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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 24(2014), No. 1, 5-16

EINSTEIN EQUATIONS IN LIE ALGEBROIDS

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Abstract. A Lie algebroid endowed with a Riemannian metric has a canonical connection of Levi-Civita type. We associate to it the Einstein tensor field and using it we construct the Einstein equations. Some particular cases are discussed.

Introduction

When the physical space-time is modeled by a four Lorentz manifold (M, g), the Einstein equation $Ric - 1/2Rg = 8k\pi T$, where Ric means the Ricci tensor and R is the scalar curvature, provides a relationship between the geometry of space-time given by g and the matter and energy encoded by the tensor T. The Einstein equation can be written for any dimension and any signature. Moreover, there are a lot of other generalizations of it obtained replacing the Levi-Civita connection with a connection with torsion or replacing M with the manifold TM in the so-called Finsler-Lagrange field theories ([6]) or with the total space of a vector bundle,[1].

Keywords and phrases: Lie algebroids, pseudo-Riemannian metrics, metrical connections, Einstein equations .
(2010) Mathematics Subject Classification: 53C15, 53C07, 53Z05

In the construction of the Einstein equation the main ingredient is a linear connection ∇ in TM compatible with g i.e. $\nabla g = 0$ and without torsion. When one tries to extend this construction to any vector bundle E endowed with a pseudo-Riemannian metric g, the compatibility condition $\nabla g = 0$ does not uniquely determine ∇ since is not possible to associate to ∇ a torsion. This difficulty is overcame if we confine ourselves to a particular class of vector bundles : Lie algebroids.

The notion of Lie algebroid had appeared in several contexts, see [5] and the various geometrical structures associated to it were studied, for instance, in [3, 4, 7].

In this paper we describe the construction of the Einstein equation for any Lie algebroid and we discuss some particular cases. Although of geometrical interest, the simplicity of this construction diminishes the hope for strong applications to Physics. More elaborate constructions applied to particular Lie algebroids with supplementary structures proved to be more useful to this aim, see [8], [9]

1. Lie algebroids

1.1. Preliminaries on vector bundles. Let $\xi = (E, \pi, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$, and $\pi : E \to M$ is a smooth submersion. The fibres $E_x = \pi^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}$ with the bijections $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha(u) = (\pi(u), \varphi_{\alpha,\pi(u)})$, where $\varphi_{\alpha,\pi(u)} : E_{\pi(u)} \to \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}_{\alpha \in A}$ on E.
Here $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\Phi_\alpha(u) = (\psi_\alpha(\pi(u)), \varphi_{\alpha,\pi(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^n$ and if (U_β, ψ_β) is another local chart such that $x \in U_\alpha \cap U_\beta \neq \phi$, we set $\psi_\beta(x) = \widetilde{x}^i$ and then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n), \ \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{1.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi^{-1}_{\alpha,x}(e_a) := \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ has the form $u = y^a \varepsilon_a(x)$.

We take (x^i, y^a) as coordinates on E. For the bundle chart $(U_\beta, \varphi_\beta, \mathbb{R}^m)$ we put $\varphi^{-1}_{\beta,x}(e_a) = \widetilde{\varepsilon}_a(x)$ and then $u = \widetilde{y}^a \widetilde{\varepsilon}_a(x)$. If we set $\varepsilon_a(x) = M^b_a(x)\widetilde{\varepsilon}_b$ with $\operatorname{rank}(M^b_a(x)) = m$ it follows that $\widetilde{y}^a = M^a_b(x) y^b$. Thus the mapping $\Phi_\beta \circ \Phi^{-1}_\alpha$ has the form

$$\begin{aligned} \widetilde{x}^i &= \widetilde{x}^i(x^1, \cdots, x^n), \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ \widetilde{y}^a &= M^a_b(x) y^b, \operatorname{rank}(M^a_b(x)) = m. \end{aligned} \tag{1.2}$$

The indices $i, j, k, ...$ and $a, b, c...$ will take the values $1, 2, ...n$ and $1, 2, ...m$, respectively. The Einstein convention on summation is implied.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\chi(M)$, respectively $\Gamma(E), \chi(E)$ the module of sections of the tangent bundle of M, respectively of the bundle ξ and of the tangent bundle of E. We recall that the vertical bundle $VE \to E$ is the union of the fibres $V_uE = \ker \pi_{*,u}$ over $u \in E$, where $\pi_{*,u}$ is the differential of π. A basis of local section of $VE \to E$ is given by $\left(\left.\frac{\partial}{\partial y^a}\right|_u\right)$.

1.2 Lie algebroids.

Let be $\xi = (E, \pi, M)$ a vector bundle and let us assume that

(i) The space of its sections $\Gamma(\xi)$ is endowed with a Lie algebra structure $[,]$;
(ii) There exists a bundle map $\rho : E \to TM$ (called the *anchor map*) and this induces a Lie algebra homomorphism (also denoted by ρ) from $\Gamma(\xi)$ to $\chi(M)$.
(iii) For any smooth functions f on M and any sections $s_1, s_2 \in \Gamma(\xi)$ the following identity is satisfied

$$[s_1, f s_2] = f[s_1, s_2] + (\rho(s_1) f) s_2.$$

Definition 1.1. The triplet $A = (\xi, [,], \rho)$ with the properties (i), (ii) and (iii) is called a Lie algebroid.

Examples:

(1) The tangent bundle $(TM, \tau, M)$with the usual Lie bracket and ρ equal to the identity map form a Lie algebroid.
(2) Any integrable subbundle of TM with the Lie bracket defined by restriction and ρ the inclusion map is a Lie algebroid.

(3) Let (F, q, M) be any vector bundle. On F we have the vertical distribution $u \longrightarrow V_u F = \mathrm{Ker} q_{*,u}$, $u \in F$, where q_* denotes the differential of q. This distribution is integrable. If we regard it as a subbundle of TF, accordingly to Example 2 a Lie algebroid is obtained.

Locally, we set

$$\rho(s_a) = \rho_a^i(x)\frac{\partial}{\partial x^i}, \quad [s_a, s_b] = L_{ab}^c(x)s_c. \tag{1.3}$$

A semispray S for the tangent bundle $\tau : TM \to M$ is a vector field on TM which at the same time is a section in the vector bundle $\tau_* : TTM \to TM$, that is we have $\tau_{TM}(S(u)) = u$ and $\tau_{*,u}(S(u)) = u$, $\forall u \in TM$, where τ_{TM} is the vector bundle projection $TTM \to TM$. It follows that $\tau_{*,u}(S(u)) = \tau_{TM}(S(u))$, $\forall u \in TM$.

This equation suggests the following

Definition 1.2. *Let $A = (\xi = (E, p, M), [,], \rho)$ be a Lie algebroid. A vector field S on E will be called a semispray if*

$$p_{*,u}(S(u)) = (\rho \circ \tau_E)(S(u)), \ \forall u \in E \tag{1.4}$$

where $\tau_E : TE \to E$ is the natural projection.

Let $c : I \to M$, $I \subseteq \mathbb{R}$ be a curve on M and let $\widetilde{c} : I \to E$ be any curve on E such that $p \circ \widetilde{c} = c$. Denote by $\dot{\widetilde{c}}$ the vector field that is tangent to $\widetilde{c}$.

Definition 1.3. *We say that $\widetilde{c}$ is* **admissible** *if*

$$\pi_*(\dot{\widetilde{c}}) = \rho(\widetilde{c}).$$

An admissible curve will be also called an $A-$ path. Its projection on M will be called the base path of it. The $A-$path $\widetilde{c}$ is called vertical if $\rho(\widetilde{c}(t)) = 0$. In this case the curve c reduces to a point and the curve $\widetilde{c}$ is contained in the fibre in that point.

In local charts on M and E, we have $c(t) = (x^i(t))$, $\widetilde{c}(t) = (x^i(t), y^a(t))$ and $\dot{\widetilde{c}}(t) = \dfrac{dx^i}{dt}\dfrac{\partial}{\partial x^i} + \dfrac{dy^a}{dt}\dfrac{\partial}{\partial y^a}$, $t \in I$.

It results

Lemma 1.1. *The curve $\widetilde{c}$ is admissible if and only if*

$$\frac{dx^i}{dt}(t) = \rho_a^i(x(t))y^a(t), \ \forall t \in I \tag{1.5}$$

and it is a vertical $A-$path if and only if

$$\rho_a^i(x(t))y^a(t) = 0, \qquad t \in [0,1]. \tag{1.6}$$

Again in local charts, let be $S = X^i \dfrac{\partial}{\partial x^i} + Y^a \dfrac{\partial}{\partial y^a}$ a vector field on E.

This is a semispray if and only if

$$X^i(x,y) = \rho_a^i(x)y^a. \tag{1.7}$$

By (1.7) and Lemma 1.1 easily follows that a vector field on E is a semispray if and only if all its integral curves are admissible curves.

2. Connections in Lie algebroids

Let $A = (\xi, [,], \rho)$ be a Lie algebroid with $\xi = (E,p,M)$ and let (F,q,M) be any vector bundle.

Definition 2.1 An $A-$connection in the bundle (F,q,M) is a mapping $D : \Gamma(E) \times \Gamma(F) \longrightarrow \Gamma(F)$, $(s,\sigma) \longrightarrow D_s\sigma$ with the properties:

1) $D_{s_1+s_2}\sigma = D_{s_1}\sigma + D_{s_2}\sigma$,
2) $D_{fs}\sigma = fD_s\sigma$,
3) $D_s(\sigma_1+\sigma_2) = D_s\sigma_1 + D_s\sigma_2$,
4) $D_s(f\sigma) = \rho(s)f\sigma + fD_s\sigma$,

for $s, s_1, s_2 \in \Gamma(E)$, $\sigma, \sigma_1, \sigma_2 \in \Gamma(F)$, $f \in \mathcal{F}(M)$.

Notice that a $TM-$connetion in the vector bundle (F,q,M) is nothing but a linear connection in this vector bundle. And a $TM-$connetion in the tangent bundle is a linear connection on M.

Definition 2.2 An $A-$connection in the bundle $\xi = (E,p,M)$ is called a linear connection in the Lie algebroid A.

Let (σ_α), $\alpha, \beta, \gamma, \ldots = k$:=rank of (F,q,M), a local basis in $\Gamma(F)$. Then a local section σ has the form $\sigma = z^\alpha\sigma_\alpha$ and (z^α) are the coordinates in the fibres of (F,q,M).

For $s = y^a s_a$ and $\sigma = z^\alpha\sigma_\alpha$, by the Definition 2.1 we have $D_s\sigma = y^a\left(\rho_a^i \dfrac{\partial z^\alpha}{\partial x^i} + z^\alpha D_{s_a}\sigma_\alpha\right)$ and if we put

$$D_{s_a}\sigma_\alpha = \Gamma_{\alpha a}^\beta \sigma_\beta, \tag{2.1}$$

we get

$$D_s\sigma = y^a(D_a z^\beta)\sigma_\beta, \qquad D_a z^\beta = \rho_a^i \frac{\partial z^\beta}{\partial x^i} + \Gamma_{\alpha a}^\beta z^\alpha. \tag{2.2}$$

For a linear connection $\mathcal{D}$ in the Lie algebroid $A = (\xi, [,], \rho)$ we get

$$\mathcal{D}_s\sigma = y^a(\mathcal{D}_a z^b)s_b, \qquad \mathcal{D}_a z^b = \rho^i_a \frac{\partial z^b}{\partial x^i} + \Gamma^b_{ca} z^c. \tag{2.3}$$

3. Riemannian metrics in Lie algebroids

Let $A = (\xi, [,], \rho)$ be a Lie algebroid with $\xi = (E, p, M)$ and a vector bundle (F, q, M) endowed with an $A-$ connection D whose local coefficients are $(\Gamma^\alpha_{\beta a})$.

A Riemannian metric in (F, q, M) is a mapping g that assigns to any $x \in M$ a scalar product g_x in F_x such that for any local section $\sigma_1, \sigma_2 \in \Gamma(F)$, the function $x \to g_x(\sigma_1, \sigma_2)$ is smooth. Locally, we set $g_x(\sigma_\alpha, \sigma_\beta) = g_{\alpha\beta}(x)$ and so $g_x(\sigma_1, \sigma_2) = g_{\alpha\beta}(x) z^\alpha_1 z^\beta_2$ if $\sigma_1 = z^\alpha_1 \sigma_\alpha$, $\sigma_2 = z^\beta_2 \sigma_\beta$.

The operator of covariant derivative D can be extended to the tensor algebra of (F, q, M) taking $D_\sigma f = \rho(\sigma) f$, assuming that it commutes with the contractions and behaves like a derivation with respect to tensor product. It comes out that if ω is a section in the dual (F^*, q^*, M) then

$$(D_s\omega)(\sigma) = \rho(s)\omega(\sigma) - \omega(D_s\sigma), \quad s \in \Gamma(E), \quad \sigma \in \Gamma(F)$$

and if g is a section in $L^2(F, \mathbb{R})$, then

$$\begin{aligned}(D_s g)(\sigma_1, \sigma_2) = \rho(s) g(\sigma_1, \sigma_2) - g(D_s\sigma_1, \sigma_2) - g(\sigma_1, D_s\sigma_2),\\ s \in \Gamma(E), \sigma_1\sigma_2 \in \Gamma(F).\end{aligned} \tag{3.1}$$

Definition 3.1. We say that the Riemannian metric g is compatible with the $A-$connection D if $D_s g = 0$ for every $s \in \Gamma(E)$.

By (3.1) the condition of compatibility between g and D is equivalent to

$$\begin{aligned}\rho(s) g(\sigma_1, \sigma_2) = g(D_s\sigma_1, \sigma_2) + g(\sigma_1, D_s\sigma_2),\\ s \in \Gamma(E), \quad \sigma_1, \sigma_2 \in \Gamma(F).\end{aligned} \tag{3.2}$$

Locally, (3.2) is written as follows

$$\rho^i_a(x) \frac{\partial g_{\alpha\beta}}{\partial x^i} = \Gamma^\gamma_{\alpha a}(x) g_{\gamma\beta}(x) + \Gamma^\gamma_{\beta a}(x) g_{\alpha\gamma}(x). \tag{3.3}$$

If (F, q, M) coincides with (E, p, M) we have

Theorem 3.1. *There exists an unique linear connection ∇ in the Lie algebroid A such that*

(i) $\nabla_s g = 0$,

(ii) $\nabla_{s_1} s_2 - \nabla_{s_2} s_1 = [s_1, s_2],\ s, s_1, s_2 \in \Gamma(E)$.

It is given by the formula

$$\begin{aligned} 2g\left(\nabla_{s_1} s_2, s_3\right) =& \rho(s_1)g(s_2, s_3) + \rho(s_2)g(s_1, s_3) - \rho(s_3)g(s_1, s_2) \\ &+ g([s_3, s_1], s_2) + g([s_3, s_2], s_1) + g([s_1, s_2], s_3) \end{aligned} \tag{3.5}$$

and its local coefficients are given by

(3.6)

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\rho^i_b \frac{\partial g_{cd}}{\partial x^i} + \rho^i_c \frac{\partial g_{bd}}{\partial x^i} - \rho^i_d \frac{\partial g_{bc}}{\partial x^i} + L^e_{dc} g_{eb} + L^e_{db} g_{ec} - L^e_{bc} g_{ed}\right).$$

Proof. In the condition (i) written for $s_1, s_2, s_3 \in \Gamma(E)$ we cyclically permute s_1, s_2, s_3 and so we obtain two new identities. We add these and from the result we subtract the first. Using (ii) some terms cancel each other and we get (3.5). Writing (3.5) in a local basis of sections we find (3.6). The uniqueness follows by contradiction.

If we put

$$T_\nabla(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2], \quad s_1, s_2 \in \Gamma(E) \tag{3.7}$$

we get a section in the bundle $L(E, E; E)$ that may be called the torsion of ∇.

The connection ∇ given by the Theorem 3.1 is called the Levi-Civita connection of A.

We stress that the Theorem 3.1 says that given g there exists and is unique ∇ such that $\nabla g = 0$ and $T_\nabla = 0$.

Now we give a different proof of this theorem.

Given g we may associate to it the energy function $\mathcal{E} : E \to \mathbb{R}$, $\mathcal{E}(s) = g(s, s)$, $s \in E$. Locally, $\mathcal{E}(x, y) = g_{ab}(x) y^a y^b$, $s = y^a s_a$.

The energy function $\mathcal{E}$ is a regular Lagrangian on E i.e. $\det\left(\dfrac{1}{2}\dfrac{\partial^2 \mathcal{E}}{\partial y^a \partial y^b}\right) = \det(g_{ab}(x)) \neq 0$.

In [2], we associated to any regular Lagrangian L on a Lie algebroid a semispray on E i.e. a vector field

$$S = \rho^i_a y^a \frac{\partial}{\partial x^i} - 2G^a_L(x, y)\frac{\partial}{\partial y^a}$$

with

$$G^a_L = \frac{1}{4} g^{ab}\left(\frac{\partial^2 L}{\partial y^b \partial x^i}\rho^i_c y^c - \rho^i_b \frac{\partial L}{\partial x^i} - L^c_{bd} y^d \frac{\partial L}{\partial y^c}\right), \tag{3.8}$$

where $g_{ab} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^a \partial y^b}$ and (g^{ab}) is the inverse of the matrix (g_{ab}).

Taking $L = \mathcal{E}$ in (3.8), a direct calculation in which $L^a_{cd} y^c y^d = 0$ is used, shows that the semispray associated to $\mathcal{E}$ has the form

$$S = \rho^i_a y^a \frac{\partial}{\partial x^i} - \Gamma^a_{cd}(x) y^c y^d \frac{\partial}{\partial y^a}, \tag{3.9}$$

with Γ^a_{cd} given by (3.6). These coefficients determines ∇. They are symmetric in bottom indices, hence $T_\nabla = 0$. The uniqueness of ∇ follows by contradiction.

By (3.9) it follows

Theorem 4.2 *The integral curves of S are just the geodesics of the Levi-Civita connection ∇ in the Lie algebroid A.*

4. Einstein equations in Lie algebroids

Let $A = (\xi, [.,.], \rho)$ be a Lie algebroid with $\xi = (E, p, M)$ endowed with a metric g which is non-degenerate but of arbitrary signature. The Theorem 3.1 still holds for this metric (called pseudo- Riemannian or semi-Riemannian) and we shall denote by ∇ the linear connection given by (3.5) or equivalently by (3.6). The torsion of ∇ vanishes. The curvature of ∇ is defined by

(4.1)
$$R(s_1, s_2)s_3 = \nabla_{s_1}\nabla_{s_2}s_3 - \nabla_{s_2}\nabla_{s_1}s_3 - \nabla_{[s_1,s_2]}s_3, \quad s_1, s_2, s_3 \in \Gamma(E).$$

The following properties of R_∇ are immediate :

- $R(s_1, s_2)s_3 = -R(s_2, s_1)s_3$.
- $R(s_1, s_2)s_3 + R(s_2, s_3)s_1 + R(s_3, s_1)s_2 = 0$.

In a local basis $\{\sigma_a\}$ of sections in ξ, we obtain

$$R(\sigma_a, \sigma_b)\sigma_c = R_c{}^e{}_{ab}\sigma_e, \tag{4.2}$$

where

$$R_c{}^e{}_{ab} = \rho^i_a \frac{\partial \Gamma^e_{cb}}{\partial x^i} - \rho^i_b \frac{\partial \Gamma^e_{ca}}{\partial x^i} + \Gamma^d_{cb}\Gamma^e_{da} - \Gamma^d_{ca}\Gamma^e_{db} - L^d_{ab}\Gamma^e_{cd}. \tag{4.3}$$

The contraction $C\left(\begin{smallmatrix} e \\ b \end{smallmatrix}\right)$ provides the Ricci tensor field $\mathrm{Ric}(s, \sigma)$ whose local coordinates are given by

$$\mathrm{Ric}(\sigma_c, \sigma_a) = \mathrm{Ric}_{ca} = R_c{}^b{}_{ab}. \tag{4.4}$$

The function R locally given by

$$R(\sigma_c, \sigma_a) = g^{ca}\mathrm{Ric}_{ca}, \tag{4.5}$$

is called the scalar curvature of ∇.

Definition 4.1. The equation

$$\mathrm{Ric} - \frac{1}{2}Rg = 8\pi\kappa T \tag{4.6}$$

is called the Einstein equation.

In the equation (4.6), the left hand side is the Einstein curvature tensor E that is constructed using the metric g. On the right hand side we have a tensor field T called the stress-energy-momentum tensor and represents the matter and energy that generate the gravitational field of potentials (g_{ab}). The constant κ is the gravitational constant. Locally, (4.6) looks as follows

$$\mathrm{Ric}_{ab} - \frac{1}{2}Rg_{ab} = 8\pi\kappa T_{ab}. \tag{4.6'}$$

In the empty space (no matter, no energy) we have $T_{ab} = 0$. Contracting (4.6') with g^{ab} one yields $R = 0$ and so it simplifies to

$$\mathrm{Ric}_{ab} = 0. \tag{4.6''}$$

We set $E_{ab} = \mathrm{Ric}_{ab} - \frac{1}{2}Rg_{ab}$ and $E^a_b = g^{ac}E_{cb}$. The divergence of E is defined by

$$\mathrm{div}E = E^a_{b;a}, \tag{4.7}$$

where ; denotes the covariant derivative and we have

Lemma 4.1. $\mathrm{div}E = 0$.

The proof is standard. It is based on the second Bianchi identity

$$(\nabla_{s_3}R)(s_1, s_2) + (\nabla_{s_1}R)(s_2, s_3) + (\nabla_{s_3}R)(s_3, s_1) = 0,$$

written in a basis (σ_a) of local sections in E.

By Lemma 4.1 necessarily we must have

$$\mathrm{div}T = 0 \text{ (assuming the Einstein equation holds).} \tag{4.8}$$

The equation (4.8) is called the continuity condition.

Remarks.

(1) The standard framework of the theory of the gravitational field is obtained when $E = TM$, $\rho =$identity, $[,]$ is the usual bracket of vector fields and g is a pseudo-Riemannian (Lorentz) metric on M.

(2) Let $\xi = (VTM, \tau_V, TM)$ be the vertical bundle over TM and $i : VTM \to TTM$ the inclusion map. Then $(\xi, i, [,])$, where $[,]$ is the usual bracket of the vertical vector fields is a Lie algebroid. A pseudo-Riemannian (Lorentz) metric g in it is nothing but a generalized Lagrange metric as this was defined in [6]. If (x^i, y^i) are local coordinates on TM, a local basis of sections of ξ is $\left(\dfrac{\partial}{\partial y^i}\right)$, $i = 1, 2, ..., n$ and the components of g are given by the matrix $(g_{ij}(x,y)) := \left(g\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right)\right)$. By the general formula (3.6), the local coefficients of the Levi-Civita connection ∇ of g are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{hj}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h} \right). \tag{4.9}$$

The curvature S of ∇ remembers the vertical curvature from the theory of generalized Lagrange metrics and the Einstein equation derived from it refers to vertical part. When g is a Finsler metric, the condition Ric$= 0$ implies the vanishing of the curvature tensor i.e. ∇ is flat. Thus a Finslerian gravitational theory using only the vertical curvature fails.

(3) Let E be a $k-$ dimensional subbundle of TM. This is called also a distribution of rank k on M. This distribution is involutive if is closed under the usual bracket $[.,,]$ of vector fields. In this case $A = (\xi, i, [.,.])$ with $\xi = (E, \pi, M)$ and $i : E \mapsto TM$ the inclusion as anchor is a Lie algebroid. By the Frobenius theorem the involutive distribution $E \subset TM$ is integrable and so every point of M is contained in a leaf of dimension k of it. A pseudo-Riemannian metric in the Lie algebroid A defines a pseudo-Riemannian metric on each leaf. The Einstein equation is then living on such a leaf immersed in M. For instance, in the dimension four if $k = 3$ the leaves could be the slices $t = constant$ and one gets a gravitational field theory in a three

dimensional space. A kind of converse of the Kaluza-Klein procedure of extending the number of dimensions,appears.

Acknowledgemets The first author was partially supported by a grant of the Romanian National Authority for Scientific Research, CNSS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0256.

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