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ON CERTAIN CLASS OF STARLIKE ANALYTIC FUNCTIONS ASSOCIATED WITH A DIFFERENTIAL OPERATOR

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Abstract. The aim of this paper is to introduce a new class of analytic functions in the unit disk, with negative coefficients, via a new generalized derivative operator. We prove coefficient inequalities for the new class of functions and for a subclass of it, as well as distortion theorems for functions in this subclass and for their derivatives. We determine the extreme points of the newly introduced subclass.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

For $x \in \mathbb{N}$, denote by $\mathcal{A}(x)$ denote the class of functions of the form

$$f(z) = z + \sum_{n=x+1}^{\infty} a_n z^n, \tag{1.2}$$

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which are analytic in the unit disk $\mathbb{U}$. Here $\mathbb{N}$ is the set of positive integers.

Also let $\mathcal{T}(x)$ denote subclass of $\mathcal{A}(x)$ consisting of functions f of the form

$$f(z) = z - \sum_{n=x+1}^{\infty} |a_n| z^n, \; x \in \mathbb{N}. \tag{1.3}$$

A function $f \in \mathcal{T}(x)$ is called a function with negative coefficients. Silverman investigated in (7) the subclasses of $\mathcal{T}(1)$ denoted by $\mathcal{S}^*_{\mathcal{T}}(\alpha)$ and $\mathcal{C}_{\mathcal{T}}(\alpha)$ for $0 \leq \alpha < 1$, that are, respectively, starlike functions of order α and convex functions of order α.

For complex parameters $a_i, b_j, (i = 1, \ldots, r, j = 1, \ldots, s, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\})$, we shall use the generalized hypergeometric function ${}_r\Phi_s(a_i, b_j; z)$

$${}_r\Phi_s(a_i, b_j; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!},$$

where $r \leq s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}, \mathbb{N}$ denotes the set of positive integers and $(x)_n$ is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n = \{1, 2, 3, \ldots\}. \end{cases}$$

Corresponding to a function ${}_r\mathcal{G}_s(a_i, b_j; z)$ defined by

$${}_r\mathcal{G}_s(a_i, b_j; z) = z\, {}_r\Phi_s(a_i, b_j; z). \tag{1.4}$$

Dziok and Srivastava (8) introduced a convolution operator on $\mathcal{A}$ such that

$$\mathcal{H}_{r,s}(a_i, b_j) : \mathcal{A} \to \mathcal{A},$$

is defined by

$$\mathcal{H}_{r,s}(a_i, b_j) f(z) = {}_r\mathcal{G}_s(a_i, b_j; z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_r)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}} \frac{a_n z^n}{(n-1)!},$$

For $\lambda_2 \geq \lambda_1 \geq 0, m, b \in \mathbb{N}_0 = \{0, 1, 2, ...\}, a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, $(i = 1, \ldots, r, j = 1, \ldots, s)$, and $r \leq s + 1; r, s \in \mathbb{N}_0$, we define the following operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f : \mathcal{A} \to \mathcal{A}$ by the following Hadamard product:

$$\mathcal{D}^{0,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) =_r \mathcal{G}_s(a_i, b_j; z) * f(z),$$

(1.5)
$$(1+b)\mathcal{D}^{1,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) = (1-(\lambda_1+\lambda_2)+b)(\varphi^b_{\lambda_2} *_r \mathcal{G}_s(a_i, b_j; z) * f(z)) + (\lambda_1 + \lambda_2)z(\varphi^b_{\lambda_2} *_r \mathcal{G}_s(a_i, b_j; z) * f(z))',$$

(1.6)
$$\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) = \mathcal{D}^{b}_{\lambda_1,\lambda_2}(\mathcal{D}^{m-1,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z)),$$

where $\varphi^b_{\lambda_2} = z + \sum_{n=2}^{\infty} \frac{z^n}{1+\lambda_2(n-1)+b}$.

From (1.5) and (1.6) we may easily deduce the following formula

(1.7)
$$\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{1 + \lambda_2(n-1) + b}\right]^m \frac{(a_1)_{n-1} \cdots (a_r)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}} \frac{a_n z^n}{(n-1)!}.$$

It should be remarked that the linear operator (1.7) is a generalization of many operators considered earlier. Let us see some of the examples:

For $m = 0$ the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f$ reduces to the well-known Dziok- Srivastava operator (8).
For $\lambda_2 = b = 0$, we get the Selvaraj derivative operator (3).
For $m = 0, r = 2$ and $s = 1$ we obtain the Hohlov derivative operator (9).
For $r = 1, s = 0, a_1 = 1, \lambda_1 = 1$ and $\lambda_2 = b = 0$, we get the Salagean derivative operator (6).
For $r = 1, s = 0, a_1 = 1$ and $\lambda_2 = b = 0$, we get the generalized Salagean derivative operator introduced by Al-Oboudi (5).
For $m = 0, r = 1, s = 0$ and $a_1 = \delta + 1$ we obtain the Ruscheweyh derivative operator (14).
For $r = 1, s = 0$ and $a_1 = \delta + 1$ we obtain the El-Yagubi and Darus

derivative operator (4).
For $m = 0, r = 2$ and $s = 1$ and $a_2 = 1$ we obtain the Carlson and Shaffer derivative operator (2).
For $r = 1, s = 0, a_1 = 1$ and $\lambda_2 = 0$ we get the Cátás derivative operator (1).

By making use of the generalized derivative operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z)$, we introduce a new subclass as follows:

Definition 1.1. *For f be defined by (1.2), $0 \le \alpha < 1$, $\lambda_2 \ge \lambda_1 \ge 0$ and $m, b \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}, (i = 1, \ldots, r, j = 1, \ldots, s)$, and $r \le s + 1; r, s \in \mathbb{N}_0$, let $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ be the subclass of $\mathcal{A}(x)$ consisting of functions f which satisfy*

$$Re\left(\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z)}\right) > \alpha, \quad (z \in \mathbb{U}), \tag{1.8}$$

where $x \in \mathbb{N}$ and $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) \neq 0$ for every $z \in \mathbb{U}$.

Note that if f given by (1.3), then we can see that

$$\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z) = \tag{1.9}$$
$$z - \sum_{n=x+1}^{\infty}\left[\frac{1 + (\lambda_1 + \lambda_2)(n - 1) + b}{1 + \lambda_2(n - 1) + b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}}\frac{|a_n|z^n}{(n - 1)!},$$

where $\lambda_2 \ge \lambda_1 \ge 0, m, b \in \mathbb{N}_0 = \{0, 1, 2, ...\}$,
$a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$,
$(i = 1, \ldots, r, j = 1, \ldots, s)$, and $r \le s + 1; r, s \in \mathbb{N}_0, x \in \mathbb{N}$.

In addition, we define the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ by

$$\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) = \mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \cap \mathcal{T}(x). \tag{1.10}$$

Note that various subclasses of $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ and $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ have been studied by many authors by suitable choices of $m, b, \lambda_1, \lambda_2$ and x. For example:

when $m = 0, r = 1, s = 0, a_1 = 1, x = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{S}^*_{\mathcal{T}}(\alpha)$, and when $m = 1, \lambda_1 = b = 0, r = 1, s = 0, a_1 = 1, x = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{C}_{\mathcal{T}}(\alpha)$, where the classes $\mathcal{S}^*_{\mathcal{T}}(\alpha)$ and

$\mathcal{C}_{\mathcal{T}}(\alpha)$ were studied by Silverman (7), for $m = 1, \lambda_1 = b = 0, r = 1, s = 0, a_1 = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{S}^*_{\mathcal{T}}(x, \alpha)$, and when $m = 0, r = 1, s = 0, a_1 = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{C}_{\mathcal{T}}(x, \alpha)$, where the classes $\mathcal{S}^*_{\mathcal{T}}(x, \alpha)$ and $\mathcal{C}_{\mathcal{T}}(x, \alpha)$ were introduced by Chatterjea (15), when $m = 1, \lambda_2 = b = 0, r = 1, s = 0, a_1 = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{P}(x, \lambda_1, \alpha)$, $(0 \leq \lambda_1 < 1)$, and for $m = 1, \lambda_2 = b = 0, r = 1, s = 0, a_1 = 1$, then $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha) \equiv \mathcal{C}(x, \lambda_1, \alpha)$, where the classes $\mathcal{P}(x, \lambda_1, \alpha)$ and $\mathcal{C}(x, \lambda_1, \alpha)$ were, respectively, studied by Altintas (12), Kamali and Akbulut (11). When $m = 0, r = 1, s = 0$ and $a_1 = \delta + 1$ in the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$, we have the class $\mathcal{R}_\delta(x, \alpha)$ introduced and studied by Ahuja (13). Finally we note that when $m = 1, \lambda_2 = b = 0, r = 1, s = 0, a_1 = \delta + 1$ in the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ we have the class $\mathcal{M}^{\delta}_{\lambda_1}(x, \alpha)$ introduced and studied by Al-Shaqsi and Darus (10).

2. Coefficient Inequalities

In this section we provide two conditions in terms of coefficient inequalities, namely a sufficient condition for a function $f \in \mathcal{A}(x)$ to belong to the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ and a necessary and sufficient condition for a function $f \in \mathcal{T}(x)$ to belong to the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$.

Theorem 2.1. *Let f be defined by (1.2). For $0 \leq \alpha < 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $m, b \in \mathbb{N}_0, x \in \mathbb{N}$, $a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}, (i = 1, \ldots, r, j = 1, \ldots, s)$, and $r \leq s + 1; r, s \in \mathbb{N}_0$. If*

(2.1)
$$\sum_{n=x+1}^{\infty} (n-\alpha) \left[\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{1 + \lambda_2(n-1) + b}\right]^m \frac{(a_1)_{n-1} \cdots (a_r)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}(n-1)!}|a_n| \leq 1-\alpha,$$

then $f \in \mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$.

Proof. Suppose that (2.1) holds true. It suffices to show that

$$\left|\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j)f(z)} - 1\right| \leq 1 - \alpha. \tag{2.2}$$

We have

$$\left|\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-1\right| = \left|\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'-\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}\right|$$

$$= \left|\frac{\sum_{n=x+1}^{\infty}(n-1)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}a_nz^n}{z+\sum_{n=x+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}a_nz^n}\right|, \ |z|<1,$$

$$\le \frac{\sum_{n=x+1}^{\infty}(n-1)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|}{1-\sum_{n=x+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|}. \tag{2.3}$$

This expression (2.3) is bounded by $(1-\alpha)$. We have

$$\sum_{n=x+1}^{\infty}(n-1)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|$$

$$\le (1-\alpha)\left[1-\sum_{n=x+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|\right], \tag{2.4}$$

which is equivalent to

$$\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n| \le (1-\alpha),$$

by (2.1). Thus $f\in\mathcal{M}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$.

Theorem 2.2. *Let f be defined by (1.3). Then $f\in\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$ if and only if (2.1) is satisfied.*

Proof. We only prove the necessity, since the sufficiency can be obtained by using similar arguments in proof of Theorem 2.1. Let $f\in\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$ by condition (1.8), we have the

$$\Re\left(\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}\right)$$

(2.5)
$$= \Re\left\{\frac{z-\sum_{n=x+1}^{\infty} n\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|z^n}{z-\sum_{n=x+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|z^n}\right\} > \alpha.$$

Choose values of z on real axis so that $\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}$ is real. Letting $z\to 1$ through real values, we have

$$\frac{1-\sum_{n=x+1}^{\infty} n\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|}{1-\sum_{n=x+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|} > \alpha. \tag{2.6}$$

Thus we obtain

$$\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n| \le 1-\alpha,$$

which is (2.1). Hence the proof is complete.

Finally the result is sharp with the extremal function f given by

(2.7)
$$f(z) = z-\frac{1-\alpha}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m \frac{(b_1)_x\cdots(b_s)_x\ x!}{(a_1)_x\cdots(a_r)_x} z^{x+1},\ x\in\mathbb{N}.$$

Corollary 2.1. *Let f be defined by (1.3) be in the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$. Then we have*

(2.8)
$$|a_n| \le \frac{1-\alpha}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\ (n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}},$$

where $0\le\alpha<1$, $\lambda_2\ge\lambda_1\ge 0$, $m,b\in\mathbb{N}_0, n\ge x+1$, $a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}, (i=1,\ldots,r, j=1,\ldots,s)$, $r\le s+1; r,s\in\mathbb{N}_0$ *and* $x\in\mathbb{N}$.

This equality is attained for the function f given by (2.7).

3. Distortion Theorems

A distortion property for function f to be in the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$ is given as follows:

Theorem 3.1. *Let the function f be defined by (1.3) be in the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$. Then for $0<|z|=r<1$, we have*

$$r-\frac{1-\alpha}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m\frac{(b_1)_x\cdots(b_s)_x\,x!}{(a_1)_x\cdots(a_r)_x}r^{x+1}$$

(3.1)
$$\leq|f(z)|\leq r+\frac{1-\alpha}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m\frac{(b_1)_x\cdots(b_s)_x\,x!}{(a_1)_x\cdots(a_r)_x}r^{x+1},\ x\in\mathbb{N}.$$

where $0\leq\alpha<1,\lambda_2\geq\lambda_1\geq0$ and $m,b\in\mathbb{N}_0,\ a_i\in\mathbb{C},b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$,
$(i=1,\ldots,r,j=1,\ldots,s)$, $r\leq s+1;r,s\in\mathbb{N}_0$ and $x\in\mathbb{N}$.

Proof. Since $f\in\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$ by Theorem 2.2, we have

$$\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m\frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|\leq 1-\alpha.$$

Now

$$(x+1-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)x+b}{1+\lambda_2 x+b}\right]^m\frac{(a_1)_x\cdots(a_r)_x}{(b_1)_x\cdots(b_s)_x\,x!}\left(\sum_{n=x+1}^{\infty}|a_n|\right)$$

$$=\sum_{n=x+1}^{\infty}(x+1-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)x+b}{1+\lambda_2 x+b}\right]^m\frac{(a_1)_x\cdots(a_r)_x}{(b_1)_x\cdots(b_s)_x\,x!}|a_n|$$

$$\leq\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m\frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|$$

$$\leq 1-\alpha,\ x\in\mathbb{N}.$$

Therefore,

(3.2)
$$\sum_{n=x+1}^{\infty}|a_n| \leq \frac{1-\alpha}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m \frac{(b_1)_x\cdots(b_s)_x\, x!}{(a_1)_x\cdots(a_r)_x}.$$

Since

$$f(z)=z-\sum_{n=x+1}^{\infty}|a_n|z^n,$$

we get

$$|f(z)|=|z-\sum_{n=x+1}^{\infty}|a_n|z^n|.$$

Next,

$$|z|-|z|^{x+1}\sum_{n=x+1}^{\infty}|a_n| \leq |f(z)| \leq |z|+|z|^{x+1}\sum_{n=x+1}^{\infty}|a_n|,$$

that is

$$r-r^{x+1}\sum_{n=x+1}^{\infty}|a_n| \leq |f(z)| \leq r+r^{x+1}\sum_{n=x+1}^{\infty}|a_n|.$$

By using the inequality (3.2), it yields Theorem 3.1. Thus, the proof is complete.

Theorem 3.2. *Let the function f be defined by (1.3) be in the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j,x,\alpha)$. Then for $0<|z|=r<1$, we have*

$$1-\frac{(x+1)(1-\alpha)}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m \frac{(b_1)_x\cdots(b_s)_x\, x!}{(a_1)_x\cdots(a_r)_x}r^x$$

(3.3)
$$\leq |f'(z)| \leq 1+\frac{(x+1)(1-\alpha)}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m \frac{(b_1)_x\cdots(b_s)_x\, x!}{(a_1)_x\cdots(a_r)_x}r^x,\ x\in\mathbb{N}.$$

where $0\leq\alpha<1, \lambda_2\geq\lambda_1\geq 0$ and $m,b\in\mathbb{N}_0$, $a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$,
$(i=1,\ldots,r, j=1,\ldots,s)$, $r\leq s+1; r,s\in\mathbb{N}_0$ and $x\in\mathbb{N}$.

Proof. Since $f \in \mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ by Theorem 2.2, we have

$$\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n| \leq 1-\alpha.$$

Now

$$(x+1-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)x+b}{1+\lambda_2 x+b}\right]^m \frac{(a_1)_x\cdots(a_r)_x}{(b_1)_x\cdots(b_s)_x\, x!}\left(\sum_{n=x+1}^{\infty} n|a_n|\right)$$

$$= \sum_{n=x+1}^{\infty}(x+1-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)x+b}{1+\lambda_2 x+b}\right]^m \frac{(a_1)_x\cdots(a_r)_x}{(b_1)_x\cdots(b_s)_x\, x!} n|a_n|$$

$$\leq (x+1)\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \cdot$$

$$\cdot\frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|,$$

$$\leq (x+1)(1-\alpha),\ x \in \mathbb{N}.$$

Hence,

$$\sum_{n=x+1}^{\infty} n|a_n| \leq \frac{(x+1)(1-\alpha)}{x+1-\alpha}\left[\frac{1+\lambda_2 x+b}{1+(\lambda_1+\lambda_2)x+b}\right]^m \frac{(b_1)_x\cdots(b_s)_x\, x!}{(a_1)_x\cdots(a_r)_x},\ x \in \mathbb{N}. \tag{3.4}$$

Since

$$f'(z) = 1 - \sum_{n=x+1}^{\infty} n|a_n|z^{n-1},$$

then we have that

$$1 - |z|^x \sum_{n=x+1}^{\infty} n|a_n| \leq |f'(z)| \leq 1 + |z|^x \sum_{n=x+1}^{\infty} n|a_n|,$$

therefore,

$$1 - r^x \sum_{n=x+1}^{\infty} n|a_n| \leq |f'(z)| \leq 1 + r^x \sum_{n=x+1}^{\infty} n|a_n|,\ x \in \mathbb{N}.$$

By using the inequality (3.4), we get Theorem 3.2. This completes the proof.

4. Extreme Points

We shall now determine the extreme points of the class $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$.

Theorem 4.1. *Let $x \in \mathbb{N}$ and $a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}, (i = 1, \ldots, r, j = 1, \ldots, s)$, where $r, s \in \mathbb{N}_0$ with $r \leq s + 1$. Let $0 \leq \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0$ and $m, b \in \mathbb{N}_0$. Define $f_x(z) = z$ and*

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n,$$

where $n \geq x + 1$. Then $f \in \mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=x}^{\infty} \mu_n f_n(z), \tag{4.1}$$

where $\mu_n \geq 0$ and $\sum_{n=x}^{\infty} \mu_n = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=x}^{\infty} \mu_n f_n(z) \\ &= \mu_x f_x(z) + \sum_{n=x+1}^{\infty} \mu_n f_n(z) \end{aligned}$$

$$= \mu_x f_x(z) + \sum_{n=x+1}^{\infty} \mu_n \left[z - \frac{1-\alpha}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n\right]$$

$$= \sum_{n=x}^{\infty} \mu_n z - \sum_{n=x+1}^{\infty} \frac{\mu_n(1-\alpha)}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n$$

$$= z - \sum_{n=x+1}^{\infty} \frac{\mu_n(1-\alpha)}{n-\alpha} \left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\ (n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n.$$

Now

$$f(z) = z - \sum_{n=x+1}^{\infty} |a_n| z^n = z - \sum_{n=x+1}^{\infty} \frac{\mu_n(1-\alpha)}{n-\alpha} \left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\ (n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n,$$

therefore,

$$|a_n| = \frac{\mu_n(1-\alpha)}{n-\alpha} \left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\ (n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}}.$$

Now, we have that

$$\sum_{n=x+1}^{\infty} \mu_n = 1 - \mu_x \leq 1,\ x \in \mathbb{N}.$$

Setting

$$\sum_{n=x+1}^{\infty} \mu_n =$$

$$= \sum_{n=x+1}^{\infty} \frac{\mu_n(1-\alpha)}{n-\alpha} \left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\ (n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} \frac{n-\alpha}{1-\alpha} \left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!} \leq 1,$$

we get

$$\sum_{n=x+1}^{\infty} \frac{n-\alpha}{1-\alpha} \left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!} |a_n| \leq 1.$$

Therefore,

$$\sum_{n=x+1}^{\infty}(n-\alpha)\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n| \leq$$
$$\leq 1-\alpha.$$

It follows from Theorem 2.2 that $f \in \mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$.

Conversely, we suppose that $f \in \mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$, it is easily seen that

$$f(z) = z - \sum_{n=x+1}^{\infty}|a_n|z^n = z-$$
$$-\sum_{n=x+1}^{\infty}\frac{\mu_n(1-\alpha)}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}}z^n,$$

which suffices to show that

$$|a_n| = \frac{\mu_n(1-\alpha)}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}}.$$

Now, we have that $f \in \mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$, then by previous Corollary 2.1,

$$|a_n| \leq \frac{1-\alpha}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}},$$

which is

$$\frac{n-\alpha}{1-\alpha}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n| \leq 1.$$

Since $\sum_{n=x}^{\infty}\mu_n = 1$, we see $\mu_n \leq 1$, for each $n \geq x$ and $x \in \mathbb{N}$. We can set that

$$\mu_n = \frac{n-\alpha}{1-\alpha}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}(n-1)!}|a_n|.$$

Thus,

$$|a_n| = \frac{\mu_n(1-\alpha)}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}}.$$

The proof of Theorem 4.1 is complete.

Corollary 4.1. *The extreme points of* $\mathcal{TM}^{m,b}_{\lambda_1,\lambda_2}(a_i, b_j, x, \alpha)$ *are the functions*

$$f_x(z) = z,$$

(4.2)

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha}\left[\frac{1+\lambda_2(n-1)+b}{1+(\lambda_1+\lambda_2)(n-1)+b}\right]^m \frac{(b_1)_{n-1}\cdots(b_s)_{n-1}\,(n-1)!}{(a_1)_{n-1}\cdots(a_r)_{n-1}} z^n,$$

where $0 \le \alpha < 1$, $\lambda_2 \ge \lambda_1 \ge 0$, $m, b \in \mathbb{N}_0, n \ge x+1$ *and* $x \in \mathbb{N}$, $a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, (i = 1,\ldots,r, j = 1,\ldots,s)$, $r \le s+1; r, s \in \mathbb{N}_0$.

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