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# TWO GENERAL FIXED POINT THEOREMS UNDER STRICT IMPLICIT RELATIONS IN $G$ - METRIC SPACES

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**Abstract.** In this paper two fixed point theorems are proved, which extend the main results from [31] to $G$ - metric spaces and generalize Theorem 2.1 [4] for mappings satisfying (E.A.) property under strict implicit relations.

## 1. Introduction

Let $(X, d)$ be a metric space and $S$, $T$ be two self mappings of $X$. In [10], Jungck defined $S$ and $T$ to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $(x_n)$ is a sequence in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

The concept was frequently used to prove existence theorems in common fixed point theory. The study of common fixed points of noncompatible mappings is also interesting. The work along this lines has been initiated by Pant in [21], [22], [23].

Recently, Aamri and Moutawakil [1] introduced a generalization of the concept on noncompatible mappings.

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**Definition 1.1** ([1]). Let $S$ and $T$ be two self mappings of a metric space $(X, d)$. We say that $S$ and $T$ satisfy property $(E.A)$ if there exists a sequence $(x_n)$ in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

**Remark 1.2.** *It is clear that two self mappings $S$ and $T$ of a metric space $(X, d)$ will be noncompatible if there exists a sequence $(x_n)$ in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non existent. Therefore, two noncompatible self mappings of a metric space $(X, d)$ satisfy property $(E.A)$.*

**Definition 1.3** ([11]). Two self mappings $S$ and $T$ of a metric space $(X, d)$ is said to be weakly compatible if $Su = Tu$ implies $STu = TSu$.

**Remark 1.4.** *It is known that the notions of weakly compatible mappings and mappings satisfying property $(E.A)$ are independent.*

**Definition 1.5.** Let $S$ and $T$ be two self mappings of a metric space $(X, d)$. A point $x \in X$ is said to be a coincidence point of $S$ and $T$ if $Sx = Tx$ and the point $w = Sx = Tx$ is said to be a point of coincidence of $S$ and $T$.

**Lemma 1.6** ([2]). *Let $f$ and $g$ be weakly compatible self mappings on a nonempty set $X$. If $f$ and $g$ have a unique point of coincidence $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.*

The following theorem is proved in [1].

**Theorem 1.7** ([1]). *Let $S$ and $T$ be weakly compatible mappings of a metric space $(X, d)$ such that*

*(i) $T$ and $S$ satisfy property $(E.A)$;*

$$d(Tx, Ty) < \max\{d(Sx, Sy), \tfrac{1}{2}[d(Sx, Tx) + d(Sy, Ty)], \\ \tfrac{1}{2}[d(Sx, Ty) + d(Sy, Tx)]\}$$

*for all $x, y \in X$;*

*(iii) $T(X) \subset S(X)$.*

*If $S(X)$ or $T(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.*

In [8] and [9], Dhage introduced a new class of generalized metric space, named $D$ - metric space.

Mustafa and Sims [13], [14] proved that most of the claims concerning the fundamental topological structures on $D$ - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named $G$ - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in $G$ - metric spaces under certain conditions [14], [15], [16], [17], [18], [19], [20], [34] and in other papers. In [24], [25] the study of fixed points for mappings satisfying an implicit relation was introduced.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, Tychonoff metric spaces, compact metric spaces, paracompact metric spaces, reflexive spaces, probabilistic metric spaces, convex metric spaces, in two or three metric spaces, for single valued functions, hybrid pairs of functions and set valued functions. Quite recently, the method is used in the study of fixed points for mappings satisfying an implicit relation of integral type, and fuzzy metric spaces. The method unified different type of contractive and extensive conditions. With this method, the proofs of some fixed theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

## 2. Preliminaries

**Definition 2.1** ([14])**.** Let $X$ be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$;

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

$(G_3) : G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

$(G_4) : G(x, y, z) = G(y, z, x) = ...$ (symmetry in all three variables);

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The function $G(x, y, z)$ is said to be a $G$ - metric on $X$ and the pair $(X, G)$ is said to be a $G$ - metric space.

Note that if $G(x, y, z) = 0$, then $x = y = z$ [14].

**Definition 2.2** ([14])**.** Let $(X, G)$ be a $G$ - metric space. A sequence $(x_n)$ in $X$ is said to be:

1) $G$ - convergent if for $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $x \in X$ such that for all $m, n \in \mathbb{N}$, $m, n \geq k$, $G(x_n, x_m, x) < \varepsilon$.

2) $G$ - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n, p \in \mathbb{N}$, $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $m, n, p \to \infty$.

A $G$ - metric space $(X, G)$ is said to be $G$ - complete if every $G$ - Cauchy sequence is $G$ - convergent.

**Lemma 2.3** ([14]). *Let $(X,G)$ be a $G$ - metric space. Then, the following properties are equivalent:*

1) $(x_n)$ *is* $G$ *- convergent to* $x$*;*
2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$*;*
3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$*;*
4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$*.*

**Lemma 2.4** ([14]). *Let $(X,G)$ be a $G$ - metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

**Definition 2.5** ([3]). Let $(X,G)$ be a $G$ - metric space. The functions $f,g:(X,G)\to(X,G)$ is called:

a) $G$ - compatible if $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$, whenever $(x_n)$ is a sequence in $X$ such that $(fx_n)$ and $(gx_n)$ are $G$ - convergent to some $t \in X$;

b) $G$ - non compatible if there exists at least one sequence $(x_n)$ in $X$ such that $(fx_n)$ and $(gx_n)$ are $G$ - convergent to some $t \in X$, but $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)$ is either nonzero or does not exists.

**Definition 2.6** ([1], [7], [20], [5]). Let $(X,G)$ be a $G$ - metric space. The self mappings $f$ and $g$ of $X$ are said to be satisfying condition $G-(E.A)$ property if there exists a sequence $(x_n)$ in $X$ such that $(fx_n)$ and $(gx_n)$ are $G$ - convergent to some $t \in X$.

Some fixed point theorems for self mappings of a $G$ - metric spaces with $G-(E.A)$ property are proved in [7], [20], [5].

Quite recently, in [4], the authors extend Theorem 1.7 for mappings in $G$ - metric spaces.

**Theorem 2.7** ([4]). *Let $(X,G)$ be a $G$ - metric space. Suppose $f,g: X\to\mathbb{R}$ be mappings satisfying $G-(E.A)$ property and such that for all $x,y\in X$ with $x\neq y$:*

*a)*

$$G(fx,fy,fy) < \max\{G(gx,gy,gy), \tfrac{G(fx,gx,gx)+G(fy,gy,gy)}{2}, \tfrac{G(fx,gy,gy)+G(fy,fy,gx)}{2}\},$$

*or*

*b)*

$$G(fx,fy,fy) < \max\{G(gx,gx,gy), \tfrac{G(fx,fx,gx)+G(fy,fy,gy)}{2}, \tfrac{G(fx,fx,gy)+G(fy,gx,gx)}{2}\}.$$

*If $f(X)\subset g(X)$ and one of $f(X)$ or $g(X)$ is a closed subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover,* *if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

## 3. Implicit relations

**Definition 3.1.** Let $\mathfrak{F}_6$ be the set of all real continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$(F_1)$ : $F(t, 0, 0, t, t, 0) > 0$, $\forall t > 0$,

$(F_2)$ : $F(t, t, 0, 0, t, t) \geq 0$, $\forall t > 0$.

In [26] a generalization of Theorem 1.7 for mappings satisfying implicit relation is proved.

**Theorem 3.2** ([26])**.** *Let $T$ and $S$ be two weakly compatible self mappings of a metric space $(X, d)$ such that*

*(i)* $T$ *and* $S$ *satisfy property* $(E.A)$,

*(ii)*

$$F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) < 0$$

*for all $x \neq y \in X$ and $F \in \mathfrak{F}_6$;*

*(iii)* $T(X) \subset S(X)$.

*If $S(X)$ or $T(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.*

**Definition 3.3** ([12])**.** An altering distance is a function $\psi : [0, \infty) \to [0, \infty)$ satisfying:

$(\psi_1)$ : $\psi$ is increasing and continuous;

$(\psi_2)$ : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problems involving an altering distance have been studied in [12], [28], [32], [33] and in other papers.

The following result is obtained in [31].

**Theorem 3.4** ([31])**.** *Let $S$ and $T$ be two weakly compatible self mappings of a metric space $(X, d)$ such that*

*1)* $S$ *and* $T$ *satisfy property* $(E.A)$;

*2)* $S$ *and* $T$ *satisfy the inequality*

$$F(\psi(d(Tx, Ty)), \psi(d(Sx, Sy)), \psi(d(Sx, Tx)), \psi(d(Sy, Ty)), \psi(d(Sx, Ty)), \psi(d(Sy, Tx))) < 0 \tag{3.1}$$

*for all $x \neq y \in X$, where $\psi$ is an altering distance and $F \in \mathfrak{F}_6$;*

*3)* $T(X) \subset S(X)$.

*If $S(X)$ or $T(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.*

**Definition 3.5.** Let $\mathfrak{F}_G$ be the set of all real continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$(F_1)$ : $F(u, 0, u, 0, u, 0) > 0$, $\forall u > 0$,

$(F_2)$ : For $u, v > 0$, $F(u, u, 0, 0, u, v) < 0$ implies $u < v$.

**Example 3.6.** $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e \leq 1$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u(1 - (b + d)) > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - au - du - ev < 0$. *Then* $u < \frac{e}{1-(a+d)}v$ *which implies* $u < v$.

**Example 3.7.** $F(t_1, ..., t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u - \frac{u}{2} = \frac{u}{2} > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - \max\{u, \frac{u+v}{2}\} < 0$. *If* $u > v$, *then* $0 < 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u < v$.

**Example 3.8.** $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $k \in (0, 1)$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u(1 - k) > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - k\max\{u, v\} < 0$. *If* $u > v$, *then* $u(1 - k) < 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u < kv < v$.

**Example 3.9.** $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, *where* $k \in (0, 1)$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u(1 - k) > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - k\max\{u, \frac{u+v}{2}\} < 0$.

*As in Example 3.7 it follows that* $u < v$.

**Example 3.10.** $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$, *where* $a$, $b$, $c \geq 0$, $b + c < 1$ *and* $a + c < 1$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u(1 - (b + c)) > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - au - c\max\{u, v\} < 0$. *If* $u > v$, *then* $u(1 - (a + c)) < 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u < \frac{c}{1-a}v < v$.

**Example 3.11.** $F(t_1, ..., t_6) = t_1 - at_2 - b\frac{t_5+t_6}{1+t_3+t_4}$, *where* $a$, $b \geq 0$ *and* $a + 2b < 1$.

$(F_1)$ : $F(u, 0, u, 0, u, 0) = u(1 - \frac{b}{1+u}) > 0$, $\forall u > 0$.

$(F_2)$ : *Let* $u, v > 0$ *be such that* $F(u, u, 0, 0, u, v) = u - au - b(u + v) < 0$ *which implies* $u < \frac{b}{1-(a+b)}v < v$.

**Example 3.12.** *$F(t_1,...,t_6) = t_1 - at_2 - b\sqrt{t_3t_4} - c\sqrt{t_5t_6}$, where $a,b,c \geq 0$ and $a+c<1$.*

$(F_1)$ : *$F(u,0,u,0,u,0) = u > 0$, $\forall u > 0$.*

$(F_2)$ : *Let $u,v > 0$ be such that $F(u,u,0,0,u,v) = u - au - c\sqrt{uv} < 0$. If $u > v$, then $u(1-(a+c)) < 0$, a contradiction. Hence, $u \leq v$ which implies $u < \frac{c}{1-a}v < v$.*

**Definition 3.13.** Let $\mathfrak{F}'_G$ be the set of all real functions $F(t_1,...,t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ satisfying the following conditions:

$(F'_1)$ : $F(u,0,0,u,u,0) > 0$, $\forall u > 0$,

$(F'_2)$ : For $u,v > 0$, $F(u,v,0,0,u,v) < 0$ implies $u < v$.

**Remark 3.14.** *The functions $F$ from Example 3.6 - 3.12 satisfies also the conditions $(F'_1)$ and $(F'_2)$.*

The study of fixed point theorems in $G$ - metric spaces for mappings satisfying implicit relations is initiated in [27], [29], [30].

The purpose of this paper is to prove two general fixed point theorems which extend Theorem 3.4 for $G$ - metric space generalizing Theorem 2.7 and obtained other particular results. As applications, in the last part of this paper, two general fixed point theorems in $G$ - metric spaces for mappings satisfying implicit contractive conditions of integral type are proved.

## 4. Main results

**Theorem 4.1.** *Let $(X,G)$ be a $G$ - metric space and $f,g : X \to X$ be two mappings satisfying the following inequality*

$$\begin{gathered} F(\psi(G(fx,fy,fy)),\psi(G(gx,gy,gy)),\psi(G(fx,gx,gx)), \\ \psi(G(fy,gy,gy)),\psi(G(fx,gy,gy)),\psi(G(fy,gx,gx))) < 0 \end{gathered} \tag{4.1}$$

*for all $x,y \in X$, where $F$ satisfy property $(F_2)$ and $\psi$ is an altering distance. Then, $f$ and $g$ have at least one point of coincidence.*

*Proof.* Suppose that $f$ and $g$ have two distinct points of coincidence $u = fa = ga$ and $v = fb = gb$. By (4.1) we have successively

$$\begin{gathered} F(\psi(G(fa,fb,fb)),\psi(G(ga,gb,gb)),\psi(G(fa,ga,ga)), \\ \psi(G(fb,gb,gb)),\psi(G(fa,gb,gb)),\psi(G(fb,ga,ga))) < 0, \end{gathered}$$

$$F(\psi(G(u,v,v)),\psi(G(u,v,v)),0,0,\psi(G(u,v,v)),\psi(G(v,u,u))) < 0\ .$$

By Property $(F_2)$ we have $\psi(G(u,v,v)) < \psi(G(v,u,u))$.

Similarly, we obtain $\psi(G(v,u,u)) < \psi(G(u,v,v))$. Hence,

$$\psi(G(u,v,v)) < \psi(G(v,u,u)) < \psi(G(u,v,v)),$$

a contradiction. $\square$

**Theorem 4.2.** *Let $(X,G)$ be a $G$ - metric space and $f,g : X \to X$ be two mappings satisfying the inequality (4.1) for all $x \neq y \in X$, $F \in \mathfrak{F}_G$ and $\psi$ is an altering distance. If*

*1) $f$ and $g$ satisfy $G-(E.A)$ property,*

*2) $g(X)$ or $f(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, $f$ and $g$ have an unique common fixed point.*

*Proof.* Since $f$ and $g$ satisfy $G-(E.A)$ property, there exists a sequence $(x_n)$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. Since $g(X)$ is a closed subspace of $X$, then there exists $p \in X$ such that $gp = t$. Also, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gp$. We will prove that $fp = gp$. Suppose that $fp \neq gp$. By (4.1) we have

$$F(\psi(G(fp, fx_n, fx_n)), \psi(G(gp, gx_n, gx_n)), \psi(G(fp, gp, gp)),$$
$$\psi(G(fx_n, gx_n, gx_n)), \psi(G(fp, gx_n, gx_n)), \psi(G(fx_n, gp, gp))) < 0.$$

Letting $n$ tend to infinity we obtain

$$F(\psi(G(fp, gp, gp)), 0, \psi(G(fp, gp, gp)), 0, \psi(G(fp, gp, gp)), 0) \leq 0,$$

a contradiction of $(F_1)$. Hence, $fp = gp$ and $u = fp = gp$ is a point of coincidence. By Theorem 4.1, $u$ is the unique point of coincidence. If $f$ and $g$ are weakly compatible, by Lemma 1.6, $u$ is the unique common fixed point of $f$ and $g$.

If $f(X)$ is a closed subspace of $X$, then the proof follows by $f(X) \subset g(X)$. $\square$

If $\psi(t) = t$ by Theorem 4.2 we obtain

**Theorem 4.3.** *Let $(X,G)$ be a $G$ - metric space and $f,g : X \to X$ be two mappings satisfying the inequality*

$$\begin{aligned} F(G(fx, fy, fy), G(gx, gy, gy), G(fx, gx, gx),\\ G(fy, gy, gy), G(fx, gy, gy), G(fy, gx, gx)) < 0 \end{aligned} \tag{4.2}$$

*for all $x \neq y \in X$ and $F \in \mathfrak{F}_G$. If*

*1) $f$ and $g$ satisfy $G-(E.A)$ property,*

*2) $g(X)$ or $f(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have an unique common fixed point.*

**Remark 4.4.** *1) The results from Theorems 4.2 and 4.3 are true if $f$ and $g$ are $G$ - noncompatible instead of $f$ and $g$ satisfy $G-(E.A)$ property.*

*2) By Theorem 4.3 and Example 3.7 we obtain the results from Theorem 2.7 (a).*

*3) By Examples 3.6, 3.8 - 3.12 we obtain new particular results.*

**Theorem 4.5.** *Let $(X, G)$ be a $G$ - metric space and $f, g : X \to X$ be two functions such that*

$$\begin{aligned} F(\psi(G(fx, fy, fy)), \psi(G(gx, gy, gy)), \psi(G(gx, fx, fx)), \\ \psi(G(gy, fy, fy)), \psi(G(gx, fy, fy)), \psi(G(gy, fx, fx))) < 0 \end{aligned} \tag{4.3}$$

*for all $x \neq y \in X$, where $F$ satisfy property $(F_2')$ and $\psi$ is an altering distance. Then, $f$ and $g$ have at least one point of coincidence.*

*Proof.* Suppose that $f$ and $g$ have two distinct points of coincidence $u = fa = ga$ and $v = fb = gb$. By (4.3) we have successively

$$\begin{aligned} F(\psi(G(fa, fb, fb)), \psi(G(ga, gb, gb)), \psi(G(ga, fa, fa)), \\ \psi(G(gb, fb, fb)), \psi(G(ga, fb, fb)), \psi(G(gb, fa, fa))) < 0, \end{aligned}$$

$$F(\psi(G(u, v, v)), \psi(G(u, v, v)), 0, 0, \psi(G(u, v, v)), \psi(G(v, u, u))) < 0 \,.$$

By $(F_2')$ it follows that $\psi(G(u, v, v)) < \psi(G(v, u, u))$.

Similarly, $\psi(G(v, u, u)) < \psi(G(u, v, v))$. Hence,

$$\psi(G(u, v, v)) < \psi(G(v, u, u)) < \psi(G(u, v, v)),$$

a contradiction. ☐

**Theorem 4.6.** *Let $(X, G)$ be a $G$ - metric space and $f, g : X \to X$ be two mappings satisfying the inequality (4.3) for all $x \neq y \in X$, where $F \in \mathfrak{F}_G'$ and $\psi$ is an altering distance. If*

*1) $f$ and $g$ satisfy $G-(E.A)$ property,*

*2) $g(X)$ or $f(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, $f$ and $g$ have an unique common fixed point.*

*Proof.* Since $f$ and $g$ satisfy $G-(E.A)$ property, there exists a sequence $(x_n)$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. Since $g(X)$ is a closed subspace of $X$, then there exists $p \in X$ such that $gp = t$. Also, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gp$.

Suppose that $fp \neq gp$. By (4.3) we get

$$\begin{aligned} F(\psi(G(fx_n, fp, fp)), \psi(G(gx_n, gp, gp)), \psi(G(gx_n, fx_n, fx_n)), \\ \psi(G(gp, fp, fp)), \psi(G(gx_n, fp, fp)), \psi(G(gp, fx_n, fx_n))) < 0. \end{aligned}$$

Letting $n$ tend to infinity we obtain

$$F(\psi(G(gp, fp, fp)), 0, 0, \psi(G(gp, fp, fp)), \psi(G(gp, fp, fp), 0) \leq 0,$$

a contradiction of $(F_1')$. Hence, $fp = gp$ and $u = fp = gp$ is a coincidence point of $f$ and $g$. By Theorem 4.5, $u$ is the unique point of coincidence.

If $f$ and $g$ are weakly compatible, by Lemma 1.6, $u$ is the unique common fixed point.

If $f(X)$ is a closed subspace of $X$, then the proof follows by $f(X) \subset g(X)$. □

If $\psi(t) = t$ by Theorem 4.6 we obtain

**Theorem 4.7.** *Let $(X, G)$ be a $G$ - metric space and $f, g : X \to X$ be two mappings satisfying the inequality*

$$\begin{aligned} F(G(fx, fy, fy), G(gx, gy, gy), G(gx, fx, fx), \\ G(gy, fy, fy), G(gx, fy, fy), G(gy, fx, fx)) < 0 \end{aligned} \tag{4.4}$$

*for all $x \neq y \in X$ and $F \in \mathfrak{F}_G'$. If*

*1) $f$ and $g$ satisfy $G-(E.A)$ property,*

*2) $f(X)$ or $g(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have an unique common fixed point.*

**Remark 4.8.** *1) The results from Theorems 4.6 and 4.7 are true if $f$ and $g$ are $G$ - noncompatible instead of $f$ and $g$ satisfy $G-(E.A)$ property.*

*2) By Theorem 4.7 and Example 3.7 we obtain the results from Theorem 2.7 (b).*

*3) By Examples 3.6, 3.8 - 3.12 we obtain new particular results.*

## 5. Applications

In [6], Branciari established the following theorem which opened the way to the study of mappings satisfying a contractive condition of integral type.

**Theorem 5.1** ([6]). *Let $(X, d)$ be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt \tag{5.1}$$

*where $h(t) : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, such that for $\varepsilon > 0$, $\int_0^\varepsilon h(t)dt > 0$. Then $f$ has an unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.*

**Remark 5.2.** *Theorem 5.1 has been generalized in several papers.*

In [5], Aydi initiated the study of fixed points in $G$ - metric spaces for mappings satisfying contractive conditions of integral type.

**Lemma 5.3** ([28]). *Let $h(t) : [0, \infty) \to [0, \infty)$ as in Theorem 5.1. Then, $\psi(t) = \int_0^t h(x)dx$ is an altering distance.*

Using the method from [28] we prove the following two theorems.

**Theorem 5.4.** *Let $(X, G)$ be a $G$ - metric space and $f, g : X \to X$ be two mappings satisfying the inequality*

$$\begin{aligned}(5.2)\qquad & F\Big(\int_0^{G(fx,fy,fy)} h(t)dt, \int_0^{G(gx,gy,gy)} h(t)dt, \int_0^{G(fx,gx,gx)} h(t)dt, \\ & \int_0^{G(fy,gy,gy)} h(t)dt, \int_0^{G(fx,gy,gy)} h(t)dt, \int_0^{G(fy,gx,gx)} h(t)dt\Big) < 0\end{aligned}$$

*for all $x \neq y \in X$, $F \in \mathfrak{F}_G$ and $h(t)$ is as in Theorem 5.1. If*

*1) $f$ and $g$ satisfy $G - (E.A)$ property,*

*2) $g(X)$ or $f(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have an unique common fixed point.*

*Proof.* As in Lemma 5.3

$$\begin{aligned}\psi(G(fx, fy, fy)) &= \int_0^{G(fx,fy,fy)} h(t)dt, \\ \psi(G(gx, gy, gy)) &= \int_0^{G(gx,gy,gy)} h(t)dt, \\ \psi(G(fx, gx, gx)) &= \int_0^{G(fx,gx,gx)} h(t)dt, \\ \psi(G(fy, gy, gy)) &= \int_0^{G(fy,gy,gy)} h(t)dt, \\ \psi(G(fx, gy, gy)) &= \int_0^{G(fx,gy,gy)} h(t)dt, \\ \psi(G(fy, gx, gx)) &= \int_0^{G(fy,gx,gx)} h(t)dt.\end{aligned}$$

By (5.2) we obtain

$$\begin{aligned}& F(\psi(G(fx, fy, fy)), \psi(G(gx, gy, gy)), \psi(G(fx, gx, gx)), \\ & \psi(G(fy, gy, gy)), \psi(G(fx, gy, gy)), \psi(G(fy, gx, gx))) < 0.\end{aligned}$$

Hence, the conditions of Theorem 4.2 are satisfied. Then, $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible then $f$ and $g$ have a unique common fixed point. $\square$

**Remark 5.5.** *1. If $h(t) = 1$, then by Theorem 5.4 we obtain Theorem 4.3.*

*2. By Theorem 5.4 and Examples 3.6 - 3.12 we obtain new particular results.*

**Theorem 5.6.** *Let $(X, G)$ be a $G$ - metric space and $f, g : X \to X$ be two mappings satisfying the inequality*

$$
\begin{aligned}
&F\Big(\int_0^{G(fx,fy,fy)} h(t)dt, \int_0^{G(gx,gy,gy)} h(t)dt, \int_0^{G(gx,fx,fx)} h(t)dt,\\
&\int_0^{G(gy,fy,fy)} h(t)dt, \int_0^{G(gx,fy,fy)} h(t)dt, \int_0^{G(gy,fx,fx)} h(t)dt\Big) < 0
\end{aligned}
\tag{5.3}
$$

*for all $x \neq y \in X$, $F \in \mathfrak{F}'_G$ and $h(t)$ is as in Theorem 5.1. If*

*1) $f$ and $g$ satisfy $G-(E.A)$ property,*

*2) $g(X)$ or $f(X)$ is a closed subspace of $X$,*

*3) $f(X) \subset g(X)$,*

*then $f$ and $g$ have a point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have an unique common fixed point.*

*Proof.* The proof is similar to the proof of Theorem 5.4 and follows by Theorem 4.6. □

**Remark 5.7.** *1. If $h(t) = 1$, then by Theorem 5.6 we obtain Theorem 4.7.*

*2. By Theorem 5.6 and Examples 3.6 - 3.12 we obtain new particular results.*

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