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PRICE'S THEOREM AND UNCORRELATEDNESS SETS

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Abstract. Price's remarkable theorem on Gaussian random variables was published in 1958 and distinguished by the "Information Theory Society Golden Jubilee Paper Award" in 1988. Nowadays, this theorem remains an important tool used extensively in a wide spectrum of engineering problems, such as those appearing in signal processing, radio and space sciences, as well as information theory and astrophysics. In this paper, Price's theorem is applied to the investigation of possible uncorrelatedness for powers of Gaussian random variables. For zero-mean Gaussian variables the only two possible uncorrelatedness sets has been identified and presented.

The study aims to bring into spotlight the celebrated theorem usually disregarded in standard Probability and Statistics courses as well as to initiate further interest in the theorem by demonstrating a series of new applications.

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1. Introduction

In 1958, Robert Price (7 July 1929 - 3 December 2008) published his famous paper "A Useful Theorem for Nonlinear Devices Having Gaussian Inputs" [8] introducing an attractive and applicable result known today as 'Price's Theorem', which has turned out to be an important tool in the investigation of random processes and statistics occurring in such areas as signal processing, software engineering, radio physics, and others (see, for example, [3] and [13]). In 1988, this paper received the "Information Theory Society Golden Jubilee Paper Award" which paid tribute to this celebrated discovery. Nowadays, the theorem is still used extensively in various investigations ranging from electrical engineering to space research.

The aim of the present paper is to shed light on this important theorem which is usually not covered in traditional Probability and Statistics courses for engineers and, in addition, to demonstrate its application in problem-based learning (PBL) (as recommended, for example, in [7]) by solving open problems related to Gaussian random variables.

Let us recall the theorem. For the purpose of clarity, we present only a simplified version of the theorem satisfying the needs of this paper.

Price's Theorem. Let ξ_1 and ξ_2 be *Gaussian* random variables - that is, random variables with joint probability density:

$$\rho(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}, \tag{1}$$

where r is a correlation coefficient of ξ_1 and ξ_2. By u, we denote the *covariance* of ξ_1 and ξ_2, that is:

$$u = Cov(\xi_1, \xi_2) = r\sigma_1\sigma_2.$$

For a polynomial $g(x,y)$, consider the expected value

$$\mathbf{E}\left[g(\xi_1,\xi_2)\right] = f(u).$$

Then

$$f^{(n)}(u) = \mathbf{E}\left[\frac{\partial^{(2n)} g(\xi_1, \xi_2)}{\partial \xi_1^n \partial \xi_2^n}\right]. \tag{1}$$

The proof can be found in [8] or in the textbook [6], Section 7-2, p. 161.

It should be emphasized that not only had R. Price proved the theorem for a wider class of functions $g(x, y)$, but he also showed that (1) holds *only* for Gaussian random variables ξ_1 and ξ_2 if $g(x, y)$ is allowed to be arbitrary. However, his considerations require notions beyond the scope of this paper and, hence, are omitted.

In this paper, we present a new application of this theorem to the study of possible uncorrelatedness for powers of Gaussian random variables.

2. Uncorrelatedness sets

Since the concept of independence is fundamental in Probability Theory, Mathematical Statistics, and their applications, generalizations of independence have been studied widely from different perspectives. We recall the classical Bohlmann and Bernstein examples of 3 random events that are pairwise but not mutually independent (exhibited, for example, in [11], Section 3.2, p. 13). Further research on the notion of independence has resulted in the definitions of m-wise independent random variables, independence on the k-th level, independence/dependence structure, total dependence, along with other rather specific independence properties (see, for example, [9] - [12]). The general problem to construct n random events with a prescribed independence/dependence structure was stated by J. Stoyanov, and called by him "The Italian Problem". Its solution presented in [10] provides a far-reaching generalization of Bernstein's example.

For random variables, the earliest and mostly used generalization is *uncorrelatedness* of two random variables described by the property:

$$\mathbf{E}(\xi_1 \xi_2) = \mathbf{E}(\xi_1)\, \mathbf{E}(\xi_2), \tag{2}$$

provided that all of the expected values exist. It is commonly known that if ξ_1 and ξ_2 are independent random variables, then they satisfy

(2) and, therefore, they are uncorrelated. However, random variables may be uncorrelated without being independent. Consider the following simple example:

Example 1. Let $\Omega = [0, 2\pi]$ be a sample space with the probability $\mathbf{P}(A) = \frac{1}{2\pi}\text{length}(A)$, and let ξ_1, ξ_2 be random variables on Ω given by:

$$\xi_1(x) = \sin x, \quad \xi_2(x) = \cos x.$$

The expected values of these random variables can be found easily: $\mathbf{E}(\xi_1) = \mathbf{E}(\xi_2) = 0$. The calculation of the expected value of their product $\xi_1\xi_2$ yields: $\mathbf{E}(\xi_1\xi_2) = 0$, whence we see that (2) holds - that is ξ_1 and ξ_2 are *uncorrelated*. However they cannot be independent because they are connected with the well-known identity:

$$\sin^2 x + \cos^2 x = 1.$$

In general, uncorrelatedness is a much weaker condition than independence, see, for a example, a paper by David [1] for some historic information. Nevertheless, it is an important concept of Probability Theory and Statistics, especially in regression analysis. Uncorrelatedness is measured with the help of a *correlation coefficient* r taking values from -1 to 1 with $r = 0$ if and only if random variables are uncorrelated. Although many different approaches have been developed (see, for example, [4] and [9]), there is no universal way of measuring whether random variables are "more independent" or "less independent." Here, we make one more attempt to compare the degrees of relationship between random variables, usually when they are dependent.

Our approach takes into consideration the uncorrelatedness of not only random variables ξ_1 and ξ_2 themselves but also of their positive integer powers ξ_1^j and ξ_2^l. It is customary to denote the set of all positive integers by $\mathbb{N}$, that is $\mathbb{N} = \{1, 2, 3, \dots\}$. We take j and l to be numbers from $\mathbb{N}$. The collection of points in the Cartesian plane whose both coordinates are positive integers is usually denoted by $\mathbb{N}^2$ or $\mathbb{N} \times \mathbb{N}$.

Given random variables ξ_1 and ξ_2, we check for which of their powers condition (2) holds. The following definition can be reached.

Definition. Let us have random variables ξ_1 and ξ_2. The collection of pairs (j, l) in $\mathbb{N}^2$ so that ξ_1^j and ξ_2^l are uncorrelated constitutes an *uncorrelatedness set* of ξ_1 and ξ_2.

We denote the uncorrelatedness set of ξ_1 and ξ_2 by $U(\xi_1, \xi_2)$. The definition above means that

$$(j,l) \in U(\xi_1, \xi_2) \Leftrightarrow \mathbf{E}\left(\xi_1^j \xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right).$$

An uncorrelatedness set shows which exactly powers of random variables are uncorrelated. Random variables ξ_1 and ξ_2 are uncorrelated in the usual sense (2) if and only if $(1,1) \in U(\xi_1, \xi_2)$.

Example 2. Let us find the uncorrelatedness set for random variables ξ_1 and ξ_2 considered in the Example 1. It has already been demonstrated that $(1,1) \in U(\xi_1, \xi_2)$. The behavior of sine and cosine implies that if j is odd, then $\mathbf{E}\left(\xi_1^j\right) = 0$ as well as $\mathbf{E}\left(\xi_1^j \xi_2^l\right) = 0$, whence all points with odd j are in the uncorrelatedness set. Similarly, we see that all points with odd l are also in the set. Then, what about points (j,l) with both j and l being even? Let us take $(j,l) = (2,2)$. Then, $\mathbf{E}\left(\xi_1^2\right) = \mathbf{E}\left(\xi_2^2\right) = \frac{1}{2}$, while $\mathbf{E}\left(\xi_1^2 \xi_2^2\right) = \frac{1}{8}$, which shows that ξ_1^2 and ξ_2^2 are not uncorrelated - that is, $(2,2) \notin U(\xi_1, \xi_2)$. Similar but more tedious calculations show that $(j,l) \notin U(\xi_1, \xi_2)$ whenever both coordinates are even. We have therefore obtained the following answer:

$$(j,l) \in U(\xi_1, \xi_2) \Leftrightarrow \text{ either } j \text{ or } l \text{ or both are odd.}$$

If random variables ξ_1 and ξ_2 are *independent*, then so are their powers and we have

$$\mathbf{E}\left(\xi_1^j \xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right) \quad \text{for all} \quad (j,l) \in \mathbb{N}^2.$$

In other words, for independent random variables $U(\xi_1, \xi_2) = \mathbb{N}^2$.

Warning! It should be pointed out that, in general, the condition $U(\xi_1, \xi_2) = \mathbb{N}^2$ does *not* imply independence of ξ_1 and ξ_2 (see, for example, [4], Theorem 2).

One may think that uncorrelatedness sets provide a measure of independence for random variables in the sense that the wider an uncorrelatedness set is, the more independent random variables become. The following general theorem concerning uncorrelatedness sets has been proved in [5], Section 2.

Theorem A. For any subset $U \subseteq \mathbb{N}^2$, there exist random variables ξ_1 and ξ_2 such that $U(\xi_1, \xi_2) = U$ - that is U is the uncorrelatedness set of ξ_1 and ξ_2.

Theorem A shows that, in general, an uncorrelatedness set of two random variables may be an arbitrary subset of $\mathbb{N}^2$. In other words, the uncorrelatedness of any set of powers of random variables does not imply uncorrelatedness of any other powers. For example, we fix any $(j_0, l_0) \in \mathbb{N}^2$ and set $U = \mathbb{N}^2 \setminus (j_0, l_0)$. Theorem A guarantees that there exist random variables ξ_1 and ξ_2 so that

$$\mathbf{E}\left(\xi_1^j \xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right) \Leftrightarrow (j, l) \neq (j_0, l_0).$$

Put in other terms, all points of $\mathbb{N}^2$ except one are in the uncorrelatedness set of ξ_1 and ξ_2.

The statement of Theorem A does not remain true if we prescribe the distributions of random variables. For example, if ξ_1 and ξ_2 are Gaussian random variables, then their uncorrelatedness given by $r = 0$ in formula (**??**) implies independence. In terms of uncorrelatedness sets, we may write for such random variables:

$$(1, 1) \in U(\xi_1, \xi_2) \Rightarrow U(\xi_1, \xi_2) = \mathbb{N}^2.$$

This shows that uncorrelatedness sets of Gaussian random variables are by no means arbitrary, and we face the problem of describing possible uncorrelatedness sets for them.

3. Results of Applying Price's Theorem

Throughout this section, ξ_1 and ξ_2 are Gaussian random variables with density (**??**) and a covariance of u. Since the Gaussian distribution plays a profound role in statistical theory and applications, the properties of those variables are of great importance for researchers. The results below provide new information on such random variables. To begin with, we prove the following fact:

Theorem 1. For Gaussian random variables ξ_1 and ξ_2, the expected value $\mathbf{E}(\xi_1^j \xi_2^l)$ is a polynomial in u of degree $\leq \max\{j, l\}$.

Proof. We denote $f_{(j,l)}(u) = \mathbf{E}(\xi_1^j \xi_2^l)$. Let us take a positive integer $n > \max\{j, l\}$ and apply Price's Theorem:

$$f_{(j,l)}^{(n)}(u) = \int\int_{\mathbb{R}^2} \frac{\partial^{(2n)} x^j y^l}{\partial x^n \partial y^n} \cdot \rho(x, y)\, dx dy = 0,$$

because the partial derivative in the integrand equals zero identically.

Therefore, all the derivatives of $f_{(j,l)}(u)$ of order greater than $\max\{j,l\}$ equal zero whence $f_{(j,l)}(u)$ is a polynomial of degree at most $\max\{j,l\}$.

□

It can be readily seen that

$$(j,l) \in U(\xi_1,\xi_2) \Leftrightarrow f_{(j,l)}(u) = f_{(j,l)}(0), \tag{3}$$

since $u = 0$ implies the independence of ξ_1 and ξ_2. Condition (3) reveals one way of using the properties of the polynomials $f_{(j,l)}(u)$ for describing uncorrelatedness sets. Although the expected values of random variables in the forthcoming theorems can be evaluated directly with the help of integration, the relevant calculations are immensely cumbersome while the application of Price's Theorem makes all reasoning very elegant and transparent.

Theorem 2. Let ξ_1 and ξ_2 be Gaussian random variables, and let $U(\xi_1,\xi_2)$ denote their uncorrelatedness set. If $(j,l) \in U(\xi_1,\xi_2)$ and $j+l$ is even, then ξ_1 and ξ_2 are independent and, thus, $U(\xi_1,\xi_2) = \mathbb{N}^2$.

Proof. (i) Let both j and l be *odd*. Applying Price's Theorem with $n = 1$, we obtain the following expression for the first derivative of $f(u)$:

$$f'_{(j,l)}(u) = \int\int_{\mathbb{R}^2} jlx^{j-1}y^{l-1}\rho(x,y)\,dxdy > 0,$$

since the integrand is positive for all $(x,y) \neq (0,0)$. Therefore, $f_{(j,l)}(u)$ is a strictly increasing function for all u and, hence, $f_{(j,l)}(u) = f_{(j,l)}(0)$ implies $u = 0$. It appears that, if both j and l are odd and $(j,l) \in U(\xi_1,\xi_2)$, then ξ_1 and ξ_2 are independent.

(ii) Let both j and l be *even*. Clearly, in this case $f_{(j,l)}(u) > 0$ for all $u \neq 0$. Taking $f'_{(j,l)}(u)$ by Price's Theorem, we conclude that $f'_{(j,l)}(0) = 0$. To investigate the behavior of the polynomial $f_{(j,l)}(u)$, we take the second derivative by applying Price's Theorem once more. It can be seen that $f''_{(j,l)}(u) > 0$ for all u, which means that the polynomial $f_{(j,l)}(u)$ is a convex function with a single point of absolute minimum at 0. What we have is, $f_{(j,l)}(u) = f_{(j,l)}(0) \Leftrightarrow u = 0$. That is, if $(j,l) \in U(\xi_1,\xi_2)$, then ξ_1 and ξ_2 are independent.

□

Theorem 3. Let ξ_1 and ξ_2 be zero-mean Gaussian random variables - that is $\mu_1 = \mu_2 = 0$ in formula (**??**) - and let $U(\xi_1, \xi_2)$ denote their uncorrelatedness set. If $j + l$ is odd, then $(j, l) \in U(\xi_1, \xi_2)$.

Proof. Using Price's Theorem, we conclude that $f^{(k)}_{(j,l)}(0) = 0$ for all $k = 0, 1, 2 \ldots$ Yet, a polynomial with all zero derivatives at 0 equals 0 identically, implying $f_{(j,l)}(u) \equiv 0$ or $f_{(j,l)}(u) = \mathbf{E}\left(\xi_1^j \xi_2^l\right) = 0 = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right)$ for all u.

$\square$

Summarizing the last two theorems, we obtain a complete identification of admissible uncorrelatedness sets for zero-mean Gaussian random variables. Namely, there are exactly two possible uncorrelatedness sets: $\mathbb{N}^2$ (in which case random variables are independent); and the set of pairs (j, l), where $j + l$ is odd.

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