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DIFFERENTIABILITY OF MONOTONE SOBOLEV FUNCTIONS ON METRIC SPACES

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Abstract. We prove a differentiability result for monotone Sobolev functions on doubling metric measure spaces supporting a Poincaré inequality. This generalizes a result used by Rickman in proving the differentiability of quasiregular mappings. Our main tools are a Stepanov differentiability theorem in doubling metric measure spaces supporting a Poincaré inequality, proved in 2004 by Balogh, Rogovin and Zürcher and a Sobolev embedding theorem on spheres proved by Hajłasz and Koskela.

As an application, it is shown that continuous quasiminimizers for the $p-$ energy integral with $p > Q - 1$ are almost everywhere Cheeger differentiable, where Q is the doubling exponent of the underlying metric measure space.

Keywords and phrases: doubling metric measure space, Poincaré inequality, function monotone in the sense of Lebesgue, Sobolev function, Cheeger differentiability, $p-$energy integral, quasiminimizer.
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1. Introduction

The study of Sobolev spaces and $p-$harmonic functions on metric measure spaces led to the definition of a concept of differentiability in the setting of metric measure spaces. In his seminal paper [2] Cheeger proved that every metric space with a doubling measure supporting a Poincaré inequality admits a strong measurable differentiable structure, with which Lipschitz functions can be differentiated almost everywhere. We recall that quasiconformal theory and nonlinear potential theory have been successfully extended to doubling metric measure spaces supporting a Poincaré inequality.

Balogh, Rogovin and Zürcher [1] used Cheeger's extension of Rademacher differentiability theorem to prove a generalization of this result, the following extension of Stepanov's differentiability theorem.

Theorem 1.1.[1] *Let (X, d, μ) be a doubling metric measure space. Assume that there exists a strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}$ for (X, d, μ) with respect to $LIP(X)$, such that the sets X_α are mutually disjoint. Then each function $f : X \to \mathbb{R}$ is $\mu-$a.e. differentiable in $S(f) := \{x \in X : Lip\, f(x) < \infty\}$ with respect to the structure $\{(X_\alpha, \varphi_\alpha)\}$.*

We will assume that (X, d, μ) is a doubling metric measure space, where the measure μ is Borel regular positive and finite on ball, $Q \geq 1$ is a homogeneous dimension of X and $1 \leq p < \infty$.

If a function u belongs to the Sobolev space $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain and $p > n$, then u is differentiable almost everywhere in Ω, by Cesari-Calderón theorem [16, Lemma VI.4.1]. The applications of Stepanov's differentiability theorem given in [1] include the following extension of Cesari-Calderón theorem to doubling metric spaces: if $u : X \to \mathbb{R}$ is a measurable function and $g \in L^p_{loc}(X)$, with $p > Q$, such that the pair (u, g) satisfies a weak $(1, p)-$Poincaré inequality, then $Lip\, u(x) < \infty$ at every Lebesgue point of g^p [1, Theorem 4.1]. In particular, u is differentiable almost everywhere in X. Moreover, u has a locally $(1 - Q/p)$ $-$Hölder continuous representative.

The condition $p > n$ is not necessary for the differentiability of functions $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain. Rickman [16, Lemma VI.4.4] proved that every monotone Sobolev function $u \in W^{1,p}(\Omega)$ with $p > n - 1$ is differentiable almost everywhere, using a method of Väisälä [19] which is an $n-$dimensional version of a technique used by Gehring and Lehto [4]. An important consequence of this result is the differentiability a.e. of quasiregular mappings [16, Theorem I.2.4], that was proved for the first time by Reshetnyak [14].

Onninen [13] proved a sharp integrability condition on the partial derivatives of a weakly monotone Sobolev function on a domain in $\mathbb{R}^n$, that guarantees the differentiability a.e. of the function. Unfortunately, the methods from the proofs of Rickman and Onninen are unlikely to be extended to the setting of metric measure spaces.

In this paper we extend Rickman's result to doubling metric measure spaces supporting a Poincaré inequality, using Theorem 1.1 and a Sobolev embedding theorem on spheres proved by Hajłasz and Koskela [6, Theorem 7.1].

2. Preliminaries

There are several definitions for the monotonicity of real functions in domains in $\mathbb{R}^n$, definitions that can be extended to the setting of metric measure spaces.

Let $\Omega \subset \mathbb{R}^n$ be a domain. A continuous function $u : \Omega \to \mathbb{R}$ is said to be *monotone in the sense of Lebesgue* if $\sup\limits_D u \le \sup\limits_{\partial D} u$ and $\inf\limits_D u \ge \inf\limits_{\partial D} u$ for every bounded domain D with $\overline{D} \subset \Omega$. The continuity of u implies $\sup\limits_D u \ge \sup\limits_{\partial D} u$ and $\inf\limits_D u \le \inf\limits_{\partial D} u$ for every set D with $\overline{D} \subset \Omega$. Every continuous and open real function on a domain $\Omega \subset \mathbb{R}^n$ is monotone in the sense of Lebesgue.

Onninen [13] says that a continuous function $u : \Omega \to \mathbb{R}$ is monotone in the sense of Lebesgue if

$$\sup_B u \le \sup_{\partial B} u \text{ and } \inf_B u \ge \inf_{\partial B} u. \tag{1}$$

for each ball B with $\overline{B} \subset \Omega$. This definition is less restrictive than the above one.

For every real function u on a set B from a topological space, $\sup\limits_{\overline{B}} u = \max\left\{\sup\limits_B u,\ \sup\limits_{\partial B} u\right\}$ and $\inf\limits_{\overline{B}} u = \min\left\{\inf\limits_B u,\ \inf\limits_{\partial B} u\right\}$. Inequalities (1) show that the restriction to $\overline{B}$ of a monotone continuous function u satisfies a maximum principle and a minimum principle.

Let u be a real function in a set X. For every nonempty set $A \subset X$, the oscillation of u on A is defined by $osc(u, A) = \sup_{x,y \in A} |u(x) - u(y)|$. Note that $osc(u, A) = \sup_A u - \inf_A u$.

In [16] a continuous function $u : U \to \mathbb{R}$ in an open set $U \subset \mathbb{R}^n$ is called monotone if

$$osc\,(u, D) = osc\,(u, \partial D) \tag{2}$$

for every bounded domain D with $\overline{D} \subset U$. Note that $osc\,(u, D) \geq osc\,(u, \partial D)$ for every continuous function $u : U \to \mathbb{R}$ and each set D with $\overline{D} \subset U$.

If $u : U \to \mathbb{R}$ is continuous and monotone in the sense of Lebesgue, then u satisfies (2).

Let (X, d) be a metric space. The ball $B\,(x, r)$ and the sphere $S\,(x, r)$ of center $x \in X$ and radius $r > 0$ are defined by $B\,(x, r) = \{y \in X : d\,(y, x) < r\}$, respectively $S\,(x, r) = \{y \in X : d\,(y, x) = r\}$. By the continuity of the distance $d(x, \cdot)$ on X, we have $\partial B\,(x, r) \subset S\,(x, r)$. It is possible to have $S\,(x, r)$ empty or to have $S\,(x, r)$ different from $\partial B\,(x, r)$, for example in discrete spaces.

In [6, p. 34] it is said that a function $u : X \to \mathbb{R}$ is monotone if

$$osc\,(u, B\,(x, r)) \leq osc\,(u, S\,(x, r)) \tag{3}$$

for every $x \in X$ and $r > 0$. It is implicitely assumed that $S\,(x, r)$ is non-empty.

We will say that the metric space (X, d) has the (NESS) property if small spheres are nonempty, i.e. for each $x_0 \in X$ there exists $R\,(x_0) > 0$ such that $S\,(x, r)$ is nonempty wehenever $0 < r \leq R\,(x_0)$.

In the setting of metric spaces with (NESS) property we will replace inequalities (1) by

$$\sup_{B(x,r)} u \leq \sup_{S(x,r)} u \text{ and } \inf_{B(x,r)} u \geq \inf_{S(x,r)} u, \tag{4}$$

where $x \in X$ and $0 < r < R\,(x)$.

Definition 2.1. Assume that the metric space (X, d) has (NESS) property. Let $\Omega \subset X$ be an open set and $u : \Omega \to \mathbb{R}$. We say that u is locally monotone (respectively, locally monotone in the sense of Lebesgue) at $x_0 \in \Omega$ if there exists a positive number $R\,(u, x_0)$ such that (3) holds (respectively, inequalities (4) hold for all $0 < r <$

$R(u, x_0)$. We say that $u : \Omega \to \mathbb{R}$ is locally monotone (locally monotone in the sense of Lebesgue) if it is locally monotone (locally monotone in the sense of Lebesgue) at each point $x_0 \in X$.

Obviously, every function that is locally monotone in the sense of Lebesgue is also locally monotone.

We will need a connection between strong extremum principles and the property of a function to be locally monotone in the sense of Lebesgue.

Lemma 2.2. *Let X be a proper metric space with (NESS) property. Let $\Omega \subset X$ be an open set and $u : \Omega \to R$ be a continuous function. If the restriction of u to an arbitrary ball $B \subset \Omega$ has neither a global maximum, nor a global minimum, then $u : \Omega \to R$ is locally monotone in the sense of Lebesgue.*

Proof. Let $x_0 \in X$ and let $R = R(x_0)$ be a positive number such that, for each $0 < r < R$, the sphere $S(x_0, r)$ is nonempty and the closed ball $D(x_0, r) = B(x_0, r) \cup S(x_0, r) \subset \Omega$, for . Since X is proper, $D(x_0, r)$is compact.

The restriction of the continuous function u to $D(x_0, r)$ has a global maximum $u(x_M)$ and a global minimum $u(x_m)$. If $x_M \in B(x_0, r)$, then the restriction of u to $B(x_0, r)$ has a global maximum, namely $u(x_M)$. This contradiction shows that $x_M \in S(x_0, r)$, therefore

$$\sup_{B(x_0,r)} u \leq \sup_{D(x_0,r)} u = u(x_M) = \sup_{S(x_0,r)} u.$$

Similarly, $\inf\limits_{B(x_0,r)} u \geq \inf\limits_{S(x_0,r)} u$. $\square$

In what follows, we assume that (X, d, μ) is a metric measure space, i.e. the metric space (X, d) is equipped with a measure μ, that is assumed to be Borel regular, positive and finite on balls. If $B := B(x, r)$ and $\sigma > 0$ then σB stands for $B(x, \sigma r)$.

In order to describe the scaled oscillations of a function $u : X \to \mathbb{R}$ which is not necessarily Lipschitz, we use the upper Lipschitz constant defined by

$$Lip\, u(x) = \limsup_{r \to 0} \frac{1}{r} \sup_{y \in B(x,r)} |u(x) - u(y)|.$$

Note that $Lip\,u(x) = \limsup\limits_{\substack{y\to x\\ y\neq x}} \frac{|u(x)-u(y)|}{d(x,y)}$ if x is a limit point of X and $Lip\,u(x) = 0$ if x is isolated ([2], [1]).

Let $1 \leq p < \infty$. The $p-$modulus of a family of paths Γ in X, denoted by $Mod_p(\Gamma)$, is the number $\inf\limits_{\rho} \int\limits_X \rho^p d\mu$, where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ we have $\int\limits_{\gamma} \rho ds \geq 1$ [17].

The concept of upper gradient is a basic one in the "first-order calculus" on metric measure spaces, playing the role of a substitute for the length of the gradient of a real-valued C^1-function on an Euclidean domain. A non-negative Borel measurable function g is said to be an *upper gradient* of $u : X \to \mathbb{R}$ in X if for all rectifiable paths $\gamma : [a, b] \to X$ the following inequality holds

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int\limits_{\gamma} g ds. \tag{5}$$

A non-negative Borel measurable function g is called a *$p-$weak upper gradient* of u if (5) holds for all rectifiable paths $\gamma : [a, b] \to X$ except a family of zero $p-$modulus [17].

The Newtonian spaces $N^{1,p}(X)$, $1 \leq p < \infty$ [17] are Sobolev-type spaces on metric measure spaces, based on the notion of (weak) upper gradient. Let $\widetilde{N}^{1,p}(X)$ be the collection of all real-valued $p-$integrable functions u on X that possess a $p-$integrable $p-$weak upper gradient. This space can be endowed with the seminorm $\|u\|_{\widetilde{N}^{1,p}} := \|u\|_p + \inf \|g\|_p$, where the infimum is taken over all $p-$integrable $p-$weak upper gradients g of u. If u and v are functions in $\widetilde{N}^{1,p}$, we set $u \sim v$ if $\|u - v\|_{\widetilde{N}^{1,p}} = 0$. Then $\sim$ is an equivalence relation. The quotient space $N^{1,p}(X) := \widetilde{N}^{1,p}(X)/\sim$ equipped with the norm $\|u\|_{N^{1,p}} := \|u\|_{\widetilde{N}^{1,p}}$, is the *Newtonian space* corresponding to the index p. Let Ω be an open subset of X. The Newtonian space $N^{1,p}(\Omega)$ is defined in an obvious way. As in [11], we say that a function $u : \Omega \to \mathbb{R}$ belongs to the *local Newtonian space* $N^{1,p}_{loc}(\Omega)$ if $u \in L^p_{loc}(\Omega)$ and u has a $p-$weak upper gradient $g \in L^p_{loc}(\Omega)$.

The metric measure space (X, d, μ) is said to be *doubling* if there is a constant $C_d \geq 1$ so that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)) \tag{6}$$

for every ball $B(x, r)$ in X. By the doubling condition (6) there exist some constants $C_b > 0$ and $Q \geq 0$ such that

$$\frac{\mu(B(x, r))}{\mu(B(x_0, r_0))} \geq C_b \left(\frac{r}{r_0}\right)^Q \tag{7}$$

whenever $x \in B(x_0, r_0)$ and $0 < r \leq r_0$. Every such Q is called a *homogeneous dimension (a doubling exponent)* of the given metric measure space.

The metric space (X, d) is said to be *proper* if every closed bounded subset of X is compact. Under the assumption that μ is doubling, X is proper if and only if it is complete [5, Theorem 4.5].

Let $u \in L^1_{loc}(X)$ and g be a measurable non-negative function on X. Let $p > 0$. The pair (u, g) is said to *satisfy a weak* $(1, p)-$*Poincaré inequality* if there exist some constants $C_P > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C_P r \left(\frac{1}{\mu(\tau B)} \int_{\tau B} g^p d\mu \right)^{1/p} \tag{8}$$

for every ball $B = B(x, r)$ in X. Here $u_B = \frac{1}{\mu(B)} \int_B u d\mu$ [6, page 9].

We say that the metric measure space (X, d, μ) *supports a weak* $(1, p)-$*Poincaré inequality* (for locally integrable functions) if there exist some constants $C_P > 0$ and $\tau \geq 1$ such that a pair (u, g) satisfies (8) whenever $u \in L^1_{loc}(X)$ and g is an upper gradient of u.

If (X, d, μ) supports a weak $(1, p)-$Poincaré inequality and $u \in L^1_{loc}(X)$ has a $p-$integrable $p-$weak upper gradient g then (8) holds, since g is the limit in $L^p(X)$ of a sequence of upper gradients (g_n) of u and each pair (u, g_n) satisfies (8).

Remark 2.3. If (X, d, μ) supports a weak $(1, p)-$Poincaré inequality for some $p > 0$ then (X, d, μ) supports a weak $(1, q)-$Poincaré inequality for every $q > p$, by Hölder inequality.

Remark 2.4. In a metric measure space supporting a Poincaré inequality every ball whose complement is non-empty has a non-empty boundary, in particular the metric space has (NESS) property [18, p. 25].

The essence of Cheeger's study on the infinitesimal behavior of Lipschitz functions on doubling metric measure spaces can been synthetized in the following theorem, that extends Rademacher differentiability theorem.

Theorem 2.5. [8], [1] *Let (X, d, μ) be a doubling metric measure space supporting a weak $(1, p)-$Poincaré inequality for some $1 \leq p < \infty$. Then there exists a countable collection $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ of measurable sets $X_\alpha \subset X$ with positive measure and Lipschitz coordinates $\varphi_\alpha = (\varphi_\alpha^1, ..., \varphi_\alpha^{N(\alpha)}) : X \to R^{N(\alpha)}$, with the following properties:*

(i) $\mu(X \setminus \bigcup_{\alpha\in\Lambda} X_\alpha) = 0$;

(ii) There exists a non-negative integer N such that $N(\alpha) \leq N$ for each $\alpha \in \Lambda$;

(iii) If $f : X \to R$ is Lipschitz , then for each $(X_\alpha, \varphi_\alpha)$ there exists a unique (up to a set of zero measure) measurable bounded vector valued function $d^\alpha f : X_\alpha \to R^{N(\alpha)}$ such that

$$\lim_{\substack{y\to x\\ y\neq x}} \frac{|f(y) - f(x) - d^\alpha f(x) \cdot (\varphi_\alpha(y) - \varphi_\alpha(x))|}{d(y, x)} = 0, \tag{9}$$

for $\mu-$almost every $x \in X_\alpha$, where " $\cdot$ " is the usual inner product on $R^{N(\alpha)}$.

If a metric measure space (X, d, μ) satisfies the conclusion of Theorem 2.5, it is said that the space admits a strong measurable differentiable structure and the collection $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is said to be a *strong measurable differentiable structure* for the space.

A function $f : X \to \mathbb{R}$ (not necessarily Lipschitz) is said to be *Cheeger differentiable* at $x \in X_\alpha$, with respect to the strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ if there exists a vector $d^\alpha f(x) \in \mathbb{R}^{N(\alpha)}$ such that (9) holds for f at x [1].

3. Differentiability of monotone Sobolev functions

The main tool we use in extending Rickman's lemma to doubling metric measure spaces is a Sobolev embedding theorem on spheres proven by Hajłasz and Koskela ([6], Theorem 7.1):

Lemma 3.1. [6] *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension Q. Assume that the pair (u, g) satisfies a weak $(1,p)-$Poincaré inequality for some $p > Q-1$, where u is in $L^1_{loc}(X)$ and g is a measurable non-negative function on X. Let $x_0 \in X$ and let $r_0 > 0$ such that $S(x_0, r)$ is nonempty for every $0 < r \leq r_0$. Then:*

(i) The restriction of u to $S(x_0, r)$ is uniformly $(1-(Q-1)/p)-$Hölder continuous for almost every $0 < r < r_0$;

(ii) There exist a constant $C_1 > 0$, depending only on p, Q, C_P, C_b, C_d and a radius $r_0/2 < r < r_0$ such that:

(10)
$$|u(x) - u(y)| \leq C_1 d(x,y)^{1-(Q-1)/p} r_0^{(Q-1)/p} \left(\frac{1}{\mu(B(x_0, 5\tau r_0))} \int_{B(x_0, 5\tau r_0)} g^p d\mu \right)^{1/p}$$

for every $x, y \in S(x_0, r)$.

In what follows, we say that a function $u : X \to \mathbb{R}$ is monotone if
$$osc(u, B(x_0, r)) \leq osc(S(x_0, r)),$$
for every $x_0 \in X$ and every $r > 0$ such that $S(x_0, r)$ is non-empty (see [6], page 36).

Theorem 3.2. *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension Q. Assume that for every point $x_0 \in X$ there exists $R(x_0) > 0$ such that $S(x_0, r)$ is non-empty for every $0 < r \leq R(x_0)$. Let $u \in L^1_{loc}(X)$ be a locally monotone function and $g \in L^p_{loc}(X)$ be a nonnegative function, where $1 \leq p < \infty$. If $p > Q-1$ and the pair (u, g) satisfies a weak $(1,p)-$Poincaré inequality, then $Lip\, u(x) < \infty$ for $\mu-$ a.e. $x \in X$.*

If X admits a strong measurable differentiable structure, then u is differentiable $\mu-$ a.e. with respect to this structure.

Proof. Fix $x_0 \in X$. If x_0 is isolated, then $Lip\, u(x_0) = 0 < \infty$. Assume that x_0 is a limit point of X.

Let $R := R(u, x_0)$ be a positive number as in Definition 2.

For every $z \in B(x_0, R/2)$ there exists a positive integer $k = k(z)$ such that

$$2^{-k-1}R \leq d(z, x_0) < 2^{-k}R. \tag{11}$$

Apply to u Lemma 3.1 with $r_0 = 2^{-k+1}R$. There exists a radius r with $2^{-k}R < r < 2^{-k+1}R$ such that (10) holds for every x, $y \in S(x_0, r)$. It follows that

$$osc(u, S(x_0, r)) \leq 2C_1 r \left(\frac{1}{\mu(B(x_0, 5\tau r_0))} \int\limits_{B(x_0, 5\tau r_0)} g^p d\mu \right)^{1/p}. \tag{12}$$

Since u is a locally monotone function, $|u(z) - u(x_0)| \leq osc(u, B(x_0, r)) \leq osc(u, S(x_0, r))$. Using this inequality together with (11) and (12) we get

$$\frac{|u(z) - u(x_0)|}{d(z, x_0)} \leq 2^{2k+1} C_1 \left(\frac{1}{\mu(B(x_0, 5\tau r_0))} \int\limits_{B(x_0, 5\tau r_0)} g^p d\mu \right)^{1/p}.$$

From (11) and the choice of r_0 it follows that
$B(x_0, 5\tau r_0) \subset \mu(B(x_0, \lambda d(z, x_0))) \subset B(x_0, 10\tau r_0)$, where $\lambda = 20\tau$. By the doubling property of μ we obtain $\mu(B(x_0, \lambda d(z, x_0))) \leq C_d \mu(B(x_0, 5\tau r_0))$. Therefore,

$$\frac{|u(z) - u(x_0)|}{d(z, x_0)} \leq C_3 \left(\frac{1}{\mu(B(x_0, \lambda d(z, x_0)))} \int\limits_{B(x_0, \lambda d(z, x_0))} g^p d\mu \right)^{1/p}, \tag{13}$$

where $C_3 = 2^{2k+1} C_1 \left(C_d\right)^{-1/p}$.

Assume that x_0 is a Lebesgue point of g^p. Taking lim sup for $z \to x_0$, $z \neq x_0$ in (13) we obtain

$$Lip\, u(x_0) \leq C_3 g(x_0) < \infty.$$

The second claim follows by Stepanov differentiability theorem, Theorem 1.1. □

Corollary 3.3. *Assume that* (X, d, μ) *is a doubling metric measure space supporting a weak* $(1, p)-$*Poincaré inequality with* $1 \leq p < \infty$. Let Q *be a homogeneous dimension of* (X, d, μ). *If* $p > Q - 1$ *then every locally monotone function* $u \in N^{1,p}_{loc}(X)$ *satisfies* $Lip\, u(x) < \infty$ *for* $\mu-$ *a.e.* $x \in X$, *in particular it is differentiable* $\mu-$*a.e. with respect to any strong measurable differentiable structure for* (X, d, μ).

Proof. Let $u \in N^{1,p}_{loc}(X)$, $1 \leq p < \infty$. Then $u \in L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$ and there exists a weak $p-$ upper gradient $g \in L^p_{loc}(X)$ of u. Since (X, d, μ) supports a weak $(1, p)-$Poincaré inequality, the metric space (X, d) has the (NESS) property and the pair (u, g) satisfies a weak $(1, p)-$Poincaré inequality. If, in addition, u is locally monotone and $p > Q - 1$, then all the assumptions of Theorem 3.2 are met. □

4. On the regularity of quasiminimizers

If $1 < p < \infty$ then each function $u \in N^{1,p}(X)$ has a minimal $p-$integrable $p-$weak upper gradient in X, denoted by g_u, in the sense that if g is another $p-$weak upper gradient of u, then $g_u \leq g$ $\mu-$a.e.in X [2, Theorem 2.18].

Let Ω be an open subset of X. If $u \in N^{1,p}_{loc}(\Omega)$ with $1 < p < \infty$, then u has a minimal $p-$weak upper gradient g_u in Ω, in the following sense: whenever D is an open set with $\overline{D} \subset \Omega$, if $g_{u,D}$ is a minimal $p-$weak upper gradient of u in D, then $g_u = g_{u,D}$ $\mu-$a.e.in D [10].

The $p-$*capacity* of a set $E \subset X$ is defined by $C_p(E) = \inf_u \|u\|^p_{N^{1,p}}$, where the infimum is taken over all functions $u \in N^{1,p}(X)$ with $u = 1$ on E. The *Newtonian space with zero boundary values* $N^{1,p}_0(E)$ is the set of functions $u : E \to \mathbb{R}$ for which there exists a function $\widetilde{u} \in N^{1,p}(X)$ such that $\widetilde{u} = u$ $\mu-$almost everywhere in E and

$$C_p(\{x \in X \setminus E : \widetilde{u}(x) \neq 0\}) = 0.$$

A representative in $N^{1,p}_{loc}(\Omega)$ of a function $u \in N^{1,p}_{loc}(\Omega)$ is obtained by modifying u on a set of zero $p-$capacity.

In what follows, $1 < p < \infty$ and $\Omega \subset X$ is an open set.

A function $u \in N^{1,p}_{loc}(\Omega)$ is said to be a *quasiminimizer* (for the Dirichlet $p-$ energy integral) on Ω if there exists a constant $K \geq 1$

such that for all bounded open sets Ω' with $\overline{\Omega'} \subset \Omega$ and for each $v \in N^{1,p}(\Omega')$ with $u - v \in N_0^{1,p}(\Omega')$ we have

$$\int_{\Omega' \cap \{u \neq v\}} g_u^p d\mu \leq K \int_{\Omega' \cap \{u \neq v\}} g_v^p d\mu,$$

where g_u and g_v are the minimal $p-$weak upper gradients of u and v respectively.

In particular, every $p-$harmonic function is a quasiminimizer with $K = 1$. A representative in $N_{loc}^{1,p}(\Omega)$ of a quasiminimizer $u \in N_{loc}^{1,p}(\Omega)$ is also a quasiminimizer.

Kinnunen and Shanmugalingam [11] studied the regularity of quasiminimizers of the $p-$energy integral, under the assumptions that the space (X, d, μ) supports a weak $(1, q)-$Poincaré inequality with $1 < q < p$. Kinnunen and Shanmugalingam proved that, for every quasiminimizer $u \in N_{loc}^{1,p}(\Omega)$ in Ω, the functions u and $(-u)$ belong to the De Giorgi class $DG_p(\Omega)$ [11, Proposition 3.3] and derived from this result several regularity properties of quasiminimizers. Each function u with u, $(-u) \in DG_p(\Omega)$ is essentially locally bounded [11, Theorem 4.3] and has a representative in $N_{loc}^{1,p}(\Omega)$ that is locally $\alpha-$Hölder continuous for some $0 < \alpha \leq 1$ not depending on u [11, Theorem 5.2]. Moreover, quasiminimizers satisfy the following strong extremum principle [11, Corollary 6.4].

Lemma 4.1. *Assume that the doubling metric measure space (X, d, μ) supports a weak $(1, q)-$Poincaré inequality with $1 < q < p$. Let Ω be an open subset of X and suppose that u is a continuous non-constant quasiminimizer for the $p-$energy integral in Ω. Then u does attain neither its maximum nor its minimum in Ω.*

Remark 4.2. A deep theorem by Keith and Zhong [9] shows that for every metric measure space (X, d, μ) that is *complete*, doubling and supports a weak $(1, p)-$Poincaré inequality for some $1 < p < \infty$ there exists $\varepsilon > 0$ such that (X, d, μ) supports a weak $(1, q)-$Poincaré inequality for every $q > p - \varepsilon$. Therefore, the results from [11] remain valid relaxing the assumption that (X, d, μ) supports a weak $(1, q)-$Poincaré inequality with $1 < q < p$ to the assumption that (X, d, μ) supports a weak $(1, p)-$Poincaré, but supposing that the metric space (X, d) is complete.

From Lemma 4.1 and Lemma 2.2 we obtain the following lemma.

Lemma 4.3. *Assume that the doubling metric measure space X is complete and supports a weak $(1,p)$ −Poincaré inequality with $1 < p < \infty$. Let $\Omega \subset X$ be open. Every continuous quasiminimizer $u \in N^{1,p}_{loc}(\Omega)$ of the $p-$energy integral is locally monotone in the sense of Lebesgue.*

Proof. Let u be a continuous quasiminimizer of the $p-$energy integral. If u is constant, then it is obviously locally monotone in the sense of Lebesgue. Assume that u is non-constant. (X, d, μ) supports a weak $(1,q)$ −Poincaré inequality for some $1 < q < p$, see Remark 4.2. Then Lemma 4.1 implies that u does not attain neither its maximum nor its minimum in Ω.

X is doubling and complete, hence X is proper. Since X supports a weak $(1,p)$ −Poincaré inequality, X has (NESS) property, see Remark 2.4. Then u is locally monotone in the sense of Lebesgue, according to Lemma 2.2. □

In [12] it is proved a higher integrability property for the minimal weak upper gradient of a quasiminimizer for the $p-$energy integral, u-sing an extension of Gehring lemma to doubling metric measure spaces, proved by Zatorska-Goldstein [20]. Assume that $Q > 1/2$, X satisfies weak $(1,q)$ −Poincaré inequality for some $1 < q < \infty$ and that $q < p \leq 2Qq$ and, in addition, $p < Qq/(Q-q)$ when $q < Q$. Then there exists a positive constant ε_0 such that for every quasiminimizer $u \in N^{1,p}_{loc}(X)$ and for each $0 \leq \varepsilon < q\varepsilon_0$ we have $g_u \in L^{p+\varepsilon}_{loc}(X)$ [12, Theorem 5].

Assume that $Q > 1$ and X satisfies a weak $(1,Q)$ −Poincaré inequality.We may assume that X satisfies a weak $(1,q)$ −Poincaré inequality for some q such that $1 < q < Q$, see Remark 4.2. Then the above result for $p = Q$ implies that for every quasiminimizer $u \in N^{1,Q}_{loc}(X)$ we have $g_u \in L^{Q+\varepsilon}_{loc}(X)$ for some $\varepsilon > 0$. By Cesari-Calderón-type theorem [1, Theorem 4.1], $Lip\, u(x) < \infty$ for $\mu-$a.e. $x \in X$.

Obviously, every $u \in N^{1,p}_{loc}(X)$ with $p > Q$ satisfies the condition $Lip\, u(x) < \infty$ for $\mu-$a.e. $x \in X$.

It remains to consider the case $p < Q$. We will give a simple proof for the fact that, when $p > Q-1$, every continuous quasiminimizer $u \in N^{1,p}_{loc}(X)$ of the $p-$energy integral satisfies the condition $Lip\, u(x) < \infty$ for $\mu-$a.e. $x \in X$.

Proposition 4.4. *Let (X, d, μ) be a complete doubling metric measure space, with a homogeneous dimension Q, and supporting a weak $(1,p)$ −Poincaré inequality, where $1 < p < \infty$. If $p > Q-1$, then every continuous quasiminimizer $u \in N^{1,p}_{loc}(X)$ of the $p-$energy integral satisfies*

$$Lip\,(u) < \infty \ \mu - a.e.\ in\ X. \tag{14}$$

In particular, u is differentiable with respect to any strong differentiable structure on X.

Proof. By Lemma 4.3, u is locally monotone in the sense of Lebesgue, hence it is locally monotone. Since $p > Q-1$, the claim follows from Corollary 4.3. □

Proposition 4.4 can be derived from a deeper known result. Very recently, under the above assumptions on (X, d, μ), where $1 < p < Q$, Gong and Hajłasz [3, Theorem 3.1] proved that every quasiminimizer $u \in N^{1,p}_{loc}(X)$ of a certain functional more general than the $p-$energy functional satisfies (14).

Remark 4.5. For $p = Q$ Proposition 4.4 gives [12, Corollary 6].

It would be interesting to extend Proposition 4.4 to the case when $u \in N^{1,p}_{loc}(\Omega)$, where Ω is an open subset of X.

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