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VERIFICATION OF THE NORMAL GRAPH CONJECTURE ON PARTICULAR CLASSES OF GRAPHS

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Abstract. A graph is normal if there exists a cross-intersecting pair of set families one of which consists of cliques while the other one consists of stable sets, and furthermore every vertex is obtained as one of these intersections. It is known that perfect graphs are normal. Korner and de Simone observed that C_5, C_7 and $\overline{C}_7$ are minimal not normal and conjectured, as generalization of the Strong Perfect Graph Theorem, that every $(C_5, C_7, \overline{C}_7)$-free graph is normal (Normal Graph Conjecture). In this paper we prove this conjecture for the class of minimal Asteroidal Triple (AT) graphs.

1. Introduction

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph ([1]), without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G.

A graph is called *triangulated* if it does not contain chordless cycles having the length greater or equal four.

A *circulant* C_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most $k(mod\ n)$ and $i \neq j$.

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When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally non-adjacent.

The structure of the paper is the following. In Section 2 we give an algorithm to find one. In Section 3 we present a new characterization of the AT graphs. In Section 4 we prove that the Normal Graph Conjecture is true for the minimal AT graphs.

2. Preliminary results

In [6], the authors study the linearity property of the Asteroidal Triple-free (AT-free) graphs, property that is also satisfied by the following classes: interval graphs, permutation graphs, and comparable graphs. AT-free graphs generalize the interval graphs and permutation graphs, which are comparable with respect to linearity. Interval graphs (extensively presented in [11]) appear in models related to real life situations that imply time dependencies or other linear restrictions. Such graphs arise in archeology, molecular biology, genetics, sociology. In [3], some mathematical models of population biology use interval graphs. Recent applications have been found in protein sequencing [13], DNA mapping [20], and macro substitution [9]. Other applications, on various complex optimization problems, are presented in ([3], [12], [17]). While Dirac [8] gives a theoretical characterization of triangulated graphs, Fulker-son and Gross [10] give a characterization with algorithms of triangulated graphs. For the recognition of weakly triangulated graphs, Berry, Bordat and Heggernes [2] ex-tended the characterization presented in [16] by Lekkerkerker and Boland.

We recall the notion of weakly component decomposition.

Definition 1. ([18], [19]) *A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A.*

Definition 2. ([18], [19]) *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.*

Below we remind a characterization of the weakly decomposition of a graph and give a new characterization for the unbreakable graphs.

In order to make the paper as clear as possible, we also present proofs of Theorems 1 and 2.

The name of "*weakly component*" is justified by the following result.

Theorem 1. ([18], [19]) *Every connected and non-complete graph $G = (V, E)$ admits a weakly component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 2. ([18], [19]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weakly decomposition (A, B, C), such that $G(A)$ is a weakly component and $G(V - A) = G(B) + G(C)$.*

Theorem 2 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

3. Characterization of AT graphs using neighborhoods

Definition 3. ([18]) *A graph G=(V,E) with at least three vertices is called confidentially connected if $\forall$ $(a, b, c) \in V^3$ three distinct vertices, there exists P an a, b-path in G such that $N[c] \cap V(P) \subseteq \{a, b\}$.*

Definition 4. *A graph , where every induced subgraph of does not have the confidential connectivity property is called CC-free.*

Definition 5. ([6]) *Three vertices in a graph determine an asteroidal triple if every two of them are joined through a path, avoiding the neighborhood of the third.*

Definition 6. ([6]) *A graph is called AT if any three vertices determine an asteroidal triple.*

Remark 1. *In an asteroidal triple, every two vertices are joined by a path whose intersection with the neighborhood of any other third vertex is empty, that is $\forall(a, b, c) \in V^3$: $N[c] \cap V(P_{ab}) = \emptyset$.*

The confidential connectivity property can be translated as follows: every two vertices are joined by a path whose intersection with the neighborhood of any other third vertex is either empty or an extremity or both extremities of the path, that is $\forall(a, b, c) \in V^3 :\ N[c] \cap V(P_{ab}) \subseteq \{a, b\}$.

Remark 2. *Every graph with the property that any three of its vertices form an asteroidal triple is confidentially connected.*

Remark 3. *A confidentially connected graph with the property that the extremities of every path do not intersect the neighborhood of any* *other third vertex, called in what follows is a graph where every three vertices form an asteroidal triple.*

Definition 7. *A graph is called if its vertices can by put in a one-to-one correspondence with a set of intervals on the real line such that two vertices are adjacent if and only if they correspond to non-disjoint intervals.*

Definition 8. *A graph is chordal if each of its cycles of four or more nodes has a chord, which is an edge joining two nodes that are not adjacent in the cycle.*

Theorem 3. *A graph is an interval graph if and only if it is chordal and strongly CC-free.*

Proof. In [15] Lekkerkerker and Boland demonstrated that a graph is an interval graph if and only if it is chordal and asteroidal triple-free. The conclusion of theorem 1 follows considering remarks 2 and 3. The following result is true for AT graphs. Some similar results are stated in [18], but for confidentially connected graphs.

Theorem 4. *Let be a connected and non-complete graph, with at least three vertices. Then is AT if and only if*
(i) $\forall v \in V\ [\overline{N}(v)]_G$ *is connected*
(ii) $\forall v \in V\ N(\overline{N}(v)) = N(v)$.

Proof. Let be AT. Then G is CC, that is (i) and (ii) holds. Converse, Suppose that (i) and (ii) holds and G is not AT. Then $\exists a, b, c \in V(G)$ such that $N[c] \cap V(P_{ab}) \neq \emptyset$, that is $a, b \in \overline{N}(c)$. Then $G(\overline{N}(c))$ is non-connected, we obtain again a contradiction to (i) for v=c.

4. The Normal Graph Conjecture for minimal AT graphs

In this section we give a new characterization of the AT graphs and show that the Normal Graph Conjecture is true for the minimal AT graphs.

As Theorem 2 provides an $O(n+m)$ algorithm for building a weakly decomposition for a non-complete and connected graph, it follows that step 1 of the algorithm above is $O(n \cdot (n + m))$. Because steps 2 and 3 perform in smaller time, it follows that the complexity of the recognition algorithm for unbreakable graphs is $O(n \cdot (n + m))$.

Definition 9. *A graph G is called normal if G admits a clique cover $\mathcal{C}$ and a stable set cover $\mathcal{S}$ such that every clique in $\mathcal{C}$ intersects every stable set in $\mathcal{S}$.*

Normal graphs form an superclass of perfect graphs and can be seen as closure of perfect graphs by means of co-normal products ([14]) and graph entropy ([7]). Perfect graphs have been characterized as those graphs without odd holes and odd antiholes (**Strong Perfect Graph Theory** ([4])). By analogy, Korner and de Simone stated that the non-existence of the three smallest odd holes and odd antiholes implies normality ([15]) and formulated the following conjecture.

Normal Graph Conjecture. Graphs without any C_5, C_7 or $\overline{C}_7$ as induced subgraphs are normal.

Proving or disproving the Normal Graph Conjecture is subject of investigations.

Wagler proved that Normal Graph Conjecture is true for the circulant graphs ([21]).

In this paper we show that Normal Graph Conjecture is true for the minimal AT graphs.

We denote:

$$P(G,v): \begin{cases} i)\overline{G}(N_G(v))\, and\, G(N_{\overline{G}}(v))\, are\, connected\, graphs\,, \forall v \in V(G) \\ ii)N_G(N_{\overline{G}}(v)) = N_G(v)\, si\, N_{\overline{G}}(N_G(v)) = N_{\overline{G}}(v), \forall v \in V(G). \end{cases}$$

Corollary 2. *A graph G is unbreakable if and only if $P(G,v)$ holds for every $v \in V(G)$.*

Definition 10. *A graph G is called minimal AT if G is AT and none of its proper induced subgraphs is AT.*

Theorem 5. *If G is a minimal AT graph with at least eight vertices then G is a normal graph.*

Proof. Let G be a minimal AT graph. We show that G is C_k or $\overline{C}_k$ for some $k \geq 5$. Let $v \in V(G)$ and $\mathcal{H} = \{H | H$ is a subgraph induced of G, containing v such that $P(H,v)$ holds $\}$. Then $\mathcal{H} \neq \emptyset$, because $G \in \mathcal{H}$, from the hypothesis and from Theorem 4. Let H be a minimal member of $\mathcal{H}$.

Let $A = \{a | a \in N_{\overline{H}}(v),\ H(N_{\overline{H}}(v)) - a$ is connected $\}$.

Similarly, let $B = \{b | b \in N_H(v),\ \overline{H}(N_H(v)) - b$ is connected $\}$.

Then, $|A| = |B| = 2$, which means that $H(N_{\overline{H}}(v))$ and $\overline{H}(N(H(v))$ are isomorphic to P_k and P_l respectively, with $k, l \geq 2$. Let $P_k = [a_1, \ldots, a_k]$, $P_l = [b_1, \ldots, b_l]$, such that a_1 is adjacent only to b_1, a_k is adjacent only to b_l. If $k \geq 3$ and $l \geq 3$ simultaneously, then $C = [b_1, a_1, \ldots, a_k, b_l]$ is a cycle induced in G, with at least five vertices, which means it is unbreakable and because $v \notin C$ we have that G is not minimal unbreakable. We conclude that either $l = 2$ or $k = 2$. For $l = 2$ it follows that $G = C_{k+3} = [v, b_1, a_1, \ldots, a_k, b_l]$. For $k = 2$ it follows that $G = C_{l+3}$.

If $G = C_{2p}$ then, taking $\mathcal{S} = \{\{1, 3, \ldots, 2p-1\}, \{2, 4, \ldots, 2p\}\}$, a covering with stable sets and $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2p-1, 2p\}\}$, a clique covering, we conclude that C_{2p} is a normal graph.

It is well known that $G = C_{2p+1}$ is a normal graph for $p \geq 4$ ([17]). It follows that C_s $(s \geq 8)$ is a normal graph. As $\overline{C}_s$ $(s \geq 8)$ is a normal graph, it follows that G is a normal graph.

5. Conclusions and future work

In this paper we proved that Normal Graph Conjecture is true for minimal Asteroidal Triple graphs. Our future work concerns the verification of Normal Graph Conjecture for other classes of graphs, which are particular classes of (α, ω)-partitionable graphs.

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