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COMMON FIXED POINTS FOR MAPS SATISFYING A NEW RATIONAL INEQUALITY IN ORDERED FUZZY METRIC SPACES

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Abstract. In this paper, we prove some common fixed point results for four and two mappings satisfying a rational contractive condition in a partially ordered fuzzy metric space.

1. Introduction and preliminaries

The theory of fuzzy sets was introduced by L.Zadeh [9] in 1965, George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7], Grabiec [11] proved the contraction principle in the setting of fuzzy metric spaces introduced in [1]. For fixed point theorems in fuzzy metric spaces some of the interesting references are [1,3,4,5,11,12-19]. In the sequel, we need the following

Definition 1.1. ([2]). *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if it satisfies the following conditions*

(1) $*$ *is associative and commutative,*
(2) $*$ *is continuous,*
(3) $a * 1 = a$ *for all* $a \in [0,1]$,
(4) $a*b \leq c*d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

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Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = min\{a, b\}$.

Definition 1.2. ([1]). *A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and each $t, s > 0$,*
(M_1). *$M(x, y, t) > 0$,*
(M_2). *$M(x, y, t) = 1$ if and only if $x = y$,*
(M_3). *$M(x, y, t) = M(y, x, t)$,*
(M_4). *$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,*
(M_5). *$M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.3. ([11]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.4. *Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e., whenever

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.5. ([8]).*Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.6. ([1]). *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$*

if $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$. *The sequence* $\{x_n\}$ *is said to be Cauchy if* $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$ *for all* $t > 0$. *The space* $(X, M, *)$ *is said to be complete if every Cauchy sequence in* X *is convergent in* X.

Definition 1.7. ([6]). *Let* f *and* g *be self mappings on a fuzzy metric space* (X, d). *Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is,* $fx = gx$ *implies that* $fgx = gfx$.

Very recently M. Abbas, T. Nazir and S. Radenović [10] introduced the new concepts in a partial ordered set as follows:

Definition 1.8. ([10]). *Let* $(X, \preceq)$ *be a partially ordered set. A mapping* $f : X \to X$ *is called dominating if* $x \preceq fx$ *for all* $x \in X$.

Definition 1.9. ([10]). *Let* $(X, \preceq)$ *be a partially ordered set and* $f, g : X \to X$. *The mapping* f *is called a weak annihilator of* g *if* $fgx \preceq x$ *for all* $x \in X$.

Definition 1.10. *Let* $(X, \preceq)$ *be a partially ordered set and* $(X, M, *)$ *be a fuzzy metric space. Then* $(X, M, *, \preceq)$*is called an ordered fuzzy metric space.*

Several authors [3, 4, 5, 11, 13, 16 - 19] obtained fixed point theorems in fuzzy metric spaces for a single map using one of the following contraction conditions.
There exists $k \in (0, 1)$ such that for all $x, y \in X$ and for all $t > 0$,
(1) $M(Tx, Ty, kt) \geq M(x, y, t)$,
(2) $M(Tx, Ty, kt) \geq \min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, 2t), M(y, Tx, t) \end{array} \right\}$,
(3) $M(Tx, Ty, kt) \geq \min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, 2t), M(y, Tx, 2t) \end{array} \right\}$,
(4) $M(Tx, Ty, kt) \geq \min \left\{ \begin{array}{c} M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ M(x, Ty, \alpha t), M(y, Tx, (2-\alpha)t) \end{array} \right\}$,
$\forall\ \alpha \in (0, 2)$.
In all these types of theorems, the authors assumed that

$$\lim_{t\to\infty} M(x, y, t) = 1,\ \forall x, y \in X. \tag{1}$$

In this paper, we prove fixed point theorems for maps satisfying a new rational inequality without using the condition (1).

2. Main results

Theorem 2.1. *.Let $(X, M, *, \preceq)$ be an ordered fuzzy metric space such that $a * b \geq ab$ for every $a, b \in [0, 1]$. Let $f, g, S, T : X \to X$ be mappings satisfying*
(2.1.1) *f and g are dominating maps and f and g are weak annihilators of T and S respectively,*
(2.1.2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and at least one of $T(X)$ and $S(X)$ is a complete sub-space of X,*
(2.1.3) *the pairs (f, S) and (g, T) are weakly compatible ,*
(2.1.4) *there exists some $0 \leq \delta < 1$ such that $M(fx, gy, t) \geq L^{\delta}(x, y, t)$, for all comparable elements $x, y \in X$, $\forall\ t > 0$, where*

$$L(x, y, t) = \min\left\{M(Sx, Ty, t), \frac{M(fx, Sx, t) * M(gy, Ty, t)}{M(Sx, Ty, t)},\right. \\ \left. \frac{M(fx, Ty, 2t) * M(gy, Sx, 2t)}{M(Sx, Ty, t)}\right\},$$

(2.1.5) *if for a non decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies that $x_n \preceq u$ for all n.*

Then f, g, S and T have a common fixed point in X. Further, if we assume that the set of common fixed points of f, g, S and T is well ordered then f, g, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. From (2.1.2), there exist sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, $n \geq 0$.
From (2.1.1), we have $x_{2n} \preceq fx_{2n} = Tx_{2n+1} \preceq fTx_{2n+1} \preceq x_{2n+1}$ and $x_{2n+1} \preceq gx_{2n+1} = Sx_{2n+2} \preceq gSx_{2n+2} \preceq x_{2n+2}$. Thus $x_n \preceq x_{n+1}$ for all $n \geq 0$.

Case(a): Suppose $y_{2m} = y_{2m+1}$ for some m, so that $M(y_{2m+1}, y_{2m}, t) = 1$. Then

$$\begin{aligned} M(y_{2m+2}, y_{2m+1}, t) &= M(fx_{2m+2}, gx_{2m+1}, t) \\ &\geq L^{\delta}(x_{2m+2}, x_{2m+1}, t), \end{aligned}$$

where

$$L(x_{2m+2}, x_{2m+1}, t) = \min \left\{ \begin{array}{c} M(Sx_{2m+2}, Tx_{2m+1}, t), \\ \frac{M(fx_{2m+2}, Sx_{2m+2}, t) * M(gx_{2m+1}, Tx_{2m+1}, t)}{M(Sx_{2m+2}, Tx_{2m+1}, t)}, \\ \frac{M(fx_{2m+2}, Tx_{2m+1}, 2t) \, * \, M(gx_{2m+1}, Sx_{2m+2}, 2t)}{M(Sx_{2m+2}, Tx_{2m+1}, t)} \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} M(y_{2m+1}, y_{2m}, t), \\ \frac{M(y_{2m+2}, y_{2m+1}, t) \, * \, M(y_{2m+1}, y_{2m}, t)}{M(y_{2m+1}, y_{2m}, t)}, \\ \frac{M(y_{2m+2}, y_{2m}, 2t) \, * \, M(y_{2m+1}, y_{2m+1}, 2t)}{M(y_{2m+1}, y_{2m}, t)} \end{array} \right\}$$

Since

$$\begin{aligned} M(y_{2m+2}, y_{2m}, 2t) &\geq M(y_{2m+2}, y_{2m+1}, t) * M(y_{2m+1}, y_{2m}, t) \\ &= M(y_{2m+2}, y_{2m+1}, t), \end{aligned}$$

we have $L(x_{2m+2}, x_{2m+1}, t) = M(y_{2m+2}, y_{2m+1}, t)$.
Thus

$$M(y_{2m+2}, y_{2m+1}, t) \geq M^{\delta}(y_{2m+2}, y_{2m+1}, t),$$

which implies that $y_{2m+2} = y_{2m+1}$.
Suppose $y_{2m-1} = y_{2m}$ for some m. Then

$$M(y_{2m}, y_{2m+1}, t) = M(fx_{2m}, gx_{2m+1}, t) \geq L^{\delta}(x_{2m}, x_{2m+1}, t), \tag{2}$$

where

$$\begin{aligned} L(x_{2m}, x_{2m+1}, t) &= \min \left\{ \begin{array}{c} M(y_{2m-1}, y_{2m}, t), \\ \frac{M(y_{2m}, y_{2m-1}, t) \, * \, M(y_{2m+1}, y_{2m}, t)}{M(y_{2m-1}, y_{2m}, t)}, \\ \frac{M(y_{2m}, y_{2m}, 2t) \, * \, M(y_{2m+1}, y_{2m-1}, 2t)}{M(y_{2m-1}, y_{2m}, t)} \end{array} \right\} \\ &= M(y_{2m}, y_{2m+1}, t), \end{aligned}$$

since $a * b \geq ab$ and (M_4).
Thus

$$M(y_{2m}, y_{2m+1}, t) \geq M^{\delta}(y_{2m}, y_{2m+1}, t),$$

which implies that $y_{2m} = y_{2m+1}$.

Continuing in this way we can conclude that $\{y_n\}$ is eventually constant. Hence $\{y_n\}$ is a Cauchy sequence in X.

Case(b): Suppose that $y_n \neq y_{n+1}$ for each n. Then

$$M(y_{2n}, y_{2n+1}, t) = M(fx_{2n}, gx_{2n+1}, t) \geq L^{\delta}(x_{2n}, x_{2n+1}, t), \tag{3}$$

where

$$\begin{aligned} L(x_{2n}, x_{2n+1}, t) &= \min \left\{ \begin{array}{c} M(y_{2n-1}, y_{2n}, t), \\ \frac{M(y_{2n}, y_{2n-1}, t) \, * \, M(y_{2n+1}, y_{2n}, t)}{M(y_{2n-1}, y_{2n}, t)}, \\ \frac{M(y_{2n}, y_{2n}, 2t) \, * \, M(y_{2n+1}, y_{2n-1}, 2t)}{M(y_{2n-1}, y_{2n}, t)} \end{array} \right\} \\ &\geq \min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t)\}, \end{aligned}$$

since $a * b \geq ab$ and (M_4).

If $M(y_{2n+1}, y_{2n}, t) \leq M(y_{2n-1}, y_{2n}, t)$, then

$$L(x_{2n}, x_{2n+1}, t) \geq M(y_{2n+1}, y_{2n}, t),$$

and from (3), we now have

$$M(y_{2n+1}, y_{2n}, t) \geq M^{\delta}(y_{2n+1}, y_{2n}, t), \tag{4}$$

a contradiction.

Hence from (3) , we have

$$M(y_{2n+1}, y_{2n}, t) \geq M^{\delta}(y_{2n-1}, y_{2n}, t). \tag{5}$$

Now

$$M(y_{2n}, y_{2n-1}, t) = M(fx_{2n}, gx_{2n-1}, t) \geq L^{\delta}(x_{2n}, x_{2n-1}, t), \tag{6}$$

where

$$\begin{aligned} L(x_{2n}, x_{2n-1}, t) &= \min \left\{ \begin{array}{c} M(y_{2n-1}, y_{2n-2}, t), \\ \frac{M(y_{2n}, y_{2n-1}, t) \, * \, M(y_{2n-1}, y_{2n-2}, t)}{M(y_{2n-1}, y_{2n-2}, t)}, \\ \frac{M(y_{2n}, y_{2n-2}, 2t) \, * \, M(y_{2n-1}, y_{2n-1}, 2t)}{M(y_{2n-1}, y_{2n-2}, t)} \end{array} \right\} \\ &\geq \min\{M(y_{2n-1}, y_{2n-2}, t), M(y_{2n}, y_{2n-1}, t)\}, \end{aligned}$$

since $a * b \geq ab$ and (M_4).

If $M(y_{2n}, y_{2n-1}, t) \leq M(y_{2n-1}, y_{2n-2}, t)$, then

$$L(x_{2n}, x_{2n-1}, t) \geq M(y_{2n}, y_{2n-1}, t)$$

and from (6), we have

$$M(y_{2n}, y_{2n-1}, t) \geq M^{\delta}(y_{2n}, y_{2n-1}, t),$$

a contradiction. Hence from (6), we obtain

$$M(y_{2n}, y_{2n-1}, t) \geq M^{\delta}(y_{2n-1}, y_{2n-2}, t). \tag{7}$$

From (5) and (7), we get

$$M(y_n, y_{n+1}, t) \geq M^{\delta}(y_{n-1}, y_n, t)$$

for all $n \geq 1$. Hence

$$M(y_n, y_{n+1}, t) \geq M^{\delta^n}(y_0, y_1, t) \tag{8}$$

for all $n \geq 1$.

Now for $m > n$ and $t > 0$, there exists $t_1 > 0$ such that $t_1 \leq \frac{t}{m-n}$. Then

$$\begin{aligned} M(y_m, y_n, t) &\geq M(y_m, y_n, (m-n)t_1) \\ &\geq M(y_n, y_{n+1}, t_1) * M(y_{n+1}, y_{n+2}, t_1) * \cdots * M(y_{m-1}, y_m, t_1) \\ &\geq M^{\delta^n}(y_0, y_1, t_1) M^{\delta^{n+1}}(y_0, y_1, t_1) \cdots M^{\delta^{m-1}}(y_0, y_1, t_1), \end{aligned}$$

from (8) and so

$$\begin{aligned} M(y_m, y_n, t) &\geq M^{\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}}(y_0, y_1, t_1) \\ &\geq M^{\frac{\delta^n}{1-\delta}}(y_0, y_1, t_1) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

Now assume that $T(X)$ is complete. Since $\{y_{2n}\} = \{Tx_{2n+1}\} \subseteq T(X)$, there exists $y \in T(X)$ such that $y_{2n} \to y$. Hence there exists $u \in X$ such that $y = Tu$ and since $\{y_n\}$ is Cauchy, we also have $y_{2n+1} \to y$. Now since $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to y$ as $n \to \infty$, we have from (2.1.5) that $x_{2n} \preceq y$. Since the dominating map f is weak annihilator of T, we obtain

$$x_{2n} \preceq y = Tu \preceq fTu \preceq u.$$

Now we consider

$$M(y_{2n}, gu, t) = M(fx_{2n}, gu, t) \geq L^{\delta}(x_{2n}, u, t). \tag{9}$$

We have

$$
\begin{aligned}
L(x_{2n},u,t) &= \min\left\{\begin{array}{c} M(Sx_{2n},Tu,t), \\ \\ \frac{M(fx_{2n},Sx_{2n},t)\ *\ M(gu,Tu,t)}{M(Sx_{2n},Tu,t)}, \\ \\ \frac{M(fx_{2n},Tu,2t)\ *\ M(gu,Sx_{2n},2t)}{M(Sx_{2n},Tu,t)} \end{array}\right\} \\
&= \min\left\{M(y_{2n-1},y,t), \frac{M(y_{2n},y_{2n-1},t)\ *\ M(gu,y,t)}{M(y_{2n-1},y,t)},\right. \\
&\qquad\qquad \left.\frac{M(y_{2n},y,2t)\ *\ M(gu,y_{2n-1},2t)}{M(y_{2n-1},y,t)}\right\} \\
&\to M(gu,y,t), \text{ as } n\to\infty.
\end{aligned}
$$

Letting $n \to \infty$ in (9), we get

$$M(y,gu,t) \geq M^{\delta}(gu,y,t),$$

which implies that $gu = y$ and since g and T are weakly compatible, we have $gy = Ty$.

Now

$$M(y_{2n},gy,t) = M(fx_{2n},gy,t) \geq L^{\delta}(x_{2n},y,t), \tag{10}$$

where

$$
\begin{aligned}
L(x_{2n},y,t) &= \min\left\{\begin{array}{c} M(Sx_{2n},Ty,t), \\ \\ \frac{M(fx_{2n},Sx_{2n},t)\ *\ M(gy,Ty,t)}{M(Sx_{2n},Ty,t)}, \\ \\ \frac{M(fx_{2n},Ty,2t)\ *\ M(gy,Sx_{2n},2t)}{M(Sx_{2n},Ty,t)} \end{array}\right\} \\
&= \min\left\{M(y_{2n-1},gy,t), \frac{M(y_{2n},y_{2n-1},t)\ *\ 1}{M(y_{2n-1},gy,t)},\right. \\
&\qquad\qquad \left.\frac{M(y_{2n},gy,2t)\ *\ M(gy,y_{2n-1},2t)}{M(y_{2n-1},gy,t)}\right\} \\
&\to M(y,gy,t), \text{ as } n\to\infty.
\end{aligned}
$$

Letting $n \to \infty$ in (10), we get

$$M(y,gy,t) \geq M^{\delta}(y,gy,t),$$

which implies that $gy = y$. Thus

$$Ty = gy = y. \tag{11}$$

Since $g(X) \subseteq S(X)$, there exists $v \in X$ such that $y = gy = Sv$. Since the dominating map g is weak annihilator of S, we obtain

$$y \preceq gy = Sv \preceq gSv \preceq v$$

and so $y \preceq v$.

Now

$$M(fv, Sv, t) = M(fv, gv, t) \geq L^{\delta}(v, y, t),$$

where

$$\begin{aligned} L(v,y,t) \;&=\; \min\left\{M(Sv,Ty,t), \frac{M(fv,Sv,t) \;*\; M(gy,Ty,t)}{M(Sv,Ty,t)},\right. \\ &\qquad\qquad \left.\frac{M(fv,Ty,2t) \;*\; M(gy,Sv,2t)}{M(Sv,Ty,t)}\right\} \\ &=\; \min\{1, M(fv,Sv,t), M(fv,Sv,2t)\} \\ &=\; M(fv,Sv,t). \end{aligned}$$

Thus $M(fv, Sv, t) \geq M^{\delta}(fv, Sv, t)$ which implies that $fv = Sv$. Since f and S are weakly compatible, we have $fy = Sy$.

Now suppose $fy \neq y$. Then

$$M(fy, y, t) = M(fy, gy, t) \geq L^{\delta}(y, y, t),$$

where

$$\begin{aligned} L(y,y,t) \;&=\; \min\left\{M(Sy,Ty,t), \frac{M(fy,Sy,t) \;*\; M(gy,Ty,t)}{M(Sy,Ty,t)},\right. \\ &\qquad\qquad \left.\frac{M(fy,Ty,2t) \;*\; M(gy,Sy,2t)}{M(Sv,Ty,t)}\right\} \\ &=\; \min\left\{M(fy,y,t), \frac{1 \;*\; 1}{M(fy,y,t)},\right. \\ &\qquad\qquad \left.\frac{M(fy,y,2t) \;*\; M(y,fy,2t)}{M(fv,y,t)}\right\} \\ &=\; M(fy,y,t), \text{ since } M(fy,y,2t) > M(fy,y,t). \end{aligned}$$

Thus we have $M(fy, y, t) \geq M\delta(fy, y, t)$, a contradiction.
Hence $fy = y$. Thus

$$Sy = fy = y \tag{12}$$

and from (11) and (12), we now see that y is a common fixed point of f, g, S and T.

Similarly we can prove that y is a common fixed point of f, g, S and T when $S(X)$ is complete.

If the set of common fixed points of f, g, S and T is well ordered, then the uniqueness of the common fixed point follows easily from (2.1.4).

Now, we give an example to illustrate Theorem 2.1.

Example 2.2. *Let* $X = [0, \infty)$ *and* $M(x, y, t) = e^{-\frac{|x-y|}{t}}$, $\forall x, y \in X$ *and* $t > 0$ *with* $a * b = ab$, $\forall a, b \in [0, 1]$. *Suppose that* $\leq$ *is the usual ordering on* R.

We define a new ordering " $\preceq$ " *on* X *as follows:*

$$x \preceq y \iff y \leq x, \ \forall x, y \in X.$$

Then $(X, M, *, \preceq)$ *is an ordered complete fuzzy metric space.*

Define $f, g, S, T : X \to X$ *as* $fx = \ln\left(1 + \frac{x}{2}\right)$, $gx = \ln(1 + x)$, $Sx = e^x - 1$ *and* $Tx = e^{2x} - 1$.

Then $f(X) = g(X) = [0, \infty) = S(X) = T(X)$ *and* $S(X)$ *is complete.*

Since $fx = \ln\left(1 + \frac{x}{2}\right) \leq x$ *and* $gx = \ln(1 + x) \leq x$ *we have* $x \preceq fx$ *and* $x \preceq gx$ *respectively. Since*

$$fTx = \ln\left(\frac{e^{2x} + 1}{2}\right) = \ln\left(e^x \frac{e^x + e^{-x}}{2}\right) = x + \ln(\cosh x) \geq x,$$

and

$$gSx = \ln(e^x) = x,$$

we have $fTx \preceq x$ *and* $gSx \preceq x$ *respectively.*

Since $fx = Tx$ *implies* $x = 0$ *and* $fT0 = Tf0$ *and* $gx = Sx$ *implies* $x = 0$ *and* $gS0 = Sg0$, *it follows that* (f, S) *and* (g, S) *are weakly compatible pairs.*

Example 1. *From the Mean Value Theorem, we have*

$$\begin{aligned} M(fx, gy, t) &= e^{-\frac{|f(x)-g(y)|}{t}} \\ &= e^{-\frac{\left|\log\left(1+\frac{x}{2}\right)-\log(1+y)\right|}{t}} \\ &\geq e^{-\frac{\left|\frac{x}{2}-y\right|}{t}} \\ &= \left(e^{-\frac{|x-2y|}{t}}\right)^{\frac{1}{2}} \\ &\geq M^{\frac{1}{2}}(Sx, Ty, t) \\ &\geq L^{\frac{1}{2}}(x, y, t) \end{aligned}.$$

Thus all conditions of Theorem 2.1 *are satisfied and clearly '0' is the unique common fixed point of* f, g, S *and* T.

Now we prove a similar theorem for Jungck type maps.

Theorem 2.3. *Let* $(X, M, *, \preceq)$ *be an ordered fuzzy metric space such that* $a * b \geq ab \ \forall \ a, b \in [0, 1]$. *Let* f *and* S *be self mappings on* X *satisfying*
(2.3.1) f *is a dominating map which is an annihilator of* S,
(2.3.2) $f(X) \subseteq S(X)$ *and at least one of* $f(X)$ *and* $S(X)$ *is a complete sub-space of* X,
(2.3.3) *the pair* (f, S) *is weakly compatible,*
(2.3.4) $M(fx, fy, t) \geq L^\delta(x, y, t)$, *for every two comparable elements* $x, y \in X, \ t > 0$ *and* $0 \leq \delta < 1$, *where*

$$L(x,y,t) \ = \ \min\left\{M(Sx,Sy,t), \frac{M(fx,Sx,t) * M(fy,Sy,t)}{M(Sx,Sy,t)}, \frac{M(fx,Sy,t) * M(fy,Sx,2t)}{M(Sx,Sy,t)}\right\},$$

(2.3.5) *Further, suppose that for a non-decreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \preceq u$ *for all* n.
Then f *and* S *have a common fixed point in* X. *If further we assume that the set of common fixed points of* f *and* S *is well ordered then* f *and* S *have a unique common fixed point.*

Proof: Let $x_0 \in X$. From (2.3.2), there exists a sequence $\{x_n\}$ in X such that $fx_n = Sx_{n+1}, \ n = 0, 1, 2, \cdots$ Then, from(2.3.1), we have

$$x_n \preceq fx_n = Sx_{n+1} \preceq fSx_{n+1} \preceq x_{n+1}$$

for all $n \geq 0$.

Case(a): Suppose that $fx_m = fx_{m+1}$ for some m. Then $Sx_{m+1} = fx_{m+1}$. Hence x_{m+1} is a coincidence point of f and S.
Let $\alpha = Sx_{m+1} = fx_{m+1}$. Since the pair (f, S) is weakly compatible, we have $f\alpha = S\alpha$.
From (2.3.4), we have

$$M(\alpha, S\alpha, t) = M(fx_{m+1}, f\alpha, t) \geq L^\delta(x_{m+1}, \alpha, t),$$

where

$$L(x_{m+1},\alpha,t) = \min\left\{\begin{array}{c} M(Sx_{m+1},S\alpha,t), \\ \frac{M(fx_{m+1},Sx_{m+1},t)\ *\ M(f\alpha,S\alpha,t)}{M(Sx_{m+1},S\alpha,t)}, \\ \frac{M(fx_{m+1},S\alpha,t)\ *\ M(f\alpha,Sx_{m+1},2t)}{M(Sx_{m+1},S\alpha,t)} \end{array}\right\}$$

$$= \min\left\{M(\alpha,S\alpha,t),\frac{1\ *\ 1}{M(\alpha,S\alpha,t)},\right.$$
$$\left.\frac{M(\alpha,S\alpha,t)\ *\ M(S\alpha,\alpha,2t)}{M(\alpha,S\alpha,t)}\right\}$$
$$= M(\alpha,S\alpha,t),$$

since $a*b \geq ab,\ \forall a,b\in[0,1]$.

Thus $M(\alpha,S\alpha,t)\geq M^{\delta}(\alpha,S\alpha,t)$. This implies that $S\alpha=\alpha$ and so $f\alpha=S\alpha=\alpha$. Hence α is a common fixed point of f and S.

Case(b): Suppose that $fx_n\neq fx_{n+1}$ for all n. Then

$$M(fx_n,fx_{n+1},t)\geq L^{\delta}(fx_{n-1},fx_n,t), \tag{13}$$

where

$$L(x_n,x_{n+1},t) = \min\left\{\begin{array}{c} M(fx_{n-1},fx_n,t), \\ \frac{M(fx_n,fx_{n-1},t)\ *\ M(fx_{n+1},fx_n,t)}{M(fx_{n-1},fx_n,t)}, \\ \frac{M(fx_n,fx_n,t)\ *\ M(fx_{n+1},fx_{n-1},2t)}{M(fx_{n-1},fx_n,t)} \end{array}\right\}$$
$$= \min\left\{M(fx_{n-1},fx_n,t),M(fx_{n+1},fx_n,t)\right\},$$

since $a*b\geq ab$ and (M_4).

If $M(fx_{n+1},fx_n,t)\leq M(fx_{n-1},fx_n,t)$, then from (13), we obtain

$$M(fx_{n+1},fx_n,t)\geq M^{\delta}(fx_{n+1},fx_n,t),$$

a contradiction. Hence

$$M(fx_{n+1},fx_n,t)\geq M^{\delta}(fx_{n-1},fx_n,t).$$

The rest of the proof follows as in Theorem 2.1.

The following example illustrates Theorem 2.3.

Example 2.4. *Let* $X = [0, \infty)$ *and* $M(x, y, t) = e^{-\frac{|x-y|}{t}}$, $\forall x, y \in X$ *and* $t > 0$ *with* $a * b = ab$, $\forall a, b \in [0, 1]$. *Suppose that* $\leq$ *is the usual ordering on* R.

We define a new ordering " $\preceq$ " *on* X *as follows:*

$$x \preceq y \iff y \leq x, \ \forall x, y \in X.$$

Then $(X, M, *, \preceq)$ *is an ordered complete fuzzy metric space.*

Define $f, S : X \to X$ *as* $fx = \ln\left(1 + \frac{x}{2}\right)$ *and* $Sx = e^{2x} - 1$.
Then $f(X) = [0, \infty) = S(X)$ *and* $S(X)$ *is complete.*

Since $fx = \ln\left(1 + \frac{x}{2}\right) \leq x$, *we have* $x \preceq fx$ *and since*

$$fSx = \ln\left(\frac{e^{2x}+1}{2}\right) = \ln\left(e^x \frac{e^x + e^{-x}}{2}\right) = x + \ln(\cosh x) \geq x,$$

we have $fSx \preceq x$.

Since $fx = Sx$ *implies* $x = 0$ *and* $fS0 = Sf0$, *it follows that* (f, S) *is a weakly compatible pair.*

From the Mean Value Theorem, we have

$$\begin{aligned} M(fx, fy, t) &= e^{-\frac{|f(x)-f(y)|}{t}} \\ &= e^{-\frac{\left|\log\left(1+\frac{x}{2}\right) - \log\left(1+\frac{y}{2}\right)\right|}{t}} \\ &\geq e^{-\frac{|x-y|}{2t}} \\ &\geq \left(e^{-2\frac{|x-y|}{t}}\right)^{\frac{1}{4}} \\ &= M^{\frac{1}{4}}(Sx, Sy, t) \\ &\geq L^{\frac{1}{4}}(x, y, t) \end{aligned}.$$

Thus all conditions of Theorem 2.3 *are satisfied and clearly '0' is the unique common fixed point of* f *and* S.

3. Conclusions

In this paper our results for four maps and two maps satisfying new rational inequalities without condition (1), generalize and improve some known results in existing literature in fuzzy metric spaces. We also provided two examples to illustrate our main two theorems

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