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ON CONFORMAL TRANSFORMATION OF A QUARTIC FINSLER SPACE

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Abstract. In this paper we consider the conformal transformation of a quartic Finsler space. We obtain the conformal change of Cartan's connection.

1. Introduction

Let M be an n-dimensional, real C^∞ manifold and $F : TM \to R_+$, $F(x,y) = \sqrt[4]{a_{hijk}(x)\, y^h y^i y^j y^k}$ be the fundamental function of the Finsler space $QF^n = (M, F)$, called quartic Finsler space. In [2] Knebelman initiated the conformal theory of Finsler spaces and Matsumoto and Hashiguchi developed this theory. The conformal change is defined as $\overline{F}(x,y) = e^{\sigma(x)} F(x,y)$, where $\sigma(x)$ is a function of position only, known as conformal factor.

In the present paper we investigate the conformal change for the quartic Finsler spaces. We obtain the relations between Cartan connection associated to $QF^n = (M,F)$ and Cartan connection associated to $Q\overline{F}^n = \left(M, \overline{F}\right)$.

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2. **Preliminaries**

Let M be an n-dimensional, real C^{∞} manifold. Denote by (TM, τ, M) the tangent bundle of M and let $F^n = (M, F(x,y))$ be a Finsler space, with the fundamental function

(2.1) $$F : TM \to R_+,\ F(x,y) = \sqrt[4]{a_{hijk}(x)\, y^h y^i y^j y^k}\ ,$$

where $a_{hijk}(x)$ are the components of a symmetric covariant tensor field of order 4.The manifold M equipped with the metric(1.1) is called a quartic Finsler space and is denoted by QF^n. We define the tensors $a_{ijk}(x,y)$, $a_{jk}(x,y)$ and $a_k(x,y)$ as follows:

(2.2) $$\begin{cases} a_{hijk}(x)\, y^h = F a_{ijk}(x,y) \\ a_{hijk}(x)\, y^h y^i = F a_{jk}(x,y) \\ a_{hijk}(x)\, y^h y^i y^j = F a_k(x,y) \end{cases} .$$

The normalized supporting element $l_i = \frac{\partial F}{\partial y^i}$ is

(2.3) $$l_i = a_i$$

and the angular metric tensor $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$ is

(2.4) $$h_{ij} = 3\,(a_{ij} - a_i a_j)\,.$$

The fundamental tensor of the QF^nspace is

(2.5) $$g_{ij} = 3a_{ij} - 2a_i a_j.$$

The contravariant tensor g^{ij} is given by

(2.6) $$g^{ij} = \tfrac{1}{3}\left(a^{ij} + 2a^i a^j\right),$$

where $a^{ij} = (a_{ij})^{-1}$and $a^i = a^{ir} a_r = l^i$.

The Cartan covariant tensor C with the components $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is written as

(2.7) $$C_{ijk} = \tfrac{3}{F}\left(a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_i a_j a_k\right).$$

The generalized Christoffel symbols of the metric (1.1) are

(2.8) $$\Gamma^s_{hijk} = \tfrac{1}{6} a^{sp}\left(\frac{\partial a_{ijkp}}{\partial x^h} + \frac{\partial a_{jkph}}{\partial x^i} + \frac{\partial a_{kphi}}{\partial x^j} + \frac{\partial a_{iphj}}{\partial x^k} - \frac{\partial a_{hijk}}{\partial x^p}\right)$$

From (1.5) we obtain

(2.9) $$F^2 = g_{ij} y^i y^j = 3a_{ij} y^i y^j - 2a_i y^i a_j y^j.$$

We also have

(2.10) $$a_i y^i = \tfrac{1}{F^3} a_{ijkh} y^j y^k y^h y^i = \tfrac{1}{F^3} F^4 = F.$$

So,

(2.11) $$F^2 = 3a_{ij} y^i y^j - 2F^2,$$

or,

(2.12) $$F^2 = a_{ij} y^i y^j.$$

The tensor $a_{ij}(x,y)$ is called the basic tensor, because we can construct a Finsler connection based on a_{ij} instead of g_{ij}.

Theorem 1. *In a quartic Finsler space QF^n there exists an unique connection $C\Gamma = \left(G^i_j, F^i_{jk}, U^i_{jk}\right)$ which verifies the following axioms:*

a) $\nabla^h a_{ij} = 0$;
b) $\nabla^v a_{ij} = 0$;
c) $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$;
d) $S^i_{jk} = U^i_{jk} - U^i_{kj} = 0$;
e) $D^i_j = y^i \mid_j = 0$.

This connection has the coefficients expressed by

$$U^i_{jk} = C^i_{jk} + \frac{2}{3F}\left(a_{jk} - a_j a_k\right) a^i \tag{2.13}$$

and

$$F^i_{jk} = a^{is}\left[f_{sjk} - \frac{1}{F}\left(G^i_k\left(a_{jsi} - a_{js}a_i\right) + G^i_j\left(a_{skj} - a_{sk}a_j\right) - G^i_s\left(a_{kjs} - a_{kj}a_s\right)\right)\right], \tag{2.14}$$

where

$$f_{sjk} = \frac{1}{2}\left(\frac{\partial a_{sj}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^s} - \frac{\partial a_{ks}}{\partial x^j}\right). \tag{2.15}$$

3. **Main results**

Let $QF^n = (M, F)$ and $Q\overline{F}^n = \left(M, \overline{F}\right)$ be two quartic Finsler space on the same manifold M. If there exists a scalar field $\sigma(x)$ satisfying $\overline{F}(x, y) = e^{\sigma(x)} F(x, y)$, then the change $F \longrightarrow \overline{F}$ is called conformal.

Proposition 2. *By a conformal transformation, the Finsler quartic space QF^n is transformed to a Finsler quartic space $Q\overline{F}^n$ and we have*

$$\begin{cases} \overline{a}_{hijk} = e^{4\sigma(x)} a_{hijk} \\ \overline{a}_{ijk} = e^{3\sigma(x)} a_{ijk} \\ \overline{a}_{jk} = e^{2\sigma(x)} a_{jk} \\ \overline{a}_k = e^{\sigma(x)} a_k \end{cases} \tag{3.1}$$

and

$$\begin{cases} \overline{a}^{hijk} = e^{-4\sigma(x)} a^{hijk} \\ \overline{a}^{ijk} = e^{-3\sigma(x)} a^{ijk} \\ \overline{a}^{jk} = e^{-2\sigma(x)} a^{jk} \\ \overline{a}^k = e^{-\sigma(x)} a^k \end{cases} \tag{3.2}$$

From (2.8), (3.1) and (3.2), by a direct calculation we get the following result

Proposition 3. *Under the given conformal transformation $F \longrightarrow \overline{F}$, the Christoffel symbols change as follows:*

$$\overline{\Gamma}^p_{hijk} = e^{\sigma}\Gamma^p_{hijk} + \frac{4}{6}\left(\frac{\partial\sigma}{\partial x^h} a_{ijkp} + \frac{\partial\sigma}{\partial x^i} a_{jkph} + \frac{\partial\sigma}{\partial x^j} a_{kphi} + \frac{\partial\sigma}{\partial x^k} a_{iphj} - \frac{\partial\sigma}{\partial x^p} a_{hijk}\right). \tag{3.3}$$

We give the main result concerning the conformal change of Cartan's connection.

Theorem 4. *The conformal change of Cartan's connection of a Finsler quartic space is given by*

$$\left\{\begin{array}{l}\overline{F}^i_{jk} = F^i_{jk} + U^i_{jk}\\ \overline{C}^i_{jk} = C^i_{jk}\end{array}\right. , \tag{3.4}$$

where

$$\begin{aligned} U^i_{jk} = & \left(\delta^i_j \frac{\partial\sigma}{\partial x^k} + \delta^i_k \frac{\partial\sigma}{\partial x^j} - a_{jk}\sigma^i\right) + \\ & + \frac{1}{12}\left[4\left(a^t_{rj}\sigma^r a^i_{kt} + a^t_{rk}\sigma^r a^i_{jt} - a^i_{rt}\sigma^r a^t_{jk}\right) - \right. \\ & -8\left(\frac{\sigma_0 a^i_{jk}}{F} - a^i_{rj}\sigma^r a_k - a^i_{rk}\sigma^r a_j + a_{rjk}\sigma^r a^i\right) + \\ & \left. +2\left(\delta^i_j a_r\sigma^r a_k + \delta^i_k a_{rj}\sigma^r - a_{jk}\sigma^i\right)\right] \end{aligned} \tag{3.5}$$

Proof. From (1.15), (2.1) and (2.2) we find the conformal transformation of f_{ijk}:

$$\overline{f}_{ijk} = e^{2\sigma}\left(f_{ijk} + a_{ij}\tfrac{\partial\sigma}{\partial x^k} + a_{jk}\tfrac{\partial\sigma}{\partial x^i} - a_{ki}\tfrac{\partial\sigma}{\partial x^j}\right). \tag{3.6}$$

Contracting this expression with y^k, y^j and y^i successively we get

$$\overline{f}_{ij0} = e^{2\sigma}\left(f_{ij0} + \sigma_0 a_{ij} + F\tfrac{\partial\sigma}{\partial x^i}a_j - F\tfrac{\partial\sigma}{\partial x^j}a_i\right) \tag{3.7}$$

$$\overline{f_{i00}} = e^{2\sigma}\left(f_{i00} + F^2\tfrac{\partial\sigma}{\partial x^i}\right) \tag{3.8}$$

$$\overline{f_{000}} = e^{2\sigma}\left(f_{000} + F^2\sigma_0\right) \tag{3.9}$$

Contracting (3.6) with $y^i y^k$ we obtain

$$\overline{f}_{0j0} = e^{2\sigma}\left(f_{0j0} + 2\sigma_0 a_j - F^2\tfrac{\partial\sigma}{\partial x^j}\right) \tag{3.10}$$

From (3.6), (3.7), (3.8), (3.9), (3.10), (3.1) and (3.2) we obtain the conformal change of F^i_{jk} and C^i_{jk}. ∎

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