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REMARKS ON THE EXPONENT FUNCTION ASSOCIATED TO A FINITE GROUP

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Abstract. This note deals with the exponent function associated to a finite group. Some classes of finite groups determined by basic properties of this function are investigated.

1. Introduction

The relation between the structure of a group and the structure of its lattice of subgroups constitutes an important domain of research in group theory. The topic has enjoyed a rapid development starting with the first half of the '20 century. Many characterizations and classifications have been obtained for groups for which the subgroup lattice has certain lattice-theoretic properties. We refer to [6, 8] for more information about this theory.

In the following, given a finite group G of order n, we will denote by $L(G)$ the subgroup lattice of G and by L_n the lattice of divisors of n. Recall that $L(G)$ is a complete bounded lattice with respect to set inclusion, having initial element the trivial subgroup 1 and final element G, and its binary operations $\wedge, \vee$ are defined by

$$H \wedge K = H \cap K,\ H \vee K = \langle H \cup K \rangle,\ \forall\, H, K \in L(G).$$

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Several integer valued functions on $L(G)$ have been studied. From these the most important is probably the *order function*

$$\text{ord} : L(G) \longrightarrow L_n,\ H \mapsto |H|,\ \forall\, H \in L(G),$$

whose basic properties are strongly connected with the structure of G. For example, it is well-known that the following conditions are equivalent:

(o1) ord is injective;
(o2) ord is a semilattice homomorphism from $(L(G), \wedge)$ to (L_n, gcd);
(o3) ord is a semilattice homomorphism from $(L(G), \vee)$ to (L_n, lcm);
(o4) ord is a lattice isomorphism from $(L(G), \wedge, \vee)$ to $(L_n, \text{gcd}, \text{lcm})$;
(o5) G is cyclic.

Recall also that the surjectivity of ord leads to an interesting class of finite groups: the CLT groups, i.e. finite groups satisfying the Converse of Lagrange's Theorem (see e.g. [1, 4, 5]). Notice that CLT groups are solvable and that supersolvable groups are CLT.

The main goal of the current note is to investigate similar properties for another integer valued function on $L(G)$, namely the *exponent function*

$$\exp : L(G) \longrightarrow L_n,\ H \mapsto \exp(H),\ \forall\, H \in L(G).$$

In this case (o1), (o2), (o4) and (o5) remain equivalent, while (o3) induces a new class of finite groups that has the same intersection with p-groups as the class CP_2, studied in [9]. We also determine the finite groups for which exp is surjective, namely the ZM-groups. A more natural version of the exponent function and some related open problems will be also presented.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on group theory can be found in [3, 7].

2. Main results

Let G be a finite group of order n. First of all, we recall that $\exp(H) \mid \text{ord}(H)$, $\forall\, H \in L(G)$. Moreover, if H is abelian, then we have $\exp(H) = \text{ord}(H)$ if and only if H is cyclic. We easily infer that the functions exp and ord associated to a finite abelian group G coincide if and only if G is cyclic. Notice that there are also finite non-abelian groups G for which the functions exp and ord coincide, such as S_3.

Clearly, exp is order-preserving, i.e. if $H, K \in L(G)$ satisfy $H \leq K$, then $\exp(H) \mid \exp(K)$. Another easy but very important property of exp is that it is a multiplicative function: if H_i, $i = 1, 2, ..., k$, are subgroups of coprime orders of G and if the product $\prod_{i=1}^{k} H_i$ is also a subgroup of G, then

$$\exp(\prod_{i=1}^{k} H_i) = \prod_{i=1}^{k} \exp(H_i).$$

In particular, this equality shows that the study of the exponent function for finite nilpotent groups is reduced to p-groups.

Our first result proves that the above conditions (o1), (o2), (o4) and (o5) remain also equivalent by replacing ord with exp.

Theorem 1. *For a finite group G of order n the following conditions are equivalent:*

(e1) exp *is injective*;
(e2) exp *is a semilattice homomorphism from* $(L(G), \wedge)$ *to* (L_n, gcd);
(e4) exp *is a lattice isomorphism from* $(L(G), \wedge, \vee)$ *to* $(L_n, \mathrm{gcd}, \mathrm{lcm})$;
(e5) *G is cyclic.*

Proof. If G is cyclic, then exp coincides with ord and so this is a lattice isomorphism between $L(G)$ and L_n, i.e. (e4) is satisfied. On the other hand, it is obvious that (e4) implies both (e1) and (e2). Therefore we must show only that (e1) implies (e5) and (e2) implies (e5).

Assume that exp is injective. Then G has a unique Sylow p-subgroup S_p for every prime p, because Sylow p-subgroups are of the same exponent. In other words, G is nilpotent. We also infer that every S_p possesses exactly one subgroup of order p. Thus, it is either cyclic or a generalized quaternion 2-group by (4.4) of [7], II. If $S_p \cong Q_{2^k}$ for some $k \geq 3$, then it contains at least two distinct isomorphic maximal subgroups, contradicting our assumption. Hence G is cyclic, as a direct product of cyclic Sylow p-subgroups.

Finally, assume that exp is a $\wedge$-homomorphism. One obtains again that G has a unique subgroup of order p, for all $p \in \pi(G)$. Consequently, the Sylow p-groups of G are cyclic for p odd, while the Sylow 2-subgroups of G are either cyclic or generalized quaternion. Since in each Q_{2^k}, $k \geq 3$, there are distinct cyclic subgroups of order 4 and for two such subgroups H and K the condition $\exp(H \wedge K) = \gcd(\exp(H), \exp(H))$ is obviously not verified, it follows that all Sylow subgroups of G must be cyclic. Let $S \in \mathrm{Syl}_p(G)$. Then for all $x \in G$ we have

$$\mathrm{ord}(S \wedge S^x) = \exp(S \wedge S^x) = \gcd(\exp(S), \exp(S^x)) =$$

$$= \gcd(\mathrm{ord}(S), \mathrm{ord}(S^x)) = \mathrm{ord}(S),$$

which implies $S^x = S$. In this way, the Sylow subgroups of G are also normal and hence G is cyclic. This completes the proof. ∎

In the following let us denote by $\mathcal{C}_1$ the class consisting of all finite groups G whose exponent function satisfies the condition (e3), that is

$$\exp(H \vee K) = \mathrm{lcm}(\exp(H), \exp(K)),\ \forall\, H, K \in L(G). \quad (1)$$

We observe that (1) can be rewritten equivalently as

$$\exp(\langle x, y\rangle) = \mathrm{lcm}(\exp(\langle x\rangle), \exp(\langle y\rangle)) = \mathrm{lcm}(o(x), o(y)),\ \forall\, x, y \in G, \quad (2)$$

because the subgroups of G are the join of their cyclic subgroups.

It is clear that $\mathcal{C}_1$ contains all finite abelian groups, but some non-abelian groups, as Q_8, also belong to $\mathcal{C}_1$. Therefore it must be investigated more carefully.

Lemma 2. *$\mathcal{C}_1$ is contained in the class of finite nilpotent groups.*

Proof. Let G be a group in $\mathcal{C}_1$ and S be a Sylow p-subgroup of G. For every $x \in G$, we have $\exp(S \vee S^x) = \exp(S)$ and so $S \vee S^x$ is a p-subgroup of G. Since $S \subseteq S \vee S^x$, we infer that $S \vee S^x = S$. Clearly, this leads to $S^x = S$, i.e. S is normal in G. Hence G is nilpotent. ∎

Lemma 2 allows us to restrict the study to p-groups. The containment of these groups to $\mathcal{C}_1$ can be easily characterized.

Lemma 3. *For a finite p-group G the following conditions are equivalent:*

i) *G belongs to $\mathcal{C}_1$;*
ii) $o(xy) \le \max\{o(x), o(y)\}$, $\forall\, x, y \in G$;
iii) $\Omega_n(G) = \{x \in G \mid x^{p^n} = 1\}$, $\forall\, n \in \mathbb{N}$.

Proof. By (2) we infer that G belongs to $\mathcal{C}_1$ if and only if

$$\exp(\langle x, y\rangle) = \max\{o(x), o(y)\},\ \forall\, x, y \in G,$$

and it is a simple exercise to show that this condition is equivalent with ii). In other words, G belongs to $\mathcal{C}_1$ if and only if it belongs to the class CP_2 defined by ii). The equivalence of ii) and iii) follows from Theorem D of [9]. ∎

Clearly, the above two results lead to a characterization of groups in $\mathcal{C}_1$.

Theorem 4. *A finite group is contained in* $\mathcal{C}_1$ *if and only if it is a nilpotent group all of whose Sylow subgroups belong to* CP_2.

In [9] we proved that large classes of p-groups, such as regular p-groups, modular p-groups or powerful p-groups for p odd, are contained in CP_2. Consequently, they are contained in $\mathcal{C}_1$, too.

Corollary 5. *Every finite nilpotent group whose Sylow subgroups are either regular, modular or powerful* p*-groups for* p *odd belongs to* $\mathcal{C}_1$. *In particular, every finite abelian group belongs to* $\mathcal{C}_1$.

Remark. The classes $\mathcal{C}_1$ and CP_2 are distinct, even they have the same intersection with p-groups. Indeed, $\mathcal{C}_1$ contains only nilpotent groups, while in CP_2 there are certain non-nilpotent groups, as A_4.

Next we will investigate the surjectivity of the exponent function. This also leads to a well-known class of finite groups.

Theorem 6. *For a finite group* G *of order* n *the following conditions are equivalent:*

1. $\exp : L(G) \longrightarrow L_n$ *is surjective*;
2. G *is a* ZM*-group, that is all Sylow subgroups of* G *are cyclic.*

Proof. Assume first that exp is surjective, take a prime divisor p of n and put $n = p^k m$ with $p \nmid m$. Then G contains a subgroup H of the exponent p^k. We infer that H is a p-subgroup and so there is $x \in H$ such that $o(x) = p^k$. Clearly, one obtains $H = \langle x \rangle$, i.e. H is cyclic.

Conversely, assume that G is a ZM-group. By [3] such a group is of type

$$\mathrm{ZM}(u, v, r) = \langle a, b \mid a^u = b^v = 1,\ b^{-1}ab = a^r \rangle,$$

where the triple (u, v, r) satisfies the conditions

$$\gcd(u, v) = \gcd(u, r - 1) = 1 \text{ and } r^v \equiv 1 \pmod{u}.$$

The subgroups of $\mathrm{ZM}(u, v, r)$ have been completely described in [2]. Set

$$L = \left\{ (u_1, v_1, s) \in \mathbb{N}^3 \ \mid\ u_1 \mid u,\ v_1 \mid v,\ s < u_1,\ u_1 \mid s\frac{r^v - 1}{r^{v_1} - 1} \right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\mathrm{ZM}(u,v,r))$ of $\mathrm{ZM}(u,v,r)$, namely the function that maps a triple $(u_1,v_1,s) \in L$ into the subgroup $H_{(u_1,v_1,s)}$ defined by

$$H_{(u_1,v_1,s)} = \bigcup_{k=1}^{\frac{v}{v_1}} \alpha(v_1,s)^k \langle a^{u_1} \rangle = \langle a^{u_1}, \alpha(v_1,s) \rangle,$$

where $\alpha(x,y) = b^x a^y$, for all $0 \le x < v$ and $0 \le y < u$. Remark also that

$$\exp(H_{(u_1,v_1,s)}) = \frac{uv}{u_1 v_1}\,, \text{ for any } s \text{ such that } (u_1,v_1,s) \in L. \tag{3}$$

It is obvious that if $G = \mathrm{ZM}(u,v,r)$, then $n = uv$ and therefore all divisors of n are of the form $\frac{uv}{u_1 v_1}$ with $u_1 \mid u$ and $v_1 \mid v$. Hence the equality (3) completes the proof. ∎

Since every ZM-group is supersolvable, and consequently a CLT group, an unexpected implication follows from Theorem 6.

Corollary 7. *Let G be a finite group of order n and* $\mathrm{ord}, \exp : L(G) \longrightarrow L_n$ *be the order function and the exponent function associated to G, respectively. If* $\exp$ *is surjective, then so is* ord.

We end this note by indicating two open problems concerning the exponent function.

Problem 1. Let G be a finite group and $m = \exp(G)$. Denote by $\mathcal{C}_2$ the class consisting of all finite groups G for which $\exp : L(G) \longrightarrow L_m$ is surjective. Obviously, $\mathcal{C}_2$ contains $\mathcal{C}_1$. We observe that if G has elements of order m, then it belongs to $\mathcal{C}_2$. In other words, $\mathcal{C}_2$ also contains the class $\mathcal{C}$ studied in [10]. These remarks make it natural to ask for a precise description of the groups in $\mathcal{C}_2$.

Problem 2. It is a usual technique to consider an equivalence relation $\sim$ on an algebraic structure and then to study the factor set with respect to $\sim$, partially ordered by certain ordering relations. Following this technique, let P be the set of equivalence classes of subgroups of G with respect to the kernel of exp, that is

$$P = \{[H] \mid H \in L(G)\}, \text{ where } [H] = \{K \in L(G) \mid \exp(K) = \exp(H)\},$$

and define

$$[H_1] \le [H_2] \text{ if and only if } K_1 \subseteq K_2 \text{ for some } K_1 \in [H_1] \text{ and } K_2 \in [H_2].$$

Determine the finite groups G for which $(P, \leq)$ is a poset and study its properties. When is $(P, \leq)$ a lattice?

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