"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 24(2014), No. 1, 75-80

SOME GENERALIZATIONS OF LOCAL CONTINUITY IN IDEAL TOPOLOGICAL SPACES

ALI ÖZKURT

Abstract. In this paper, we consider generalizations of two forms of local continuity of maps between topological spaces, with respect to an ideal of subsets either in the domain or in the codomain. In particular given an ideal I of subsets of a space Y, one-to-one mappings $f : X \to Y$ are investigated such that $f|_{X_0}$ is continuous for some closed $X_0 \subset X$. Also we investigate continuity of mappings $f : X \to Y$ if X is an ideal topological space. Our results provide some corollaries for Baire spaces.

1. Introduction and preliminaries

Let (X, τ) be a topological space. As it is well known a nonempty family I of subsets of X is called an ideal if the two following conditions are fulfilled: (1) If $A \in I$ and $B \subset A$ then $B \in I$. (2) If $A \in I$ and $B \in I$ then $A \cup B \in I$. Observe that a family of sets is a filter if and only if the family of the complements of these sets is an ideal. A topological space (X, τ) with an ideal I is called an ideal topological space which we denote by (X, τ, I).

Keywords and phrases: $I-$continuous, Baire spaces, Meager sets.
(2010) Mathematics Subject Classification: 54C05, 54C08, 54C10, 54E52

One connection between an ideal and the topology on a given space arises through the concept of the local function on a subset with respect to the ideal.

For a subset $A \subset X$, $A^* = \{x \in X : U \cap A \notin I$, for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ [4]. Furthermore in an ideal space (X, τ, I) the following statements are equivalent [6]

$$
\begin{array}{rl}
(i) & X^* = X \\
(ii) & \tau \cap \ I = \{\emptyset\} \\
(iii) & A \subseteq A^*, \forall A \in \tau
\end{array}
$$

The ideal spaces which satisfy $X^* = X$ are called Hayashi-Samuels spaces [3]. For an ideal topological space (X, τ, I) there exists a topology τ^*, finer than, τ, which has $\mathbb{B} = \{U - V : U \in \tau$ and $V \in I\}$ as a base. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for τ^* [3]. Ideals in general topological spaces were considered in [3], and a more modern study can be found in [2].

2. $I-$continuity at a point

Given an ideal I of subsets of a space (Y, φ), we are interested in characterizing one-to-one mappings, $f : (X, \tau) \to (Y, \varphi, I)$, that admit continuous restriction to a closed set $X_0 \subset X$ with an extra property $f(X - X_0) \in I$.

A notion of $I-$continuity was introduced by M.E. Abd El-Monsef et al. [5]. For a newer paper dealing with $I-$continuity see [1].

In the present paper another version of $I-$ continuity is introduced.

Definition 2.1. *Let us say that f is $I-$continuous at x if, for every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $f(U) - V \in I$. Let $f : (X, \tau) \to (Y, \varphi, I)$, we say that f is $I-$continuous if f is $I-$continuous at every point of X.*

Observe that if f is continuous function then f is $I-$continuous function, since $\emptyset \in I$.

On the other hand $I-$continuity does not imply continuity.

Example 2.2. *$f : (R, \tau_{cof}) \to (R, \tau_{cof}, I)$, $f(x) = sinx$, where τ_{cof} is cofinite topology on R and I denotes the ideal of finite subsets of R. It is easy to see that f is $I-$continuous but not continuous function.*

Theorem 2.3. *Let Y be a regular space and $f : (X, \tau) \to (Y, \varphi, I)$ be an $I-$continuous and one-to-one map. Then $f|_{X_0}$ is continuous for some closed $X_0 \subset X$.*

Proof. Let $G = \bigcup\{U : U \text{ is open in } X \text{ and } f(U) \in I\}$ and $X_0 = X - G$. Clearly X_0 is a closed subset of X.

We will show that $f|_{X_0}$ is continuous. Consider any point $x \in X_0$ and any neighborhood V of $f(x)$ in Y. We need a neighborhood U of x in X such that $f(U \cap X_0) \subset V$.

First, using regularity of Y, we can select an open set W containing $f(x)$ with $ClW \subset V$. Since f is $I-$continuous, there exists a neighborhood U of x such that $f(U) - W \in I$.

We will show that $f(U \cap X_0) \subset V$. Take any $x' \in U \cap X_0$; since $ClW \subset V$, it will be enough to prove that $f(x') \in ClW$.

Suppose that $f(x')$ is not an element of clW. Then there exists an open neighborhood W' of $f(x')$ such that $W \cap W' = \emptyset$. Since f is $I-$continuous, there exists a neighborhood U' of x' such that $f(U') - W' \in I$. Hence

$f(U) \cap f(U') = f(U) \cap f(U') - (W \cap W') \subset (f(U) - W) \cup (f(U') - W') \in I$. Since f is one-to-one map, $f(U \cap U') = f(U) \cap f(U') \in I$. On the other hand, we have $f(U \cap U') \subset Y - f(X_0)$, by definition of X_0.

We have reached a contradiction, because on the other hand, $f(x') \in f(U \cap U') \cap f(X_0)$. ∎

Remark 2.4. *In Theorem 2.3 we need X_0 to be non-empty. If we assume that f is bijective (one-to-one and onto), then X_0 is non-empty if and only if $\tau \cap \{f^{-1}(A) : A \in I\}$ is not a cover of X.*

Corollary 2.5. *Let Y be a regular space and $f : (X, \tau) \to (Y, \varphi, I)$ be an $I-$continuous and one-to-one map. If $f(U) \notin I$ for every non-empty open set U in X then f is continuous.*

Proof. Since $f(U) \notin I$ for every nonempty open set U in X, we have $G = \bigcup\{U : U \text{ is open in } X \text{ and } f(U) \in I\} = \emptyset$, therefore, by Theorem 2.3, f is continuous. ∎

Definition 2.6. *An ideal I is said to be a σ-ideal if it is countably additive; that is, if $I_n \in I$ for each $n \in \mathbb{N}$, then $\cup\{I_n : n \in \mathbb{N}\} \in I$. For example the collection of all meager subsets (i.e. countable union of nowhere dense sets) of a space is a σ-ideal.*

Corollary 2.7. *Let Y be a regular and hereditarily Lindelöf space and I be a $\sigma-$ideal of subsets of Y. Then $f(X - X_0) \in I$.*

Proof. $f(X - X_0) = f(G) = \bigcup\{f(U) : U$ is open in X and $f(U) \in I\}$. Since Y is hereditarily Lindelöf, $f(G)$ is a Lindelöf subspace of Y. This means that $f(G) = \bigcup\{f(U_n) : n \in \mathbb{N}, U_n$ is open in X and $f(U_n) \in I\}$. Since I is a $\sigma-$ideal of subsets Y, it is countably additive. Therefore $f(X - X_0) = f(G) \in I$. ■

Corollary 2.8. *Let Y be a regular Baire space and I be the $\sigma-$ideal of all meager subsets of Y. If $f : X \to Y$ is a bijection, open and $I-$continuous map, then f is a homeomorphism.*

Proof. We will only show that f is continuous. First observe that a meager set in a Baire space has empty interior. Since Y is Baire space and I is a $\sigma-$ideal of all meager subsets of Y, we have $Y^* = Y$. Therefore $\emptyset$ is the only member of I that is open in Y. In addition; since f is an open map, $G = \bigcup\{U : U$ is open in X and $f(U) \in I\} = \emptyset$. Therefore, by Theorem 2.3, we have $X_0 = X$ and hence that of f is continuous. Thus f is a homeomorphism. ■

3. Continuity of Maps on Ideal Spaces

In this section we investigate continuity of mappings, $f : (X, \tau, I) \to (Y, \varphi)$, where I is an ideal of subsets of X and Y is any topological space.

Definition 3.1. *A network $\mathcal{N}$ for a space X is a collection of subsets N of X such that whenever $x \in U$ with U open, then there exists $N \in \mathcal{N}$, with $x \in N \subset U$.*

So, a network is like a base, but its elements need not be open. In some case a network can be "fattened up" to a base for a space, e.g., a compact space with a countable network has a countable base (See [7]).

Let (X, τ, I) be an ideal space and given a space Y having a network $\mathcal{N}$ and let $f : (X, \tau, I) \to (Y, \varphi)$ be a function.

We will denote $C(f, \mathcal{N}) = \{x \in X :$ for every $N \in \mathcal{N}$, we have $f(x) \in N$ implies $x \in Int(f^{-1}(N))^*\}$.

Proposition 3.2. *If $\mathfrak{B}$ is a base for Y, then $C(f, \mathcal{N}) \subset C(f, \mathfrak{B})$ for any network $\mathcal{N}$,* *Proof.* Let $x \in C(f, \mathcal{N})$. Assume that $f(x) \in B$ for some $B \in \mathfrak{B}$. Since $\mathcal{N}$ is a network for X, there exists an $N \in \mathcal{N}$ such that $f(x) \in N \subset B$. This implies $x \in Int(f^{-1}(N))^*$ because of x being an element of $C(f, \mathcal{N})$. Since $(f^{-1}(N))^* \subset (f^{-1}(B))^*$, it can be easily seen that $x \in Int(f^{-1}(B))^*$ and hence $x \in C(f, \mathfrak{B})$. ■

Theorem 3.3. *Let $\mathfrak{B}$ be a base for Y and $C(f)$ be the set of points of continuity of f. If $X = X^*$, then $C(f) \subset C(f, \mathfrak{B})$.*

Proof. Let $x \in C(f)$. Then $x \in f^{-1}(B)$, for any $B \in \mathfrak{B}$ such that $f(x) \in B$. Since $f^{-1}(B)$ is open, it suffices to show that $f^{-1}(B) \subset (f^{-1}(B))^*$. Let $y \in f^{-1}(B)$. Since $X = X^*$, there is no nonempty open set in I. This implies $f^{-1}(B) \cap V \notin I$ for any neighborhood V of y. Thus $y \in (f^{-1}(B))^*$. ■

The fact that $X = X^*$ is necessary. For example; let I be the σ-ideal of meager sets in $\mathbb{Q}$ (the rational set) and let $f : \mathbb{Q} \to R$ be any constant function. Clearly $C(f) = \mathbb{Q}$, whereas $C(f, \mathfrak{B}) = \emptyset$.

Corollary 3.4. *Let (X, τ, I) be an ideal topological space such that $X = X^*$. Let $f : (X, \tau, I) \to (Y, \varphi)$ be a function. If Y has a base $\mathfrak{B}$ and for some $x_0 \in X$ there exists an element B of $\mathfrak{B}$ containing $f(x_0)$ such that $(f^{-1}(B))^*$ has empty interior, then f is discontinuous at x_0.*

Proof. Since $(f^{-1}(B))^*$ has no interior for some $B \in \mathfrak{B}$ containing $f(x_0)$, this implies $C(f, \mathfrak{B})$, hence that of, $C(f)$ is empty. ■

This corollary is applicable for Baire space X and $\sigma-$ideal of all meager subsets of X.

Corollary 3.5. *Let X be a Baire space and I be the $\sigma-$ideal of all meager subsets of X. Since a meager set has no interior in a Baire space, this implies that $X = X^*$. On the other hand if $\mathfrak{B}$ is a base for Y such that there exists an element B of $\mathfrak{B}$ containing $f(x)$ such that $(f^{-1}(B))^* \in I$ for each $x \in X$, then $C(f, \mathfrak{B})$, hence that of, $C(f)$ is empty.*

ACKNOWLEDGEMENT

The author wish to thank the referee for useful comments and suggestions.

References

[1] A. Keskin, S. Yuksel and T. Noiri, **Decompositions of I-continuity and continuity**, Commun. Fac. Sci. Univ. Ank. Series 53, 2(2004), 67-75.
[2] D. Janković and T. R. Hamlett, **New topologies from old via ideals**, Amer. Math. Monthly 97 (1990), 295-310
[3] E. Hayashi, **Topologies defined by local properties**, Math. Ann. 156 (1964), 205-215
[4] K. Kuratowski, **Topology** (New York), Academic Press, 1966.
[5] M. E. Abd El-Monsef, E.F. Lashien and A. A. Nasef, **On I-open sets and I-continuous functions**, Kyungpook Math. J. 32, 1 (1992), 21-30
[6] P. Samuels, **A topology formed from a given topology and ideal**, J. London. Math.Soc. 10, 2 (1975), 409-416
[7] R. Engelking, **General Topology** (Berlin), Heldermann Verlag, 1989.

Çukurova University
Department of Mathematics
Address: Çukurova University, Department of Mathematics, 01330-Adana, Turkey
aozkurt@cu.edu.tr