}^{*s}$.

Theorem 1.3. *The connection $C\Gamma\left(N^*\right)$ has the coefficients given by generalized Christoffel symbols:*

$$\begin{gathered} L^{*i}{}_{jk} = \frac{1}{2} g^{ih}\left(\frac{\delta^* g_{hj}}{\delta x^k} + \frac{\delta^* g_{hk}}{\delta x^j} - \frac{\delta^* g_{jk}}{\delta x^h}\right) \\ C^{*i}{}_{jk} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{hj}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right) \end{gathered} \tag{1.8.}$$

This N^*- linear connection $C\Gamma\left(N^*\right)$ will be called the N^* - canonical metrical connection of Lagrangian nonholonomic mechanical system Σ.

Now, we can use the h- and v-covariant derivatives with respect to connection $C\Gamma\left(N^*\right)$. Also, we observe that in the particular case of nonholonomic mechanical systems, which have the properties:

$$g_{ij}(x, y) = g_{ij}(x), F_i(x, y) = F_i(x).$$

2. GRAVITATIONAL FIELD AND H- AND V-ELECTROMAGNETICS TENSORS

The canonical metrical connection $C\Gamma\left(N^*\right)$ of the nonholonomic mechanical system Σ allows to determine the h- and v-deflection tensor fields:

$$(2.1.)\quad \begin{aligned} D^i_j &= y^i_{\overset{*}{|}y} = y^h L^{*i}_{hj} - N^{*i}_j \\ d^i_j &= y^i\,|_j = \delta^i_j + y^h C^i{}_{hj}. \end{aligned}$$

Using the formulas of the coefficients L^{*i}_{hj} and C^i_{hj} we obtain

$$D^i_j = y^s L^i_{sj} + \frac{1}{2} y^s g^{ih}\left[K^l_j \frac{\partial g_{hs}}{\partial y^l} + K^l_s \frac{\partial g_{hj}}{\partial y^l} - K^l_h \frac{\partial g_{sj}}{\partial y^l}\right] =$$
$$= y^s L^i_{sj} + y^s g^{ih}\left[K^l_j C_{hsl} + K^l_s C_{hjl} - K^l_h C_{sjl}\right]$$

and

$$d^i_j = \delta^i_j + y^s C^i_{sj} = \delta^i_j + y^s g^{ih} C_{shj}.$$

Then, the covariant deflection tensors are given by:

$$(2.2.)\quad \begin{cases} D_{ij} = g_{ir} D^r_j = g_{ir} y^s L^i_{sj} + y^s\left(K^l_j C_{isl} + K^l_s C_{ijl} - K^l_i C_{sjl}\right) \\ d_{ij} = g_{ir} d^r_j = g_{ij} + y^s C_{sij}. \end{cases}$$

But, these tensors satisfy fundamental identities for Liouville vector field y^i.
Then, we have:

Theorem 2.1. *h- and v-covariant deflection tensors D_{ij} and d_{ij} satisfy*

$$(2.3.)\quad \begin{cases} D_{ij}\overset{*}{|}_k - D_{ik}\overset{*}{|}_j = y^s R^{*}_{s|jk} - d_{ir} R^{*r}_{jk} \\ D_{ij}\,|_k - d_{ik}\overset{*}{|}_j = y^s P^{*}_{sijk} - D_{is} C^s_{jk} - d_{is} P^{*s}_{jk} \\ d_{ij}\,|_k - d_{ik}\,|_j = y^s S_{sijk}. \end{cases}$$

These identities give the Lorentz equations for electromagnetic tensorial fields for Lagrangian nonholonomic mechanical system Σ.

Definition 2.1. The following d-tensors

$$(2.4.)\quad \begin{aligned} \mathrm{F}_{ij} &= \frac{1}{2}\left(D_{ij} - D_{ji}\right) \\ f_{ij} &= \frac{1}{2}\left(d_{ij} - d_{ji}\right) \end{aligned}$$

are h- and v-electromagnetic tensors of Σ.

From (2.2) we see that v-deflection tensor d_{ij} is symmetric. Therefore we have:

Proposition 2.1. *The v-electromagnetic tensor f of nonholonomic mechanical system Σ vanishes.*

We study only h-electromagnetic tensor F_{ij} for determining the Lorentz equations that are satisfied.

We observe that the d-tensor F_{ij} do not coincide with the elicoidal tensor F_{ij} from (3.1).

So, from (2.2), F_{ij} is given by

(2.5.) $$\mathrm{F}_{ij} = \frac{1}{2}\left\{\left(g_{ir}L^r_{sj} - g_{ir}L^r_{si}\right)y^s + 2\left(K^l_j C_{isl} - K^l_j C_{jsl}\right)y^s\right\}.$$

We deduce:

$$K^j_i = \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i} + \lambda^\sigma \frac{\partial a^j_\sigma}{\partial y^i}\right)$$

and we obtain

$$\mathrm{F}_{ij} = \frac{1}{2}y^s\left(g_{ir}L^r_{sj} - g_{jr}L^r_{si}\right) + \frac{1}{4}y^s\left[\left(\frac{\partial F^l}{\partial y^j} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^j}\right)C_{isl} - \left(\frac{\partial F^l}{\partial y^i} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^i}\right)C_{jsl}\right]$$

where $\mathrm{F}^{\,l} = g^{il}F_i$ and $C_{isl} = g_{il}C^h_{sl}$ and we obtain

(2.6.) $$\mathrm{F}_{ij} = \frac{1}{2}y^s\left(g_{ir}L^r_{sj} - g_{jr}L^r_{si}\right) + \frac{1}{4}\left[\left(\frac{\partial F^l}{\partial y^j} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^j}\right)g_{ir} - \left(\frac{\partial F^l}{\partial y^i} + \lambda^\sigma \frac{\partial a^l_\sigma}{\partial y^i}\right)g_{jr}\right].$$

From Theorem 2.1, (2.4) and Bianchi identities for $C\Gamma\left(N^*\right)$ we have

Theorem 2.2. *The h-electromagnetic tensor* F_{ij} *of nonholonomic mechanical system* Σ *with respect to* $C\Gamma\left(N^*\right)$ *satisfies the following generalized Maxwell equations:*

(2.7.) $$\mathrm{F}_{\underset{*}{ij|k}} + \mathrm{F}_{\underset{*}{jk|i}} + \mathrm{F}_{\underset{*}{ki|j}} = -\sum_{isr}^{C} y^s R^{*r}_{jk}$$

$$\mathrm{F}_{ij}\,|_k + \mathrm{F}_{jk}\,|_i + \mathrm{F}_{ki}\,|_j = 0.$$

We remark that, if the electromagnetic tensor F_{ij} does not depend of F_i and a_σ, it is given by

(2.8.) $$\mathrm{F}_{ij} = \frac{1}{2}y^s\left(g_{ir}L^r_{sj} - g_{jr}L^r_{si}\right).$$

Then, we have $\overset{*}{N}{}^{i}_{j} = N^i_j$ and the Maxwell equations are those that appear, in general, in Lagrange spaces theory.

The nonholonomic mechanical system $\Sigma = (M, L(x,y), F(x,y), Q(x,y))$ has *the gravitational potentials* given by the system of functions

(2.9.) $$\overset{*}{g}_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

We remark that this field do not depend on the external forces $F_i(x,y)$ and on the nonholonomic constrains $Q(x,y) = a_{\sigma_i}(y)y^i$. So that, the gravitational potentials $\overset{*}{g}_{ij}(x,y)$ do not depend on the Lagrange multipliers λ_i^σ $(\sigma = p+1,...,n)$.

These fact results from

(2.10.) $$\overset{*}{g}_{ij} = g_{ij}(x,y)$$

where $g_{ij}(x,y)$ is the fundamental tensor of the Lagrange space associates to system Σ, $L^n = (M, L(x,y))$.

The Theorem 1.2. shows that the canonical metrical connection $C\Gamma(N^*) = (L^{*i}_{jk}, C^i_{jk})$ has the properties

(2.11.) $$g_{ij\overset{*}{|}k} = 0, g_{ij}|_k = 0$$

and it is unique in the following conditions:

$$T^{*i}_{jk} = 0, S^i_{jk} = 0.$$

We rewrite the coefficients of the connection $C\Gamma(N^*)$:

(2.12.) $$\begin{cases} L^{*i}_{jk} = \dfrac{1}{2}g^{ih}\left(\dfrac{\overset{*}{\delta} g_{jh}}{\delta x^k} + \dfrac{\overset{*}{\delta} g_{kh}}{\delta x^j} - \dfrac{\overset{*}{\delta} g_{jk}}{\delta x^h}\right) \\ C^i_{jk} = \dfrac{1}{2}g^{ih}\left(\dfrac{\partial g_{jh}}{\partial y^k} + \dfrac{\partial g_{kh}}{\partial y^j} - \dfrac{\partial g_{jk}}{\partial y^h}\right) \end{cases}$$

where L^{*i}_{jk} is given by

(2.13.) $$L^{*i}_{jk} = L^i_{jk} + \mathrm{U}^i_{jk}$$
$$\mathrm{U}^i_{jk} = g^{ih}\left(K^l_k C_{hjk} + K^l_j C_{hjl} - K^l_h C_{jkl}\right)$$

We apply the Ricci identities to the fundamental tensor g_{ij} and we use (2.11.):

(2.14.) $$0 = R^*_{ijkh} + R^*_{jikh} = 0$$
$$P^*_{ijkh} + P^*_{jikh} = 0; S_{ijkh} + S_{jikh} = 0$$

where

(2.15.) $$R^*_{ijkh} = g_{jl} R^{*l}_{ikh}; P^*_{ijkh} = g_{jl} P^{*l}_{ikh};$$
$$S_{ijkh} = g_{jl} S^*_{ik}$$

We must calculate the curvature tensors $R^{*i}_{jkh}, P^{*i}_{jkh}$ with the tensors of the connection $C\Gamma(N)$.

So, we suppose that $C\Gamma(N)$ is the metrical connection of the associate Lagrange space L^n. Therefore we have:

(2.16.) $$L^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{jh}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h} \right)$$
$$C^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{jh}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^h} \right)$$

and

(2.17.) $$g_{ij|k} = 0, g_{ij}|_k = 0$$

(2.18.) $$T^h_{ij} = 0, S^h_{ij} = 0$$

(2.19.) $$R_{ijkh} + R_{jikh} = 0, P_{ijkh} + P_{jikh} = 0$$
$$S_{ijkh} + S_{jikh} = 0.$$

The Ricci tensors of $C\Gamma(N)$ are:

(2.20.) $$R_{ij} = R^h_{ijh}; {}'P_{ij} = P^h_{ijh}, {}''P_{ij} = P^h_{ihj}$$

and the curvature scalars are:

(2.21.) $$R = g^{ij} R_{ij}, S = g^{ij} S_{ij}.$$

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