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EFFICIENCY BY MATHEMATICS

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Abstract. This research report is devoted to the general concept of the efficiency presented and analysed in an original manner, the optimization being indicated as an equivalent of the efficiency, in the context of the accelerated interoperability that simplify the integration. It is a short synthesis of recent results obtained in this field.

1. INTRODUCTION

The following abstract construction highlights a mathematical equivalence between the "efficiency" concept and the "optimality" one. The reality proves that the optimality represents a particular case of the efficiency and conversely. Moreover, the optimal situations are best approximation cases for the efficient point sets.

Let X by any real or complex ordered linear space, Κ the collection of all the convex cones defined on X and A a non-empty arbitrary subset of X. Taking into consideration the previous statements, we are conducted to the next inclusion relation between the set of all efficient point of the set A and the sets of all points that generate the vectorial minimality and maximality, respectively, in relation to any convex cone $K \in$ Κ :

$$Eff(A) \supseteq \bigcup_{K \in K} MIN_K(A) \cup \bigcup_{K \in K} MAX_K(A).$$

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It is obvious that any Vectorial Optimization Program originates also from the Pareto Optimal Problem in the classical case of the ordered Euclidean spaces, with vectorial objective-functions and can be described as:

$$(P_{T,K}): MIN_K f(T) \text{ or } (P_{T,K}): MAX_K f(T)$$

where $K \in$ K, T is a non-empty set and $f: T \to X$ is an application. It is possible that the convex cone K to be replaced by a non-empty set from X, at least regarding the numerical processing that gives the efficient solutions, or to create a combination represented, for example, by the following approximate efficiency. If we denote by $S_f(T,K)$ the set of all the solutions, the mentioned equivalence could be justified by the relation:

$$\bigcup_{\emptyset \neq A \subseteq X} Eff(A) = \bigcup_{\substack{T \neq \emptyset \\ K \in K,\, f: T \to X}} S_f(T,K)$$

In reality, only the inclusion generally works, that is,

$$\bigcup_{\emptyset \neq A \subseteq X} Eff(A) \supseteq \bigcup_{\substack{T \neq \emptyset \\ K \in K,\, f: T \to X}} S_f(T,K).$$

As a conclusion, unlike the optimality, the efficiency concept can't be described yet, hence can't be totally controlled from the mathematical point of view, but it can be ordered by specific optimizations, obtaining various types of efficiency that can be selected after that.

Let X be a non-empty set, made out of possible restrictions and let E be a linear space, ordered by a pointed and convex cone K, having $f: X \to E$ as application-objective function. We consider the following vectorial optimization program:

$$(P) \begin{cases} \min f(x) \\ x \in X \end{cases},$$

To solve it means to determine all the efficient points $x_0 \in X$ in the sense that $f(X) \cap [f(x_0) - K] = \{f(x_0)\}$. Hence, it is important to specify when the set of the mentioned efficient points is non-empty and what properties does it have. The notion of proper efficiency, introduced by Kuhn H. W. and Tucker A. W. (1950) and developed, including the adequate applications, by Geofrion A. M. (1968), Borwein J. M. (1977, 1983), Benson H. P. (1977, 1979, 1983), Hening M. I. (1982), Hartley R. (1978), Németh A. B. (1989), Dauer J. P., Gallagher R. J. (1990), Isac G. (1994), etc., appears as a particular case, i.e. the set of all efficient points, possibly positive, of problem (P) is a subset of the set of all efficient points. We mention that a point $x_o \in X$ is proper efficient for the program (P) if it is efficient and

$cl\left[cone(f(X)+K-\{f(x_o)\})\right] \cap K = \{0\}$; when there is at least a linear and continuous functional φ on E so that $\varphi(k) > 0$ for any $k \in K$ and $\varphi[f(x_o)] \leq \varphi[f(x)]$ for all $x \in X$, x_o is a positive proper solution for (P).

2. A New Approach: Approximate Efficiency, Efficiency and Optimization on Infinite Dimensional Ordered Linear Spaces with Applications

This section includes a recent generalization of the Efficiency represented by the Approximate Efficiency in Ordered Haussdorff Locally Convex Spaces (Postolică V., 2002). All the elements related to the Ordered Topological Linear Spaces are used following Peressini A. L., 1967.

Let E be a linear space, ordered by a convex cone K, K_1 a non-empty subset of K and A a non-empty set in E. The next definition introduced a new and the last until now concept of approximate efficiency that generalizes the Pareto type efficiency.

Definition 1. (Postolică V., 2002). $a_0 \in A$ is called K_1- *minimal efficient for* A, in notation, $a_0 \in eff\left(A, K, K_1\right)$ $\left(or\ a_0 \in MIN_{K+K_1}\left(A\right)\right)$ if it meets one of the following equivalent requirements:

(i) $A \bigcap \left(a_0 - K - K_1\right) \subseteq a_0 + K + K_1$;

(ii) $\left(K + K_1\right) \bigcap \left(a_0 - A\right) \subseteq -K - K_1$;

The definition of the maximal efficient points is obtained from the above one by replacing $K + K_1$ with $-\left(K + K_1\right)$. The following sequence is obvious:

$$A \bigcap \left(a_0 - K\right) \subseteq a_0 + K_1 \Rightarrow A \bigcap \left(a_0 - K - K_1\right) \subseteq a_0 + K + K_1 \Rightarrow$$
$$A \bigcap \left(a_0 - K_1\right) \subseteq a_0 + K,$$

that suggests other possible concepts of approximate efficiency in ordered linear spaces.

Remark 1. $a_0 \in eff\left(A, K, K_1\right)$ if and only if this element is a fixed point for the multifunction $F : A \to A$ defined by

$$F\left(t\right) = \left\{a \in A : A \bigcap \left(a - K - K_1\right) \subseteq t + K + K_1\right\}$$

that is, $a_0 \in F(a_0)$.

Hence, for the existence of efficient points we can apply fixed point theorems regarding multifunction (see, for example, Cardinali, T., Papalini, F., 1994, Zhang, Con-Jun, 2005 and any other related studies).

Remark 2. When cone K is pointed, i.e., $K\bigcap(-K)=\{0\}$, $a_0\in eff\,(A,K,K_1)$ means that $A\bigcap(a_0-K-K_1)=\varnothing$ or, equivalently, $(K+K_1)\bigcap(a_0-A)=\varnothing$ for $0\notin K_1$, respectively $A\bigcap(a_0-K-K_1)=\{a_0\}$, if $0\in K_1$. In the particular case $K_1=\{0\}$, we obtain the common concept of Pareto efficiency from Definition 1 and $a_0\in eff\,(A,K)$ $(or\ a_0\in MIN_K(A))$ if it satisfies (i), (ii) or any of the next equivalent properties:

(iii) $(A+K)\bigcap(a_0 - \mathrm{K})\subseteq a_0+K$;

(iv) $\mathrm{K}\bigcap(a_0-A-K)\subseteq -K$.

Consequently, a_0 is a fixed point for each of the following multifunctions:

$$F_1:A\to A,\ F_1(t)=\{\alpha\in A: A\bigcap(\alpha-K)\subseteq t+K\},$$
$$F_2:A\to A,\ F_2(t)=\{\alpha\in A: A\bigcap(t-K)\subseteq \alpha+K\},$$
$$F_3:A\to A,\ F_3(t)=\{\alpha\in A: (A+K)\bigcap(\alpha-K)\subseteq t+K\},$$
$$F_4:A\to A,\ F_4(t)=\{\alpha\in A: (A+K)\bigcap(t-K)\subseteq \alpha+K\},$$

i.e., $a_0\in F_i(a_0)$ for all $i=\overline{1,4}$. Thus, whenever the convex cone K is pointed, an element $\mathrm{a}_0\in A$ is efficient for the set A with respect to K if only if one of the following equivalent relations holds:

(v) $\mathrm{A}\bigcap(a_0 - K)=\{a_0\}$;

(vi) $A\bigcap(a_0-K\setminus\{0\})=\varnothing$;

(vii) $K\bigcap(a_0-A)=\{0\}$;

(viii) $(K\setminus\{0\})\bigcap(a_0-A)=\varnothing$;

(ix) $(A+K)\bigcap(a_0-K\setminus\{0\})=\varnothing$.

and we have $eff\,(A,K)=\bigcap\limits_{\{0\}\neq K_2\subseteq K} eff\,(A,K,K_2)$. Moreover, $a_0\in eff\,(A,K)$ only if it is *a critical point* (*point of balance*) (Isac, G.,1981,1983, Postolică, V., Scarelli, A.,Venzi, L., 2001, Postolică, V., 2004) for the generalized dynamical system $\Gamma:A\to 2^A$ defined by $\Gamma(a)=A\bigcap(a-K), a\in A.$ In this way, $eff\,(A,K)$ describes *the balance moments* for Γ which, *in the market context, expresses the competitive balance consisting of the general price-consumption*. Considering $K_1=\{\varepsilon\}$ (where $\varepsilon\in K\setminus\{0\}$), we obtain that $a_0\in eff\,(A,K,K_1)$ if and only if $A\bigcap(a_0-\varepsilon-K)=\varnothing$. In all these cases, for the set $eff\,(A,K,K_1)$

we use the notation $\varepsilon - eff\left(A,K\right)$ and it is obvious that $eff\left(A,K\right) = \bigcap_{\varepsilon \in K \setminus \{0\}} \left[\varepsilon - eff\left(A,K\right)\right].$

When referring to the existence of efficient points and significant properties of the set of these points we mention: Bucur, I. and Postolică, V. (1994), Isac, G. (1981,1983,1985,1994,1998), Isac, G., Postolică, V. (1993), Isac, G. and Bahya, A. O. (2002), Loridan, P. (1984), Luc, D.T. (1989), Németh, A. B. (1989), NG., K. F. and Zheng, X.Y. (2002), Postolică, V. (1993, 1995, 1996, 1999, 2002), Sonntag, Z. and Zălinescu, C. (2000), Sterna-Karwat, A. (1986), Truong, X. D. H. (1994) and others.

The following theorem shows an immediate connection between the approximate efficiency and the strong optimization. We use the notation

$$S\left(A,K,K_1\right) = \left\{a_1 \in A : A \subseteq a_1 + K + K_1\right\}$$

Theorem 1. (Postolică V., 2002*). If* $S\left(A,K,K_1\right) \neq \varnothing$, *then*

$$S\left(A,K,K_1\right) = eff\left(A,K,K_1\right).$$

Remark 3. The previous theorem shows that every time there is at least one strong minimum point, the set of all minimal efficient points coincides with the set of all these minimum points, the result being valid for the maximal points as well.

We also mention that, it is possible that $S\left(A,K,K_1\right) = \varnothing$ and $eff\left(A,K,K_1\right) = A.$

Thus, for example, if one considers $X = R^n (n \in N, n \geq 2)$ endowed with the separated H - locally convex topology (Precupanu, T., 1969, Kramar, E.,1981) generated by the seminorms $p_i : X \to R_+$, $p_i(x) = |x_i|, \forall\ x = (x_i) \in X$, $i = \overline{1,n}$, $K = R^n_+$, $K_1 = \{(0,...,0)\}$ and for each real number c we define $A_c = \left\{(x_i) \in X : \sum_{i=1}^{n} x_i = c\right\}$, then it is clear that $S\left(A_c, K, K_1\right)$ is empty and $eff\left(A_c, K, K_1\right) = A_c.$

In all the following cases we will consider X a Hausdorff locally convex space, with its topology generated by a family $P = \left\{p_\alpha : \alpha \in I\right\}$ of semi-norms, ordered by a convex cone K, with its dual X^*. In this context, the following theorem contains a significant criterion for the existence of the approximate efficient points, particularly, for the common efficient points, taking into account that the dual of cone K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$, and its polar is $K^0 = -K^*$. A convex cone K is called supernormal (nuclear) (Isac, G., 1981), Isac's cone (Postolică, V., 2009) if for any semi-norm $p_\alpha \in P$ there is $f_\alpha \in X^*$ so that $p_\alpha(k) \leq f_\alpha(k)$ for any $k \in K$. It is important to mention that the class of Isac's cones is the largest set of pointed, convex cones in Hausdorff locally convex spaces ensuring the existence of the efficient points under natural conditions (Isac, G., 1981, 1983, 1994, 1998, Isac, G., Bahya, A.O., 2002, Isac, G. Postolică, V, 1993, 2005, Postolică, V, 1993, 1995, 1997, 1999). For every function $\varphi : P \to K^*$, the convex cone $K_\varphi = \{x \in X : p_\alpha(x) \leq \varphi(p_\alpha)(x), \forall p_\alpha \in P\}$ represents the full nuclear cone attached to K, P and φ (Isac, G., Bahya, A. O., 2002) and K is an Isac's cone if and only if there exists

$$\varphi : \mathrm{P} \to \mathrm{K}^* \setminus \{0\} \text{ so that } \mathrm{K} \subseteq \mathrm{K}_\varphi.$$

Theorem 2. (Postolică V., 2002). *$a_0 \in eff(A, K, K_1)$ if for every $p_\alpha \in P$ and $\eta \in (0,1)$ there is x^* in the polar cone K^0 of the cone K, having the property that*

$$p_\alpha(a_0 - a) \leq x^*(a_0 - a) + \eta, \forall a \in A.$$

Theorem 3. (Isac, G. and Postolică, V. (2005)). *If $0 \in K_1$ and there is $\varphi : P \to K^* \setminus \{0\}$ cu $K \subseteq K_\varphi$, then*

$$eff(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \setminus \{0\}}} S(A \cap (a - K - K_1), K_\varphi)$$

for any non-empty subset K_1 of the cone K.

Corollary 1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary pointed, convex cone K and the dual K^* the next coincidence is valid:*

$$eff(A, K) = \bigcup_{\substack{a \in A \\ \varphi : P \to K^* \setminus \{0\}}} S(A \cap (a - K), K_\varphi)$$

Definition 2. A function $f : X \to R$ is called $(K + K_1)$- *increasing* if $f(x_1) \geq f(x_2)$ when $x_1, x_2 \in X$ and $x_1 \in x_2 + K_1 + K$.

It is clear that any increasing real function, defined on an arbitrary linear space, ordered by a convex cone *K*, is *K*+*K*$_1$ – increasing, for any non-empty subset *K*$_1$ of the cone *K*. As the notion of efficiency is fundamental for the

Multicriterial Optimization (Luc, D.T., 1989) that it generated as a first form of mathematical expression, the Choquet boundary is a fundamental concept for the Axiomatic Theory of Potential and the related applications (Boboc, N., Bucur, Gh., 1976, Choquet, G.,1955, 1957, 1962, 1963).This statement is used here to generalize the coincidence result obtained by Bucur I. and Postolică V. in 1994 between the set of all the minimal efficient points of any, non-empty set in an arbitrary, ordered Hausdorff locally convex space and the Choquet boundary of the same set with respect to the convex cone of all real continuous increasing functions, defined on that particular set, using the above new concept of approximate efficiency.

Theorem 4. (Postolică V., 2002,2008, 2009*) If A is a non-empty compact subset of X, and*

(i) K is a arbitrary closed, convex, pointed cone in X;

(ii) K_1 is a non-empty subset of K so that $K+K_1$ is a closed set.

Then, eff (A,K,K_1) coincides with the Choquet boundary of the set A in relation to the convex cone of all real functions, $K+K_1$ - increasing and continuous on A.

Therefore, eff (A,K,K_1) endowed with the trace topology is a Baire space and if (A,τ_A) is a metrical space, then eff (A,K,K_1) is a G_δ subset in X.

Corollary 2.

(i) $eff\,(A,K,K_1)=\{a\in A: f(a)=\sup\{f(a'): a'\in A\cap(a-K-K_1)\}$ *for all* $f\in C(A)\}$;

(ii) $eff\,(A,K,K_1)$ *and* $eff\,(A,K,K_1)\cap\{a\in A: s(a)\le 0\}$ $(s\in S)$ *are dense sets in relation to the Choquet topology;*

(iii) $eff\,(A,K,K_1)$ *is a dense subset in A.*

Remark 4. The previous theorem represents a significant connection between Vectorial Optimization and the Potential Theory and can't be obtained by using the Axiomatic Theory of Potential. This coincidence result offers new opportunities of determining and exploring the properties of the efficient point sets and, respectively, of Choquet boundaries. Generally speaking, determining the Choquet boundaries is difficult, while the density properties of the efficient points, in relation to various topologies, allow acceptable approximations of these.

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