"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 24(2014), No. 2, 5-24

HYPERSURFACE FAMILY WITH A COMMON ISOGEODESIC

ERGIN BAYRAM AND EMIN KASAP

Abstract. In this paper, we study the problem of finding a hypersurface family from a given spatial geodesic curve in $\mathbb{R}^4$. We obtain the parametric representation for a hypersurface family whose members have the same curve as a given geodesic curve. Using the Frenet frame of the given geodesic curve, we present the hypersurface as a linear combination of this frame and analyze the necessary and sufficient conditions for that curve to be geodesic. We illustrate this method by presenting some examples.

1. Introduction

Geodesic is a well-known notion in differential geometry. A geodesic on a surface can be defined in many equivalent ways. Geometrically, the shortest path joining any two points of a surface is a geodesic. Geodesics are curves in surfaces that play a role analogous that of straight lines in the plane. A straight line doesn't bend to left or right as we travel along it [6].

Keywords and phrases: hypersurface, Frenet frame, geodesic.
(2010)Mathematics Subject Classification: 53A04, 53A07

In recent years, there have been various researches on geodesics. Kumar et al. [20] presented a study on geodesic curves computed directly on NURBS surfaces and discrete geodesics computed on the equivalent tessellated surfaces. Wang et al. [26] studied the problem of constructing a family of surfaces from a given spatial geodesic curve and derived a parametric representation for a surface pencil whose members share the same geodesic curve as an isoparametric curve. Sanchez and Dorado [21] presented a practical method to construct polynomial surfaces from a polynomial geodesic or a family of geodesics, by prescribing tangent ribbons. Sprynski et al. [22] dealt with reconstruction of numerical or real surfaces based on the knowledge of some geodesic curves on the surface. Paluszny [19] considered patches that contain any given 3D polynomial curve as a pregeodesic (i.e. geodesic up to reparametrization). Given two pairs of regular space curves $r_1(u)$, $r_3(u)$ and $r_2(v)$, $r_4(v)$ that define a curvilinear rectangle, Farouki et al. [10] handled the problem of constructing a C^2 surface patch $\mathbf{R}(u,v)$ for which these four boundary curves correspond to geodesics of the surface. Farouki et al. [11] considered the problem of constructing polynomial or rational tensor-product Bézier patches bounded by given four polynomial or rational Bézier curves defining a curvilinear rectangle, such that they are geodesics of the constructed surface.

On the other hand, Wang et al. [26] tackled the problem of finding surfaces passing through a given geodesic. In 2011, the given curve was changed to a line of curvature and Li et al. [18] constructed a surface family from a given line of curvature. Bayram et al. [5] gave the necessary and sufficient conditions for a given curve to be an asymptotic on a surface.

However, while differential geometry of a parametric surface in $\mathbb{R}^3$ can be found in textbooks such as in Struik [24], Willmore [28], Stoker [23], do Carmo [7], differential geometry of a parametric surface in $\mathbb{R}^n$ can be found in textbook such as in the contemporary literature on Geometric Modeling [9, 16]. Also, there is little literature on differential geometry of parametric surface family in $\mathbb{R}^3$ [2, 8, 17, 26], but not in $\mathbb{R}^4$. Besides, there is an ascending interest on fourth dimension [1, 2, 8].

Furthermore, various visualization techniques about objects in Euclidean n-space $(n \geq 4)$ are presented [3, 4, 14]. The fundamental step to visualize a 4D object is projecting first into the 3-space and then into the plane. In many real world applications, the problem of visualizing three-dimensional data, commonly referred to as scalar fields arouses. The graph of a function $\mathbf{f}(x, y, z) : U \subset \mathbb{R}^3 \to \mathbb{R}$, where U is open, is a special type of parametric hypersurface with the parametrization $(x, y, z, \mathbf{f}(x, y, z))$ in 4-space. There exists a method for rendering such a 3-surface based on known methods for visualizing functions of two variables [13].

In this paper, we consider the four dimensional analogue problem of constructing a parametric representation of a surface family from a given spatial geodesic as in Wang et al. [26], who derived the necessary and sufficient conditions on the marching-scale functions for which the curve C is an isogeodesic, i.e., both a geodesic and a parameter curve, on a given surface. We express the hypersurface pencil parametrically with the help of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ of the given curve. We find the necessary and sufficient constraints on the marching-scale functions, namely, coefficients of Frenet vectors, so that both the geodesic and parametric requirements met. Finally, as an application of our method one example for each type of marching-scale functions is given.

2. Preliminaries

Let us first introduce some notations and definitions. Bold letters such as $\mathbf{a}$, $\mathbf{R}$ will be used for vectors and vector functions. We assume that they are smooth enough so that all the (partial) derivatives given in the paper are meaningful. Let $\boldsymbol{\alpha} : \mathbf{I} \subset \mathbb{R} \to \mathbb{R}^4$ be an arc-length curve. If $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ is the moving Frenet frame along $\boldsymbol{\alpha}$, then the Frenet formulas are given by

$$(1) \qquad \begin{cases} \mathbf{T}' = \kappa_1 \mathbf{N}, \\ \mathbf{N}' = -\kappa_1 \mathbf{T} + \kappa_2 \mathbf{B}_1, \\ \mathbf{B}_1' = -\kappa_2 \mathbf{N} + \kappa_3 \mathbf{B}_2, \\ \mathbf{B}_2' = -\kappa_3 \mathbf{B}_1, \end{cases}$$

where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and $\mathbf{B}_2$ denote the tangent, principal normal, first binormal and second binormal vector fields, respectively, $\kappa_i \ (i = 1, 2, 3)$ the i-th curvature functions of the curve $\boldsymbol{\alpha}$ [14].

From elementary differential geometry we have

$$\left\{\begin{array}{c}\boldsymbol{\alpha}'(s)=\mathbf{T}(s),\\ \boldsymbol{\alpha}''(s)=\kappa_1(s)\,\mathbf{N}(s),\\ \kappa_1(s)=\|\alpha''(s)\|.\end{array}\right. \tag{2}$$

Using Frenet formulas one can obtain the followings

$$\left\{\begin{array}{c}\boldsymbol{\alpha}'''(s)=-\kappa_1^2\mathbf{T}(s)+\kappa_1'\mathbf{N}(s)+\kappa_1\kappa_2\mathbf{B}_1(s),\\ \boldsymbol{\alpha}^{(iv)}(s)=-3\kappa_1\kappa_1'\mathbf{T}(s)+\left(-\kappa_1^3+\kappa_1''-\kappa_1\kappa_2^2\right)\mathbf{N}(s)\\ +\left(2\kappa_1'\kappa_2+\kappa_1\kappa_2'\right)\mathbf{B}_1(s)+\kappa_1\kappa_2\kappa_3\mathbf{B}_2(s).\end{array}\right. \tag{3}$$

The unit vectors $\mathbf{B}_2$ and $\mathbf{B}_1$ are given by

$$\left\{\begin{array}{c}\mathbf{B}_2(s)=\frac{\boldsymbol{\alpha}'(s)\otimes\boldsymbol{\alpha}''(s)\otimes\boldsymbol{\alpha}'''(s)}{\|\boldsymbol{\alpha}'(s)\otimes\boldsymbol{\alpha}''(s)\otimes\boldsymbol{\alpha}'''(s)\|},\\ \mathbf{B}_1(s)=\mathbf{B}_2(s)\otimes\mathbf{T}(s)\otimes\mathbf{N}(s),\end{array}\right. \tag{4}$$

where $\otimes$ is the vector product of vectors in $\mathbb{R}^4$.

Since the vectors $\mathbf{T},\ \mathbf{N},\ \mathbf{B}_1,\ \mathbf{B}_2$ are orthonormal, the second curvature κ_2 and the third curvature κ_3 can be obtained from (3) as

$$\left\{\begin{array}{c}\kappa_2(s)=\frac{\mathbf{B}_1(s)\bullet\boldsymbol{\alpha}'''(s)}{\kappa_1(s)},\\ \kappa_3(s)=\frac{\mathbf{B}_2(s)\bullet\boldsymbol{\alpha}^{(iv)}(s)}{\kappa_1(s)\kappa_2(s)},\end{array}\right. \tag{5}$$

where '$\bullet$' denotes the standard inner product.

Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ be the standard basis for four-dimensional Euclidean space $\mathbb{R}^4$. The vector product of the vectors $\mathbf{u}=\sum_{i=1}^4 u_i\mathbf{e}_i,\ \mathbf{v}=\sum_{i=1}^4 v_i\mathbf{e}_i,\ \mathbf{w}=\sum_{i=1}^4 w_i\mathbf{e}_i$ is defined by

$$\mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}=\left|\begin{array}{cccc}\mathbf{e}_1&\mathbf{e}_2&\mathbf{e}_3&\mathbf{e}_4\\ u_1&u_2&u_3&u_4\\ v_1&v_2&v_3&v_4\\ w_1&w_2&w_3&w_4\end{array}\right|$$

[15, 27].

If $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are linearly independent then $\mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}$ is orthogonal to each of these vectors.

3. Hypersurface family with a common isogeodesic

A curve $\mathbf{r}(s)$ on a hypersurface $\mathbf{P}=\mathbf{P}(s,t,q)\subset\mathbb{R}^4$ is called an isoparametric curve if it is a parameter curve, that is, there exists a pair of parameters t_0 and q_0 such that $\mathbf{r}(s)=\mathbf{P}(s,t_0,q_0)$. Given a parametric curve $\mathbf{r}(s)$, it is called an *isogeodesic* of a hypersurface $\mathbf{P}$ if it is both a geodesic and an isoparametric curve on $\mathbf{P}$.

Let $C : \mathbf{r} = \mathbf{r}(s)\,,\ L_1 \leq s \leq L_2$, be a C^3 curve, where s is the arc-length. To have a well-defined principal normal, assume that $\mathbf{r}''(s) \neq 0,\ L_1 \leq s \leq L_2$.

Let $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)$ be the tangent, principal normal, first binormal, second binormal, respectively; and let $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ be the first, the second and the third curvature, respectively. Since $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ is an orthogonal coordinate frame on $\mathbf{r}(s)$ the parametric hypersurface $\mathbf{P}(s,t,q) : [L_1, L_2] \times [T_1, T_2] \times [Q_1, Q_2] \to \mathbb{R}^4$ passing through $\mathbf{r}(s)$ can be defined as follows:

(6)
$$\mathbf{P}(s,t,q) = \mathbf{r}(s) + (\mathbf{u}(s,t,q), \mathbf{v}(s,t,q), \mathbf{w}(s,t,q), \mathbf{x}(s,t,q)) \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}_1(s) \\ \mathbf{B}_2(s) \end{pmatrix},$$

$$L_1 \leq s \leq L_2, T_1 \leq s \leq T_2, Q_1 \leq s \leq Q_2,$$

where $\mathbf{u}(s,t,q), \mathbf{v}(s,t,q), \mathbf{w}(s,t,q)$ and $\mathbf{x}(s,t,q)$ are all C^4 functions. These functions are called the *marching scale functions*.

We try to find out the necessary and sufficient conditions for which a hypersurface $\mathbf{P} = \mathbf{P}(s,t,q)$ has the curve C as an isogeodesic.

First, to satisfy the isoparametricity condition there should exist $t_0 \in [T_1, T_2]$ and $q_0 \in [Q_1, Q_2]$ such that $\mathbf{P}(s, t_0, q_0) = \mathbf{r}(s)$, $L_1 \leq s \leq L_2$, that is,

$$(7) \quad \begin{cases} \mathbf{u}(s,t_0,q_0) = \mathbf{v}(s,t_0,q_0) = \mathbf{w}(s,t_0,q_0) = \mathbf{x}(s,t_0,q_0) \equiv 0, \\ \qquad t_0 \in [T_1, T_2], q_0 \in [Q_1, Q_2], L_1 \leq s \leq L_2. \end{cases}$$

Secondly, the curve C is a geodesic on the hypersurface $\mathbf{P}(s,t,q)$ if and only if the principal normal $\mathbf{N}(s)$ of the curve and the normal $\overset{\wedge}{\mathbf{n}}(s,t_0,q_0)$ of the hypersurface $\mathbf{P}(s,t,q)$ are linearly dependent, that is, parallel along the curve C [25]. The normal $\overset{\wedge}{\mathbf{n}}(s,t_0,q_0)$ of the hypersurface can be obtained by calculating the vector product of the partial derivatives and using the Frenet formula as follows

$$\begin{aligned} \frac{\partial \mathbf{P}(s,t,q)}{\partial s} = & \left(1 + \frac{\partial \mathbf{u}(s,t,q)}{\partial s} - \mathbf{v}(s,t,q)\kappa_1(s)\right)\mathbf{T}(s) \\ & + \left(\mathbf{u}(s,t,q)\kappa_1(s) + \frac{\partial \mathbf{v}(s,t,q)}{\partial s} - \mathbf{w}(s,t,q)\kappa_2(s)\right)\mathbf{N}(s) \\ & + \left(\mathbf{v}(s,t,q)\kappa_2(s) + \frac{\partial \mathbf{w}(s,t,q)}{\partial s} - \mathbf{x}(s,t,q)\kappa_3(s)\right)\mathbf{B}_1(s) \end{aligned}$$

$$+\left(\mathbf{w}\left(s,t,q\right)\kappa_3\left(s\right)+\frac{\partial \mathbf{x}(s,t,q)}{\partial s}\right)\mathbf{B}_2\left(s\right),$$

$$\frac{\partial \mathbf{P}(s,t,q)}{\partial t}=\frac{\partial \mathbf{u}(s,t,q)}{\partial t}\mathbf{T}\left(s\right)+\frac{\partial \mathbf{v}(s,t,q)}{\partial t}\mathbf{N}\left(s\right)+\frac{\partial \mathbf{w}(s,t,q)}{\partial t}\mathbf{B}_1\left(s\right)+\frac{\partial \mathbf{x}(s,t,q)}{\partial t}\mathbf{B}_2\left(s\right),$$

and

$$\frac{\partial \mathbf{P}(s,t,q)}{\partial q}=\frac{\partial \mathbf{u}(s,t,q)}{\partial q}\mathbf{T}\left(s\right)+\frac{\partial \mathbf{v}(s,t,q)}{\partial q}\mathbf{N}\left(s\right)+\frac{\partial \mathbf{w}(s,t,q)}{\partial q}\mathbf{B}_1\left(s\right)+\frac{\partial \mathbf{x}(s,t,q)}{\partial q}\mathbf{B}_2\left(s\right).$$

Remark 1. *Because,*

$$\left\{\begin{array}{c}\mathbf{u}\left(s,t_0,q_0\right)=\mathbf{v}\left(s,t_0,q_0\right)=\mathbf{w}\left(s,t_0,q_0\right)=\mathbf{x}\left(s,t_0,q_0\right)\equiv 0,\\ t_0\in\left[T_1,T_2\right],q_0\in\left[Q_1,Q_2\right],L_1\leq s\leq L_2.\end{array}\right.$$

along the curve C, by the definition of partial differentiation we have

$$\left\{\begin{array}{c}\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s}=\frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s}=\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial s}=\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s}\equiv 0,\\ t_0\in\left[T_1,T_2\right],q_0\in\left[Q_1,Q_2\right],L_1\leq s\leq L_2.\end{array}\right.$$

Using (7) we have

$$\begin{aligned}\overset{\wedge}{\mathbf{n}}\left(s,t_0,q_0\right)&=\frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial s}\otimes\frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial t}\otimes\frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial q}\\&=\phi_1\left(s,t_0,q_0\right)\mathbf{T}\left(s\right)-\phi_2\left(s,t_0,q_0\right)\mathbf{N}\left(s\right)\\&\quad+\phi_3\left(s,t_0,q_0\right)\mathbf{B}_1\left(s\right)-\phi_4\left(s,t_0,q_0\right)\mathbf{B}_2\left(s\right),\end{aligned}$$

where

$$\phi_1\left(s,t_0,q_0\right)=\left|\begin{array}{ccc}\frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s}\\ \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t}\\ \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}\end{array}\right|=0,$$

$$\begin{aligned}\phi_2\left(s,t_0,q_0\right)&=\left|\begin{array}{ccc}1+\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s}\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t}\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}\end{array}\right|\\&=\left|\begin{array}{ccc}1&0&0\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t}\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}\end{array}\right|\\&=\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t}\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}-\frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q}\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t},\end{aligned}$$

$$\phi_3\left(s,t_0,q_0\right)=\left|\begin{array}{ccc}1+\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s}\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t}\\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q}&\frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}\end{array}\right|$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q} \end{vmatrix}$$
$$= \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q} - \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t},$$

$$\phi_4(s,t_0,q_0) = \begin{vmatrix} 1+\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s} & \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q} \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q} \end{vmatrix}$$
$$= \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial q} - \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial t}.$$

So, $\overset{\wedge}{\mathbf{n}}(s,t_0,q_0) \parallel \mathbf{N}(s)$ if and only if

$$\begin{aligned} \phi_3(s,t_0,q_0) &= \phi_4(s,t_0,q_0) \equiv 0,\ \phi_2(s,t_0,q_0) \neq 0, \\ t_0 &\in [T_1,T_2], q_0 \in [Q_1,Q_2], L_1 \leq s \leq L_2. \end{aligned} \tag{8}$$

Thus, any hypersurface defined by (6) has the curve C as an isogeodesic if and only if

$$\begin{cases} \mathbf{u}(s,t_0,q_0) = \mathbf{v}(s,t_0,q_0) = \mathbf{w}(s,t_0,q_0) = \mathbf{x}(s,t_0,q_0) \equiv 0, \\ \phi_3(s,t_0,q_0) = \phi_4(s,t_0,q_0) \equiv 0,\ \phi_2(s,t_0,q_0) \neq 0, \end{cases} \tag{9}$$

$$t_0 \in [T_1,T_2], q_0 \in [Q_1,Q_2], L_1 \leq s \leq L_2.$$

is satisfied. We call the set of hypersurfaces defined by (6) and satisfying (9) an *isogeodesic hypersurface family.*

To develop the method further, and for simplification purposes, we analyze some types of marching-scale functions.

4. Marching-scale functions of type I

Let marching-scale functions be

$$\begin{cases} \mathbf{u}(s,t,q) = \mathbf{l}(s)\,\mathbf{U}(t,q), \\ \mathbf{v}(s,t,q) = \mathbf{m}(s)\,\mathbf{V}(t,q), \\ \mathbf{w}(s,t,q) = \mathbf{n}(s)\,\mathbf{W}(t,q), \\ \mathbf{x}(s,t,q) = \mathbf{p}(s)\,\mathbf{X}(t,q), \end{cases}, L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, Q_1 \leq q \leq Q_2,$$

where $\mathbf{l}(s), \mathbf{m}(s), \mathbf{n}(s), \mathbf{p}(s), \mathbf{U}(t,q), \mathbf{V}(t,q), \mathbf{W}(t,q), \mathbf{X}(t,q) \in C^1$ and $\mathbf{l}(s) \neq 0 \neq \mathbf{m}(s), \mathbf{n}(s) \neq 0 \neq \mathbf{p}(s), \forall s \in [L_1, L_2]$. Using (9), the necessary and sufficient conditions for which the curve C is an isogeodesic on the hypersurface $\mathbf{P}(s,t,q)$ can be given as

$$(10) \qquad \begin{cases} \mathbf{U}(t_0,q_0) = \mathbf{V}(t_0,q_0) = \mathbf{W}(t_0,q_0) = \mathbf{X}(t_0,q_0) = 0, \\ \frac{\partial \mathbf{V}(t_0,q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial q} - \frac{\partial \mathbf{V}(t_0,q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial t} = 0, \\ \frac{\partial \mathbf{V}(t_0,q_0)}{\partial t} \frac{\partial \mathbf{W}(t_0,q_0)}{\partial q} - \frac{\partial \mathbf{V}(t_0,q_0)}{\partial q} \frac{\partial \mathbf{W}(t_0,q_0)}{\partial t} = 0, \\ \frac{\partial \mathbf{W}(t_0,q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial q} - \frac{\partial \mathbf{W}(t_0,q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial t} \neq 0 \end{cases}$$

$$t_0 \in [T_1, T_2], q_0 \in [Q_1, Q_2].$$

With a closer investigation of (10), we should have $\frac{\partial \mathbf{V}(t_0,q_0)}{\partial t} = 0$ and $\frac{\partial \mathbf{V}(t_0,q_0)}{\partial q} = 0$.

So, (10) can be simplified to

$$(11) \qquad \begin{cases} \mathbf{U}(t_0,q_0) = \mathbf{V}(t_0,q_0) = \mathbf{W}(t_0,q_0) = \mathbf{X}(t_0,q_0) = 0, \\ \frac{\partial \mathbf{V}(t_0,q_0)}{\partial t} = \frac{\partial \mathbf{V}(t_0,q_0)}{\partial q} = 0, \\ \frac{\partial \mathbf{W}(t_0,q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial q} - \frac{\partial \mathbf{W}(t_0,q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0,q_0)}{\partial t} \neq 0 \end{cases}$$

$$t_0 \in [T_1, T_2], q_0 \in [Q_1, Q_2].$$

5. Marching-scale functions of type II

Let marching-scale functions be

$$\begin{cases} \mathbf{u}(s,t,q) = \mathbf{l}(s,t)\,\mathbf{U}(q), \\ \mathbf{v}(s,t,q) = \mathbf{m}(s,t)\,\mathbf{V}(q), \\ \mathbf{w}(s,t,q) = \mathbf{n}(s,t)\,\mathbf{W}(q), \\ \mathbf{x}(s,t,q) = \mathbf{p}(s,t)\,\mathbf{X}(q), \end{cases} L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, Q_1 \leq q \leq Q_2,$$

where $\mathbf{l}(s,t), \mathbf{m}(s,t), \mathbf{n}(s,t), \mathbf{p}(s,t), \mathbf{U}(q), \mathbf{V}(q), \mathbf{W}(q), \mathbf{X}(q) \in C^1$. Also let us choose $\mathbf{V}(q_0) = \frac{d\mathbf{V}(q_0)}{dq} = \mathbf{U}(q_0) = \frac{d\mathbf{U}(q_0)}{dq} = 0$. Using (9), the curve C is an isogeodesic on the hypersurface $\mathbf{P}(s,t,q)$ if and only if the followings are satisfied

(12)
$$\begin{cases} \mathbf{n}(s,t_0)\,\mathbf{W}(q_0) = \mathbf{p}(s,t_0)\,\mathbf{X}(q_0) \equiv 0, \\ \frac{\partial \mathbf{n}(s,t_0)}{\partial t}\mathbf{W}(q_0)\,\mathbf{p}(s,t_0)\,\frac{d\mathbf{X}(q_0)}{dq} - \mathbf{n}(s,t_0)\,\frac{d\mathbf{W}(q_0)}{dq}\frac{\partial \mathbf{p}(s,t_0)}{\partial t}\mathbf{X}(q_0) \neq 0, \end{cases}$$

$$t_0 \in [T_1, T_2]\,, q_0 \in [Q_1, Q_2]\,, L_1 \le s \le L_2.$$

6. Marching-scale functions of type III

Let marching-scale functions be

$$\begin{cases} \mathbf{u}(s,t,q) = \mathbf{l}(s,q)\,\mathbf{U}(t)\,, \\ \mathbf{v}(s,t,q) = \mathbf{m}(s,q)\,\mathbf{V}(t)\,, \\ \mathbf{w}(s,t,q) = \mathbf{n}(s,q)\,\mathbf{W}(t)\,, \\ \mathbf{x}(s,t,q) = \mathbf{p}(s,q)\,\mathbf{X}(t)\,, \end{cases} \quad L_1 \le s \le L_2, T_1 \le t \le T_2, Q_1 \le q \le Q_2,$$

where $\mathbf{l}(s,q)\,, \mathbf{m}(s,q)\,, \mathbf{n}(s,q)\,, \mathbf{p}(s,q)\,, \mathbf{U}(t)\,, \mathbf{V}(t)\,, \mathbf{W}(t)\,, \mathbf{X}(t) \in C^1$. Also let us choose $\mathbf{V}(t_0) = \frac{d\mathbf{V}(t_0)}{dt} = \mathbf{U}(t_0) = \frac{d\mathbf{U}(t_0)}{dt} = 0$. Using (9) we derive the necessary and sufficient conditions for which the curve C is an isogeodesic on the hypersurface $\mathbf{P}(s,t,q)$ as

(13)
$$\begin{cases} \mathbf{n}(s,q_0)\,\mathbf{W}(t_0) = \mathbf{p}(s,q_0)\,\mathbf{X}(t_0) \equiv 0, \\ \mathbf{n}(s,q_0)\,\frac{d\mathbf{W}(t_0)}{dt}\frac{\partial \mathbf{p}(s,q_0)}{\partial q}\mathbf{X}(t_0) - \frac{\partial \mathbf{n}(s,q_0)}{\partial q}\mathbf{W}(t_0)\,\mathbf{p}(s,q_0)\,\frac{d\mathbf{X}(t_0)}{dt} \neq 0, \end{cases}$$

$$t_0 \in [T_1, T_2]\,, q_0 \in [Q_1, Q_2]\,, L_1 \le s \le L_2.$$

7. Examples

Example 1. *Let* $\mathbf{r}(s) = \left(\frac{1}{2}\cos(s), \frac{1}{2}\sin(s), \frac{1}{2}s, \frac{\sqrt{2}}{2}s\right)$, $0 \le s \le 2\pi$, *be a curve parametrized by arc-length. For this curve,*

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{r}'(s) = \left(-\frac{1}{2}\sin(s), \frac{1}{2}\cos(s), \frac{1}{2}, \frac{\sqrt{2}}{2}\right), \\
\mathbf{N}(s) &= (-\cos(s), -\sin(s), 0, 0), \\
\mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\left\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\right\|} = \left(0, 0, \frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right), \\
\mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} = \left(-\frac{\sqrt{3}}{2}\sin(s), \frac{\sqrt{3}}{2}\cos(s), -\frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{6}\right).
\end{aligned}$$

Let us choose the marching-scale functions of type I, where

$$\mathbf{l}(s) = \mathbf{m}(s) = \mathbf{n}(s) = \mathbf{p}(s) \equiv 1$$

and

$$\begin{aligned}
\mathbf{U}(t,q) &= (t-t_0)(q-q_0), \mathbf{V}(t,q) \equiv 0, \mathbf{W}(t,q) = t - t_0, \mathbf{X}(t,q) = q - q_0, \\
t_0 &\in [0,1], q_0 \in [0,1], 0 \le s \le 2\pi.
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbf{u}(s,t,q) &= (t-t_0)(q-q_0), \\
\mathbf{v}(s,t,q) &\equiv 0, \\
\mathbf{w}(s,t,q) &= t - t_0, \\
\mathbf{x}(s,t,q) &= q - q_0.
\end{aligned}$$

The hypersurface

$$\begin{aligned}
\mathbf{P}(s,t,q) &= \mathbf{r}(s)+\mathbf{u}(s,t,q)\,\mathbf{T}(s)+\mathbf{v}(s,t,q)\,\mathbf{N}(s)+\\
&\quad +\mathbf{w}(s,t,q)\,\mathbf{B}_1(s)+\mathbf{x}(s,t,q)\,\mathbf{B}_2(s)\\
&= \Bigg(\frac{1}{2}\cos(s)-\frac{1}{2}(t-t_0)(q-q_0)\sin(s)-\frac{\sqrt{3}}{2}(t-t_0)\sin(s),\\
&\quad \frac{1}{2}\sin(s)+\frac{1}{2}(t-t_0)(q-q_0)\cos(s)+\frac{\sqrt{3}}{2}(t-t_0)\cos(s),\\
&\quad \frac{1}{2}s+\frac{1}{2}(t-t_0)(q-q_0)-\frac{\sqrt{3}}{6}(t-t_0)+\frac{\sqrt{6}}{3}(q-q_0),\\
&\quad \frac{\sqrt{2}}{2}s+\frac{\sqrt{2}}{2}(t-t_0)(q-q_0)-\frac{\sqrt{6}}{6}(t-t_0)-\frac{\sqrt{3}}{3}(q-q_0)\Bigg),
\end{aligned}$$

$0\le s\le 2\pi, 0\le t\le 1, 0\le q\le 1, t_0\in[0,1], q_0\in[0,1]$, *is a member of the isogeodesic hypersurface family, since it satisfies (11).*
By changing the parameters t_0 *and* q_0 *we can adjust the position of the curve* $\mathbf{r}(s)$ *on the hypersurface. Let us choose* $t_0=\frac{1}{2}$ *and* $q_0=0$. *Now the curve* $\mathbf{r}(s)$ *is again an isogeodesic on the hypersurface* $\mathbf{P}(s,t,q)$ *and the equation of the hypersurface is*

$$\begin{aligned}
\mathbf{P}(s,t,q) &= \Bigg(\frac{1}{2}\cos(s)-\frac{1}{2}\left(t-\frac{1}{2}\right)\left(q+\sqrt{3}\right)\sin(s),\\
&\quad \frac{1}{2}\sin(s)+\frac{1}{2}\left(t-\frac{1}{2}\right)\left(q+\sqrt{3}\right)\cos(s),\\
&\quad \frac{1}{2}s+\frac{1}{2}\left(t-\frac{1}{2}\right)q-\frac{\sqrt{3}}{6}\left(t-\frac{1}{2}\right)+\frac{\sqrt{6}}{3}q,\\
&\quad \frac{\sqrt{2}}{2}s+\frac{\sqrt{2}}{2}\left(t-\frac{1}{2}\right)q-\frac{\sqrt{6}}{6}\left(t-\frac{1}{2}\right)-\frac{\sqrt{3}}{3}q\Bigg).
\end{aligned}$$

The projection of a hypersurface into 3-space generally yields a three-dimensional volume. If we fix each of the three parameters, one at a time, we obtain three distinct families of 2-spaces in 4-space. The projections of these 2-surfaces into 3-space are surfaces in 3-space. Thus, they can be displayed by 3D rendering methods.
So, if we (parallel) project the hypersurface $\mathbf{P}(s,t,q)$ *into the* $\mathbf{w}=\mathbf{0}$

subspace and fix $q = \frac{1}{8}$ *we obtain the surface*

$$\begin{aligned}
\mathbf{P_w}\left(s,t,\frac{1}{8}\right) &= \left(\frac{1}{2}\cos(s) - \frac{1+8\sqrt{3}}{16}\left(t-\frac{1}{2}\right)\sin(s),\right.\\
&\quad \frac{1}{2}\sin(s) + \frac{1+8\sqrt{3}}{8}\left(t-\frac{1}{2}\right)\cos(s),\\
&\quad \left.\frac{1}{2}s + \frac{1}{16}\left(t-\frac{1}{2}\right) - \frac{\sqrt{3}}{6}\left(t-\frac{1}{2}\right) + \frac{\sqrt{6}}{24}\right),
\end{aligned}$$

$0 \le s \le 2\pi, 0 \le t \le 1$ *in 3-space illustrated in Fig. 1.*

Example 2. *Given the curve parameterized by arc-length* $\mathbf{r}(s) = \left(\frac{1}{2}\sin(s), \frac{1}{2}\cos(s), 0, \frac{\sqrt{3}}{2}s\right)$, $0 \le s \le 2\pi$, *it is easy to show that*

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{r}'(s) = \left(\frac{1}{2}\cos(s), -\frac{1}{2}\sin(s), 0, \frac{\sqrt{3}}{2}\right),\\
\mathbf{N}(s) &= (-\sin(s), -\cos(s), 0, 0),\\
\mathbf{B}_2(s) &= \frac{\mathbf{r}'(s)\otimes\mathbf{r}''(s)\otimes\mathbf{r}'''(s)}{\|\mathbf{r}'(s)\otimes\mathbf{r}''(s)\otimes\mathbf{r}'''(s)\|} = (0,0,-1,0),\\
\mathbf{B}_1(s) &= \mathbf{B}_2\otimes\mathbf{T}\otimes\mathbf{N} = \left(\frac{\sqrt{3}}{2}\cos(s), -\frac{\sqrt{3}}{2}\sin(s), 0, -\frac{1}{2}\right).
\end{aligned}$$

Let us choose the marching-scale functions of type II, where

$$\mathbf{n}(s,t) = s+t+1, \mathbf{p}(s,t) = (s+1)(t-t_0),$$

and

$$\mathbf{U}(q) = \mathbf{V}(q) \equiv 0, \mathbf{W}(q) = q - q_0, \mathbf{X}(q) \equiv 1.$$

So, we get

$$\begin{aligned}
\mathbf{u}(s,t,q) &\equiv 0,\\
\mathbf{v}(s,t,q) &\equiv 0,\\
\mathbf{w}(s,t,q) &= (s+t+1)(q-q_0),\\
\mathbf{x}(s,t,q) &= (s+1)(t-t_0).
\end{aligned}$$

From (12), the hypersurface

$$\begin{aligned}
\mathbf{P}(s,t,q) &= \mathbf{r}(s) + \mathbf{u}(s,t,q)\,\mathbf{T}(s) + \mathbf{v}(s,t,q)\,\mathbf{N}(s) + \\
&\qquad + \mathbf{w}(s,t,q)\,\mathbf{B}_1(s) + \mathbf{x}(s,t,q)\,\mathbf{B}_2(s) \\
&= \Bigg(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{2}(s+t+1)(q-q_0)\cos(s), \\
&\qquad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{2}(s+t+1)(q-q_0)\sin(s), \\
&\qquad -(s+1)(t-t_0), \\
&\qquad \frac{\sqrt{3}}{2}s - \frac{1}{2}(s+t+1)(q-q_0) \Bigg),
\end{aligned}$$

$0 \le s \le 2\pi, 0 \le t \le 1, 0 \le q \le 1$, *is a member of the hypersurface family having the curve* $\mathbf{r}(s)$ *as an isogeodesic.*

Setting $t_0 = \frac{1}{2}$ *and* $q_0 = 0$ *yields the hypersurface*

$$\begin{aligned}
\mathbf{P}(s,t,q) &= \Bigg(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{2}(s+t+1)\,q\cos(s), \\
&\qquad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{2}(s+t+1)\,q\sin(s), \\
&\qquad -(s+1)\left(t-\frac{1}{2}\right), \\
&\qquad \frac{\sqrt{3}}{2}s - \frac{1}{2}(s+t+1)\,q \Bigg).
\end{aligned}$$

By (parallel) projecting the hypersurface $\mathbf{P}(s,t,q)$ *into the subspace* $\mathbf{w} = \mathbf{0}$ *and fixing* $q = \frac{1}{500}$ *we get the surface*

$$\begin{aligned}
\mathbf{P}_{\mathbf{w}}\left(s,t,\frac{1}{500}\right) &= \Bigg(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{1000}(s+t+1)\cos(s), \\
&\qquad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{1000}(s+t+1)\sin(s), \\
&\qquad -(s+1)\left(t-\frac{1}{2}\right)\Bigg),
\end{aligned}$$

where, $0 \leq s \leq 2\pi, 0 \leq t \leq 1$ *in 3-space, illustrated in Fig. 2.*

Example 3. *Let* $\mathbf{r}(s) = \left(\frac{1}{2}\sin(s), \frac{1}{2}\cos(s), 0, \frac{\sqrt{3}}{2}s\right)$, $\pi \leq s \leq 3\pi$, *be an arc-length curve. One can easily show that, for this curve:*

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{r}'(s) = \left(\frac{1}{2}\cos(s), -\frac{1}{2}\sin(s), 0, \frac{\sqrt{3}}{2}\right), \\
\mathbf{N}(s) &= (-\sin(s), -\cos(s), 0, 0), \\
\mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\|} = (0, 0, -1, 0), \\
\mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} = \left(\frac{\sqrt{3}}{2}\cos(s), -\frac{\sqrt{3}}{2}\sin(s), 0, -\frac{1}{2}\right).
\end{aligned}$$

If we choose the marching-scale functions of type III, where

$$\mathbf{n}(s,q) = \sin(s(q-q_0)), \mathbf{p}(s,q) = sq^2,$$

and

$$\mathbf{U}(t) = \mathbf{V}(t) \equiv 0, \mathbf{W}(t) \equiv 1, \mathbf{X}(q) = t - t_0$$

then

$$\begin{aligned}
\mathbf{u}(s,t,q) &\equiv 0, \\
\mathbf{v}(s,t,q) &\equiv 0, \\
\mathbf{w}(s,t,q) &= \sin(s(q-q_0)), \\
\mathbf{x}(s,t,q) &= sq^2(t-t_0).
\end{aligned}$$

Thus, from (13) if we take $q_0 \neq 0$ *then the curve* $\mathbf{r}(s)$ *is an isogeodesic on the hypersurface*

$$\begin{aligned}
\mathbf{P}(s,t,q) &= \mathbf{r}(s) + \mathbf{u}(s,t,q)\mathbf{T}(s) + \mathbf{v}(s,t,q)\mathbf{N}(s) + \\
&+ \mathbf{w}(s,t,q)\mathbf{B}_1(s) + \mathbf{x}(s,t,q)\mathbf{B}_2(s) \\
&= \left(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{2}\cos(s)\sin(s(q-q_0)),\right. \\
&\quad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{2}\sin(s)\sin(s(q-q_0)), \\
&\quad -sq^2(t-t_0), \\
&\quad \left.\frac{\sqrt{3}}{2}s - \frac{1}{2}\sin(s(q-q_0))\right),
\end{aligned}$$

where $\pi \leq s \leq 3\pi, 0 \leq t \leq 1, 0 \leq q \leq 1$.

By taking $t_0 = 1$ *and* $q_0 = 1$ *we have the following hypersurface:*

$$\begin{aligned}\mathbf{P}(s,t,q) &= \left(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{2}\cos(s)\sin(s(q-1)),\right.\\ &\quad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{2}\sin(s)\sin(s(q-1)),\\ &\quad -sq^2(t-1),\\ &\quad \left.\frac{\sqrt{3}}{2}s - \frac{1}{2}\sin(s(q-1))\right).\end{aligned}$$

Hence, if we (parallel) project the hypersurface $\mathbf{P}(s,t,q)$ *into the* $\mathbf{z} = \mathbf{0}$ *subspace we get the surface*

$$\begin{aligned}\mathbf{P}_z(s,q) &= \left(\frac{1}{2}\sin(s) + \frac{\sqrt{3}}{2}\cos(s)\sin(s(q-1)),\right.\\ &\quad \frac{1}{2}\cos(s) - \frac{\sqrt{3}}{2}\sin(s)\sin(s(q-1)),\\ &\quad \left.\frac{\sqrt{3}}{2}s - \frac{1}{2}\sin(s(q-1))\right),\end{aligned}$$

where $\pi \leq s \leq 3\pi, 0 \leq q \leq 1$, *in 3-space shown in Fig. 3.*

8. Conclusion

We have introduced a method for finding a hypersurface family passing through the same given geodesic as an isoparametric curve. The members of the hypersurface family are obtained by choosing suitable marching-scale functions. For a better analysis of the method we investigate three types of marching-scale functions. Also, by giving an example for each type, the method is verified. Furthermore, with the help of the projecting methods, a member of the family is visualized in 3-space with its isogeodesic.

However, there is still much work in this area. For 3-space, one possible alternative is to consider the realm of implicit surfaces $\mathbf{F}(x,y,z,t) = 0$ and try to find out the constraints for which a given curve $\mathbf{r}(s)$ is an isogeodesic on $\mathbf{F}(x, y, z, t) = 0$. Also, the analogue of the problem dealt in this paper may be considered for 2-surfaces in 4-space or another types of marching-scale functions may be investigated.

9. Acknowledgements

The authors appreciate the comments and valuable suggestions of the editor and the reviewer. Their advice helped to improve the clarity and presentation of this paper. The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

10. Figures

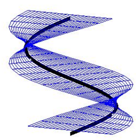

Figure 1. Projection of a member of the hypersurface family with marching-scale functions of type I and its isogeodesic.

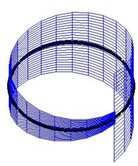

Figure 2. Projection of a member of the hypersurface family with marching-scale functions of type II and its isogeodesic.

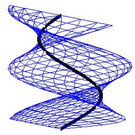

FIGURE 3. Projection of a member of the hypersurface family with marching-scale functions of type III and its isogeodesic.

REFERENCES

[1] N.H.Abdel-All, S.A.Badr, M.A.Soliman, S.A.Hassan, **Intersection curves of hypersurfaces in $\mathbb{R}^4$**, Comput. Aided Geom. Des.29,(2012),99-108.

[2] O.Aléssio, **Differential geometry of intersection curves in $\mathbb{R}^4$ of three implicit surfaces**, Comput. Aided Geom. Des.26,(2009),455–471.

[3] T.F.Banchoff, **Beyond the third dimension: geometry, computer graphics, and higher dimensions**, W.H. Freeman & Co., New York, NY, USA,1990.

[4] T.F.Banchoff, **Discovering the fourth dimension**, Prime Computer, Inc., Natick, MA, 1987.

[5] E.Bayram, F.Güler, E.Kasap, **Parametric representation of a surface pencil with a common asymptotic curve**, Comput. Aided Des.44,(2012),637-643.

[6] E.D.Bloch, **A first course in geometric topology and differential geometry**, Birkhäuser, Boston, 1997.

[7] M.P.do Carmo, **Differential geometry of curves and surfaces**, Prentice Hall, Englewood Cliffs, NJ, 1976.

[8] M.Düldül, **On the intersection curve of three parametric hypersurfaces**, Comput. Aided Geom. Des.27,(2010),118-127.

[9] G.Farin, **Curves and surfaces for computer aided geometric design: a practical guide**, Academic Press, Inc., San Diego, CA, 2002.

[10] R.T.Farouki, N.Szafran, L.Biard, **Existence conditions for Coons patches interpolating geodesic boundary curves**, Comput. Aided Geom. Des.26,(2009a),599-614.

[11] R.T.Farouki, N.Szafran, L.Biard, **Construction of Bézier surface patches with Bézier curves as geodesic boundaries**, Comput. Aided Geom. Des.41,(2009b),772-781.

[12] H.Gluck, **Higher curvatures of curves in Euclidean space**, Amer. Math. Month.73,(1966),699–704.

[13] B.Hamann, **Visualization and modeling contours of trivariate functions**, Ph.D. thesis, Arizona State University, 1991.

[14] A.J.Hanson, P.A.Heng, **Visualizing the fourth dimension using geometry and light**, In: Proceedings of the 2nd Conference on Visualization '91, IEEE Computer Society Press, Los Alamitos, CA, USA, pp. 321–328, 1991.

[15] S.R.Hollasch, **Four-space visualization of 4D objects**, Master thesis, Arizona State University, 1991.

[16] J.Hoschek, D.Lasser, **Fundamentals of computer aided geometric design**, A.K. Peters, Wellesley, MA, 1993.

[17] E.Kasap, F.T.Akyildiz, K.Orbay, **A generalization of surfaces family with common spatial geodesic**, Appl. Math. Comput.201,(2008),781-789.

[18] C.Y.Li, R.H.Wang, C.G.Zhu, **Parametric representation of a surface pencil with a common line of curvature**, Comput. Aided Des.43,(2011),1110-1117.

[19] M.Paluszny, **Cubic polynomial patches through geodesics**, Comput. Aided Des.40,(2008),56-61.

[20] G.V.V.Ravi Kumar, P.Srinivasan, V.D.Holla, K.G.Shastry, B.G.Prakash, **Geodesic curve computations on surfaces**, Comput. Aided Geom. Des.20,(2003),119-133.

[21] J.Sánchez-Reyes, R.Dorado, **Constrained design of polynomial surfaces from geodesic curves**, Comput. Aided Des.40,(2008),49-55.

[22] N.Sprynski, N.Szafran, B.Lacolle, L.Biard, **Surface reconstruction via geodesic interpolation**, Comput. Aided Des.40,(2008),480-492.

[23] J.J.Stoker, **Differential geometry**, Wiley, New York, 1969.

[24] D.J.Struik, **Lectures on classical differential geometry**, Addison–Wesley, Reading, MA, 1950.

[25] J.A.Thorpe, **Elementary topics in differential geometry**, Springer-Verlag, New York, Heidelberg-Berlin, 1979.

[26] G.J.Wang, K.Tang, C.L.Tai, **Parametric representation of a surface pencil with a common spatial geodesic**, Comput. Aided Des.36,(2004),447-459.

[27] M.Z.Williams, F.M.Stein, **A triple product of vectors in four-space**, Math. Mag.37,(1964),230–235.

[28] T.J.Willmore, **An introduction to differential geometry**, Clarendon Press, Oxford, 1959.

Ondokuz Mayıs University
Faculty of Arts and Sciences
Mathematics Department

Address: Ondokuz Mayıs University, Faculty of Arts and Sciences, Mathematics Department, Atakum, Samsun, 55139, Turkey

Email: erginbayram@yahoo.com (Ergin Bayram), kasape@omu.edu.tr (Emin Kasap)