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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 23 (2013), No. 1, 113 - 120

A NAGEL ANALOGUE OF THE STEINER-LEHMUS THEOREM

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Abstract. In this note we prove an analogue of the Steiner - Lehmus theorem from the Nagel perspective.

1. Introduction

Steiner-Lehmus's theorem states that *if two internal angle bisectors of a triangle are equal, then the triangle is isosceles.* This theorem was studied by C. L. Lehmus (1780-1863) and Jacob Steiner (1796-1863) around 1840. The standard simple proof is based by contradiction or reductio ad absurdum. For more details we refer to the monograph of H. S. M. Coxeter, S. L. Greitzer [5], and to the paper of R. Barbara [3], M. Hajja [6] and O. A. AbuArqob, H. Rabadi, J. Khitan [1].

In [10], K. Sastry gives an other version of the Steiner-Lehmus theorem using the equality of two Gergonne cevians. In this paper, we give some analogue theorems in which we consider two equal Nagel cevians. A *cevian* is a line segment that joins a vertex to a point on the opposite side.

If D, E, F are the points of tangency of the excircles of the triangle ABC and its sides (Figure 1), then the lines AD, BE and CF are concurrent, and the point of concurrency is known as the *Nagel point* (N) of the triangle ABC.

The segments $[AD], [BE]$ and $[CF]$ are called *Nagel cevians.* Denote by s the semiperimeter, and a, b, c the side lengths of triangle ABC.

Keywords and phrases: Steiner-Lehmus's theorem, Nagel cevian, Gergonne cevian.
(2010)Mathematics Subject Classification: 51M04.

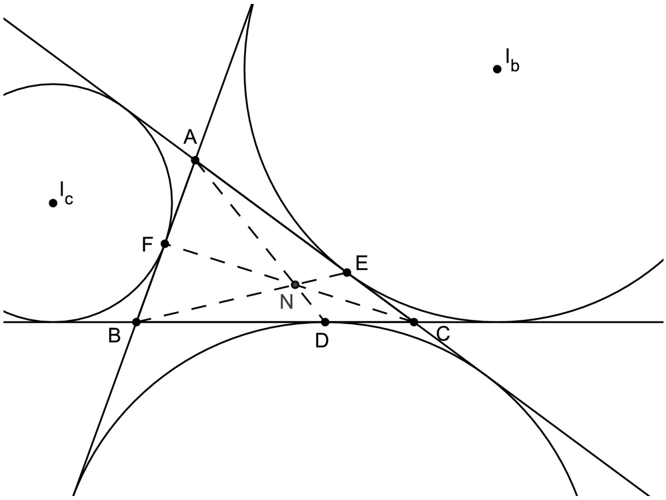

Figure 1

2. The Nagel analogue

Theorem 2.1. *If two Nagel cevians of a triangle are equal, then the triangle is isosceles.*

Proof. Let $[BE]$, $[CF]$ be the equal Nagel cevians of triangle ABC (Figure 1). It easy to see that $AE = s - c$ and $AF = s - b$. We use the Law of Cosines in the triangles ABE, ACF and obtain:

$$BE^2 = c^2 + (s-c)^2 - 2c(s-c)\cos A,$$

$$CF^2 = b^2 + (s-b)^2 - 2b(s-b)\cos A.$$

Equating the expressions for BE^2 and CF^2 we get that

$$(b-c)(b+c-a) + 2(b-c)(b+c-s)\cos A = 0,$$

hence

$$(b-c)(b+c-a)\frac{(b+c)^2 - a^2}{2bc} = 0.$$

Using the triangle inequality we have $b + c - a > 0$ and $\frac{(b+c)^2-a^2}{2bc} > 0$. Then from previous formula we obtain that $b = c$ and we are done. $\square$

Let A_b and A_c be the points of tangency of the A-excircle with AC and AB. The segments $[BA_b]$ and $[CA_c]$ are called *external Nagel cevians* corresponding to A-excircle. It is well known that external Nagel cevians corresponding to A-excircle and the line who contain the Nagel cevian AD are concurrent (Figure 2).

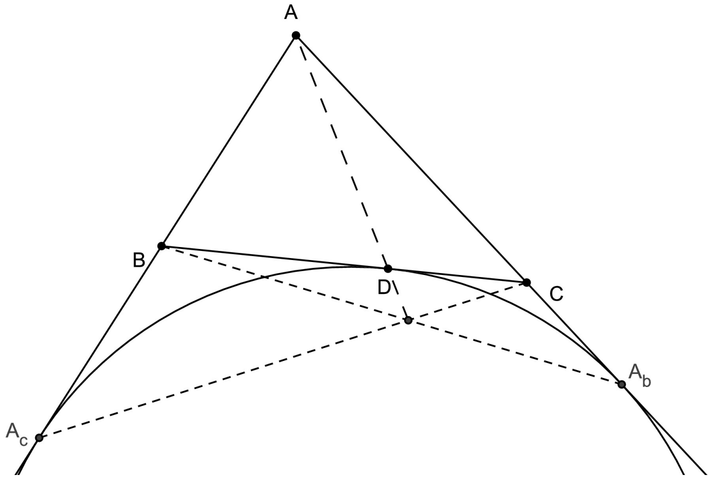

Figure 2

The external Nagel cevians are extensively studied in [9, pp.260-293] or [2]. Naturally, one may wonder whether the previous theorem is true when two external Nagel cevians are equal.

Remark 2.2. *If $[BA_b]$ and $[CA_c]$ are equal external Nagel cevians corresponding to A-excircle of the triangle ABC, then triangle ABC is not necessarily isosceles.*

It is easy to see that $AA_b = AA_c = s$. Using the Law of Cosines on the triangles ABA_b and ACA_c we obtain that

$$BA_b^2 = c^2 + s^2 - 2cs\cos A,$$

$$CA_c^2 = b^2 + s^2 - 2bs\cos A.$$

Because the segments $[BA_b]$ and $[CA_c]$ are equal it follows that $(b-c)(b+c-2s\cos A) = 0$. There are two cases to consider.

(i) $b - c = 0$ and therefore that triangle ABC is isosceles.

(ii) $b+c-2s\cos A = 0$. This can be put, after simplification, in the form

$$a^3 + (b+c)\,a^2 - \left(b^2+c^2\right)a + (b+c)\left[2bc - \left(b^2+c^2\right)\right] = 0 \tag{1}$$

Let $f(x) = x^3 + (b+c)\,x^2 - (b^2+c^2)\,x + (b+c)\,[2bc - (b^2+c^2)]$. We have $f(b) = c\,(2b^2 - c^2)$ and $f(c) = b\,(2c^2 - b^2)$. Assume that $0 < b < \frac{c}{\sqrt{2}}$. Then $f(b) < 0$ and $f(c) > 0$, hence there exists $x_0 \in (b, c)$ such that $f(x_0) = 0$. Let $a = x_0$. Obviously, $b < a + c$ and $a < b + c$. In order to have $c < a + b$, it suffices to assume that $c < 2b$, as we may.

In conclusion, if $0 < b < \frac{c}{\sqrt{2}} < b\sqrt{2}$, then there exists a triangle ABC with side lengths $b < a < c$ such that the cevians $[BA_b]$ and $[CA_c]$ are equal.

3. Other variations

In this section we show that the equality of the segments determined by a Nagel cevian and various important lines of triangle implies that the triangle is isosceles.

Theorem 3.1. *The internal angle bisectors of the angles ABC and ACB of triangle ABC meet the Nagel cevian AD at E and F respectively. If $BE = CF$, then triangle ABC is isosceles.*

Proof. Because BE and CF are internal angle bisectors of the angles ABC and ACB (Figure 3), we have

$$BE = \frac{2 \cdot BD \cdot AB}{BD + AB} \cdot \cos \frac{B}{2}$$

hence

$$BE = \frac{2(s-c)c}{s}\sqrt{\frac{s(s-b)}{ac}}.$$

Similarly

$$CF = \frac{2(s-b)b}{s}\sqrt{\frac{s(s-c)}{ab}}.$$

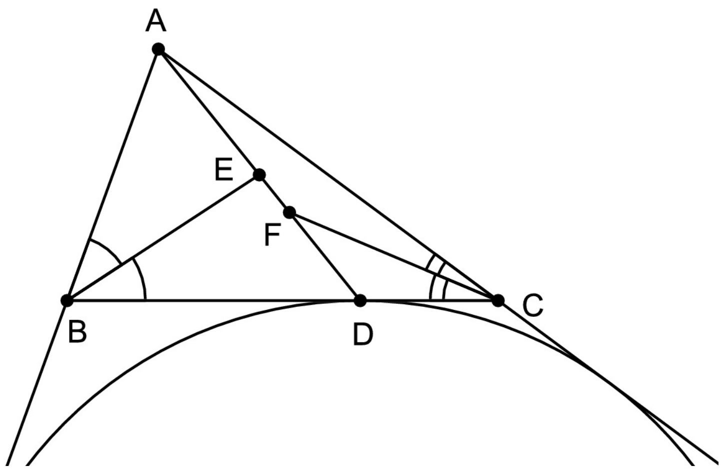

Figure 3

The equality $BE = CF$ is equivalent with

$$\frac{2(s-c)c}{s} \cdot \sqrt{\frac{p(p-b)}{ac}} = \frac{2(s-b)b}{s} \cdot \sqrt{\frac{p(p-c)}{ab}},$$

i.e.

$$\frac{a-b-c}{2} \cdot (b-c) = 0.$$

From this and by the triangle inequality we conclude that $b = c$. $\square$

Remark 3.2. *Let E, F be the projections of the points B and C respectively on the Nagel cevian AD. If $BE = CF$, then triangle ABC is isosceles.*

Triangles BED and CFD are congruent (Figure 4), then we obtain that the segments $[BD]$ and $[CD]$ are equal, i.e. $s - c = s - b$, and therefore $b = c$.

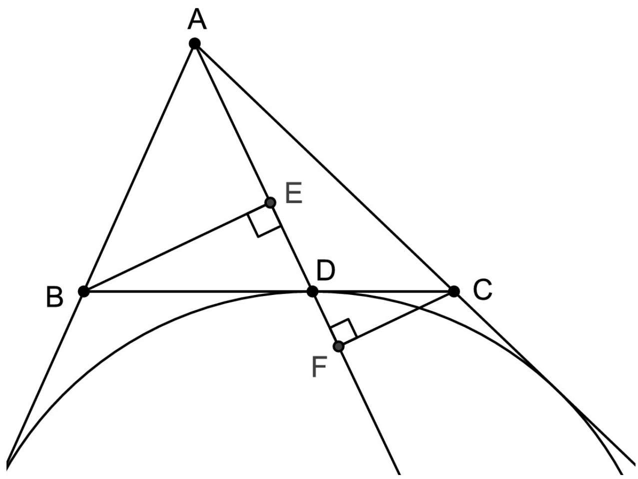

Figure 4

Theorem 3.3. *If the segments who connect two vertices of a triangle ABC with the Nagel point of the triangle ABC are equal, then the triangle is isosceles or it has sides in arithmetic progression.*

Proof. Let BN, CN be the equal segments (Figure 5). Using the Menelaus Theorem in the triangles ABE and ACF with the transversals $F{-}N{-}C$ and $E{-}N{-}B$ respectively, we obtain that $NB = \frac{b}{s} \cdot BE$ and $NC = \frac{c}{s} \cdot CF$.

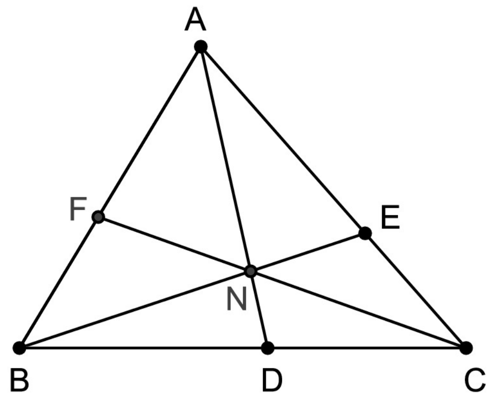

Figure 5

Because $[BN]$ and $[CN]$ are equal and using the fact that $2bc\cos A = b^2 + c^2 - a^2$, we get the following sequence of reductions:

$$\begin{aligned} b^2[c^2 + (s-c)^2 - 2c(s-c)\cos A] &= c^2[b^2 + (s-b)^2 - 2b(s-b)\cos A] \\ s(b-c)[s(b+c) + a^2 - (b+c)^2] &= 0 \\ s^2(b-c)(2a-b-c) &= 0. \end{aligned}$$

Finally, we obtain that $b = c$ or $a = \frac{b+c}{2}$. $\square$

Remark 3.4. *It is known that in a triangle with the lengths of the sides in arithmetic progression the Nagel line is parallel with the medium length side* [4].

4. Return to Gergonne cevian

In [10] K. Sastry proposed this problem: *The external angle bisectors of $\angle ABC$ and $\angle ACB$ meet the extension of the Gergonne cevian (the line segment between a vertex and the point of contact of the incircle with the opposite side) AD at the points E and F respectively. If $BE = CF$, prove or disprove that triangle ABC is isosceles.*

In the following we give an affirmative answer to the problem in question. If AD is Gergonne cevian of triangle ABC, then $BD = s - b$ and $CD = s - c$ (Figure 6).

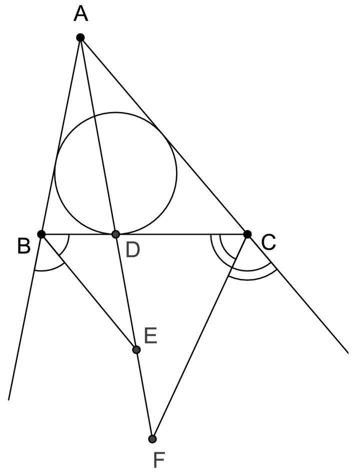

Figure 6

We have

$$Area[ABE] = Area[ABD] + Area[BDE],$$

or

$$BE \cdot c \cdot \sin\left(90^\circ + \frac{B}{2}\right) = c \cdot (s-b)\sin B + BE \cdot (s-b) \cdot \sin\left(90^\circ - \frac{B}{2}\right)$$

i.e.

$$BE \cdot c \cdot \cos\frac{B}{2} = 2c \cdot (s-b)\sin\frac{B}{2}\cos\frac{B}{2} + BE \cdot (s-b) \cdot \cos\frac{B}{2}.$$

Therefore,

$$\begin{aligned} BE &= \frac{2c \cdot (s-b)\sin\frac{B}{2}}{b+c-s} \\ &= \frac{2c \cdot (s-b)}{b+c-s} \cdot \sqrt{\frac{(s-a)(s-c)}{ac}}. \end{aligned}$$

In a similar way, we find that

$$CF = \frac{2b \cdot (s-c)}{b+c-s} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Equating the expressions for $[BE]$ and $[CF]$ we get that $\sqrt{(s-b)c} = \sqrt{(s-c)b}$, and from here we obtain that $b = c$.

Acknowledgement. The authors would like to thank the referee for useful ideas which improved the initial form of this paper.

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