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INVESTIGATIONS ON THE BISECTION PROBLEM

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Abstract. In this paper we characterize threshold graphs using the weakly decomposition, give a recognition algorithm for this class of graphs and an algorithm for the bisection problem in threshold graphs.

1. INTRODUCTION

Let $G = (V,E)$ be a graph with n vertices and m edges. Threshold graphs play an important role in graph theory as well as in several applied areas such as set-packing problem (Chvatal and Hammer [3]), parallel processing (Henderson and Zalcstein [10]), allocation problems (Ordman [16]).

The paper is organized as follows. In Section 2 we give notations and definitions. For the unity of the paper, in Section 3 we shortly remind the weakly decomposition [18]. In Section 4 we present the necessary and sufficient conditions for a graph to be a threshold, a recognition algorithm, an algorithm for the bisection problem.

2. NOTATIONS AND DEFINITIONS

Throughout this paper [1] $G = (V,E)$ is a simple (i.e. finite, undirected, without loops and multiple edges) graph. Let $co\text{-}G = \overline{G}$ denote the complement graph of G. For $U \subseteq V$ let $[U]$ (or $G(U)$) denote the subgraph of G induced by U. By G-X we mean the graph $[V$-$X]$, whenever $X \subseteq V$, but we shall often denote it simply by G-v $(\forall\, v \in V)$ when there is no ambiguity.

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A set A is totally adjacent (non adjacent) with a set B of vertices ($A \cap B = \phi$) if ab is (is not) an edge, for any a vertex in A and any b vertex in B; we denote by $A \sqcup B$ ($A \chi B$). A graph G is F-free if none of its induced subgraphs is in F.

3. The Weakly Decomposition

Here we recall the notions and the results (see [6], also [18]) that are necessary in the next section. For this we define the notion of weakly component and give a characterization for the weakly decomposition of a graph.

Definition 1. ([6], also [18]) *Let $G=(V,E)$ be a graph. A set of vertices, A, is called weakly set if $N_G(A)=V-A$ and the induced subgraph by A is connected. If A is a weakly set, maximal with respect to the inclusion, the subgraph induced by A is called weakly component. For simplification, the weakly component $G(A)$ will be denoted with A.*

The name of "weakly component" is justified by the next result.

Theorem 1. ([6], also [18]) *Any connected and incomplete graph $G=(V,E)$ admits a weakly component A such that* $G(V-A) = G(N(A)) + G(\overline{N(A)})$.

Theorem 2. ([6], also [18]) *Let $G=(V,E)$ be a connected and incomplete graph and $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim N(A)$.*

Definition 2. ([6], also [18]) *A partition $(A, N(A), V-A \cup N(A))$, where A is a weakly set, is called weakly decomposition of graph G in relation to A. We call: A the weakly component, $N(A)$ the minimal cutset, and $V-N(A)$ the remote set.*

The next result insures the existence of a weakly decomposition in a connected and incomplete graph.

Theorem 3. ([6], also [18]) *If $G=(V,E)$ is a connected and incomplete graph then the set of vertices V admits a weakly decomposition (A,B,C) such that $G(A)$ is a weakly component and $G(V-A)=G(B)+G(C)$.*

Theorem 2 provides an $O(n+m)$ algorithm for building a weakly decomposition for an incomplete and connected graph.

Algorithm for the weakly decomposition of a graph

Input: A connected graph with at least two nonadjacent vertices, $G=(V,E)$.

Output: A partition $V=(A,N,R)$ such that $G(A)$ is connected, $N=N(A)$, $A \sim R = N(A)$.

begin

A := any set of vertices such that

$A \cup N(A) \subset V$

$N := N(A)$

$R := V - A \cup N(A)$

while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
A:=$A \cup n$
N:=(N-{n}) $\cup$ ($N(n) \cap R$)
R:=R-($N(n) \cap R$)
end

One can observe that $[A]_G$ is connected, $N = N_G(A)$, $R \neq \phi$ is an invariant of the algorithm.

4. THRESHOLD GRAPHS

4.1. Basic properties

Threshold graphs

In this subsection we remind some results on threshold graphs.

A graph G is called threshold graph if $N_G(x) \subseteq N_G[y]$ or $N_G(y) \subseteq N_G[x]$ for any pair of vertices x and y in G.

Threshold graphs were first introduced by Chvatal and Hammer ([4]).

In [17], Ortiz and Villanueva-Ilufi give a structural characterization of threshold graphs for solving the following two difficult problems: enumeration of all maximal independent sets and the chromatic index problem.

Theorem 4. ([3]) A graph G is a threshold graph if and only if G does not contain a C_4, co- C_4, P_4 as an induced subgraph.

Chvatal and Hammer also showed that threshold graphs can be recognizing in $O(n^2)$ time.

Theorem 5. ([15], [3]) A graph G is a threshold graph if and only if G is a cograph and G is a split graph.

In [5] (as well as in [8] and [14]) linear algorithms for recognizing a cograph can be found.

Hammer and Simeone ([9]) give an O(n+m) algorithm for recognizing a split graph.

Therefore, an algorithm that recognizes a threshold graph is O(n(n+m)).

Characterization of a threshold graph using the weakly decomposition

In this paragraph we give a new characterization of threshold graphs using the weakly decomposition, that leads to a recognition algorithm whose complexity is O(n(n+m)).

Theorem 6. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a threshold graph if and only if:*

i) A~N~R

ii) N clique, S stable set

iii) G(A) is threshold graph.

Proof. Let G=(V,E) be a connected, incomplete graph and (A,N,R) a weakly decomposition of G, with G(A) as the weakly component.

At first, we assume that G is threshold. Then *N~R* and *A~N* also, as otherwise an in A, n in N would exists such that an$\notin$ E. Because N=N(A) it follows that there exists a_1 in A such that n a_1 $\in$ E. As G(A) is connected, a path P_{aa_1} exists. On the path from a to a_1 in P_{aa_1} , let a_2 in A the last vertex with $a_2n\notin$ E and a_3 in A the first vertex with $a_3n\in$ E. Then G({ a_2, a_3, n, r})~P_4, for every r in R, so i) holds.

If N would not be a clique then (as A~N~R) an induced C_4 would exists. This would be a contradiction, as G is threshold. So N is a clique and A~N~R. So ii) also holds.

Suppose that R is not stable. Then an edge r_1r_2 (r_1, r_2 in R) exists such that G({ r_1, r_2, a_1, a_2})~2 K_2, for every a_1 in A and every a_2 in A, as $|A| \geq 2$. Indeed, if $|A| = 1$ then because R is not stable there exists R'$\subseteq$R such that G(R') is connected. Suppose that R' is maximal with respect to inclusion. Then G(R') is a weakly component as R' is a weakly set (N_G(R')=N$\neq$ A$\cup$N$\cup$(R - R') = V - R', G(R') is connected) and R' is maximal with respect to inclusion. We have $|R'| > |A|$, contradicting the maximality of A. As A$\neq \phi$, it follows that $|A| \geq 2$. So R is stable.

As G is threshold we have that G(A) is threshold, so iii) holds, too.

Conversely, we suppose that i), ii) and iii) hold. If we suppose that X$\subset$V exists such that G(X) ~2 K_2 then, as A~N~R, N clique and R stable, it follows that X$\subset$A, contradicting that G(A) is threshold. If we suppose that G(X) ~ P_4 then X$\subseteq$A, contradicting iii). In a similar manner we can prove that G is C_4-free. So G is threshold.

Trivially perfect graphs

In this subsection we establish the necessary and sufficient conditions for a graph to be a trivially perfect graph.

Definition 3. ([12]) *A graph G is trivially perfect if for each induced subgraph H of G, the number of maximal cliques of H is equal to the maximum size an independent set of H.*

Theorem 7. ([12]) *A graph is trivially perfect if and only if it contains no vertex subset that induces P_4 or C_4.*

In [2], Brandstadt et al. establish:

Theorem 8. ([19]) *Let G=(V,E) be connected with at least two nonadjacent vertices and (A,N,R) a weakly decomposition with A weakly component. G is* $\{P_4, C_4\}$ *-free graph if and only if:*

i) A~N~R
ii) N is clique
iii) G(A), G(R) are $\{P_4, C_4\}$ *-free graphs.*

Theorem 9. *Let G=(V,E) be connected with at least two nonadjacent vertices and (A,N,R) a weakly decomposition with A weakly component. G is* $\{P_4, C_4\}$*-free graph if and only if:*
1) A~N~R
2) G(A),G(N), G(R) are $\{P_4, C_4\}$ *-free graphs.*

Proof. If *G* is $\{P_4, C_4\}$ *-free* graph then G(A), G(N), G(R) are $\{P_4, C_4\}$ *-free* graphs and *A~N~R.*

We suppose that 1) and 2) holds. Since A~N~R and G(A), G(N), G(R) are P_4-free graphs it follows that G is P_4-free. If $G \supseteq C_4$ then, because $A \square N \square R$ and $A\chi R$ it follows that either $C_4 \subseteq G(A)$ or $C_4 \subseteq G(N)$ or $C_4 \subseteq G(R)$, in contradiction with *2).*

4.2. The recognition algorithm

Remark 1. G is threshold graph if and only if G and co-G are trivial perfect graphs.

The above results (Remark 1) lead to the following recognition algorithm.

Input: A connected graph with at least two nonadjacent vertices, *G*=(*V*,*E*).

Output: An answer to the question: is G a threshold graph

begin
L:={G}; //L a list of graphs
While ($L \neq \phi$ *)*
    Extrage an element H from L;
    Find a weakly decomposition (A,N,R) for H;
    If ($A \chi N$ *or* $N \chi R$*) then*
        Return: G is not threshold
      else introduce in L, the connected components of G(A), G(N), G(R) incomplete
end;
L:={co-G};
While ($L \neq \phi$ *)*
    Extrage an element H from L;
    Find a weakly decomposition (A,N,R) for H;
    If ($A \chi N$ *or* $N \chi R$*) then*
        Return: co-G is not threshold

else introduce in L, the connected components of co-G(A), co-G(N), co-G(R) incomplete
Return: G is threshold
end

Remark 2. The most time consuming operation inside the *while* loop is the determination of the weakly decomposition (A,N,R), namely O(n+m). As the *while* body executes at most n times, it follows that the total execution time is O(n(n+m)).

4.3 An algorithm for the bisection problem in threshold graphs

A bisection of a graph G=(V,E) with an even number of vertices is a pair of disjoint subsets $V_1, V_2 \subset V$ of equal size. The cost of a bisection is the number of edges $(a,b) \in E$ such that $a \in V_1$ and $b \in V_2$. The problem of Graph Bisection takes as input a graph with an even number of vertices and return a bisection of minimum cost.

The graph partitioning problem is a well known NP-hard problem that has been successfully applied to many layout problems such as circuit board design, computer program segmentation and designing of hardware/software system architectures (see, for example, [7, 11,13]).

Considering the above results, the following algorithm solves the bisection problem for threshold graphs.

Input: G =(*V*, *E*), threshold graph with n vertices, n being even.
Output: Minimum cut size for the graph G

$X := \phi$;
$Y := \phi$;
While (G $\neq \phi$) do
If G is not complete then
Determine a weakly decomposition (A,N,R) for G with G(A) weakly component;
If $|A| < |R|$ *then*
$X := X \cup A$;
$Y := Y \cup R'$ *for* $R' \subset R$ *with* $|R'| = |A|$;
$G := G(N \cup (R - R'))$ *{G remain threshold}*
else
$X := X \cup A'$ *for* $A' \subset A$ *with* $|A'| = |R|$;
$Y := Y \cup R$;
$G := G(N \cup (A - A'))$ *{G remains threshold} ;*

Divide V(G) in N_1 and N_2 with $|N_1| = |N_2|$;

$X := X \cup N_1$;

$Y := Y \cup N_2$;

Display: (X,Y) is the bisection with minimum cut size.

Remark 3. The most time consuming operation inside the *while* loop is the determination of the weakly decomposition (A,N,R), namely O(n+m). As the *while* body executes at most n times, it follows that the total execution time is O(n(n+m)).

5. Conclusions and future work.

In this paper we characterize threshold graphs using the weakly decomposition, give a recognition algorithm for this class of graphs and an algorithm for the bisection problem in threshold graphs. Our future work concerns to give some applications of threshold.

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