"Vasile Alecsandri" University of Bacău
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UPPER AND LOWER QUASI cl-SUPERCONTINUOUS MULTIFUNCTIONS

J. K. KOHLI, C. P. ARYA AND D. SINGH

Abstract. The notion of quasi cl-supercontinuity of functions is extended to the framework of multifunctions. Basic properties of upper (lower) quasi cl-supercontinuous multifunctions are studied and their place in the hierarchy of variants of continuity of multifunctions, that already exist in the literature, is elaborated. The class of upper (lower) quasi cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions and so includes all upper (lower) perfectly continuous multifunctions; and is strictly contained in the class of upper (lower) quasi z-supercontinuous multifunctions. The upper quasi cl-supercontinuity of multifunctions is preserved under compositions, union of multifunctions, restriction to a subspace and the passage to the graph multifunction. A sufficient condition for the intersection of two upper quasi cl-supercontinuous multifunctions to be upper quasi cl-supercontinuous is formulated. The lower quasi cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, union of multifunctions and restriction to a subspace.

Keywords and phrases: Upper/lower quasi cl-supercontinuous multifunction, upper/lower (almost) cl-supercontinuous multifunction, upper/lower (almost) z-supercontinuous multifunction, upper/lower (almost) perfectly continuous multifunction, δ-embedded set, quasi zero dimensional space.
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1. Introduction

Recently there has been considerable interest in trying to extend the notions and results of weak and strong variants of continuity of functions to the realm of multifunctions (see for example [1], [2], [3], [4], [13], [21], [19], [20], [22], [32], [34], [35], [36], [37] and [39]). In the present paper we extend the notion of quasi cl - supercontinuity [18] of functions to the realm of multifunctions and introduce the notions of upper and lower quasi cl-supercontinuous multifunctions and elaborate upon their place in the hierarchy of variants of continuity of multifunctions that already exist in the literature. It turns out that the class of upper (lower) quasi cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions [21] and so includes all upper (lower) perfectly continuous multifunctions [19] and is strictly contained in the class of upper (lower) quasi z-supercontinuous multifunctions [22] which in turn is properly contained in the class of quasi upper (lower) D_δ - supercontinuous multifunctions [22].

Section 2 is devoted to preliminaries and basic definitions, wherein we define the notions of upper and lower quasi cl-supercontinuous multifunctions and discuss the interrelations that exist among them and other variants of continuity of multifunctions. Examples are included to reflect upon the distinctiveness of the notions so introduced and other variants of continuity of multifunctions that already exist in the mathematical literature.

In Section 3 we discuss characterizations and study basic properties of upper quasi cl-supercontinuous multifunctions. It turns out that upper quasi cl-supercontinuity of multifunctions is preserved under the composition, union of multifunctions, restriction to a subspace and passage to the graph multifunction. Moreover, we formulate a sufficient condition for the intersection of two upper quasi cl-supercontinuous multifunctions to be upper quasi cl-supercontinuous.

In Section 4 we study lower quasi cl-supercontinuous multifunctions, and give their characterizations. It is shown that quasi lower cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, union of multifunctions, and under restriction to a subspace.

In section 5 we study the behaviour of upper and lower quasi cl-supercontinuous multifunctions if its domain and/ or co-domain are retopologized in an appropriate way and conclude with alternative proofs of certain results of preceding sections. Throughout the paper we essentially adopt the notation and terminology of Górniecowicz [10] [1] pertaining to multifunctions.

2. Basic Definitions and Preliminaries

A subset A of a space X is called a regular G_δ-set [32] if A is the intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \cap_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called regular F_σ-set. A space X is said to be D_δ-completely regular ([25] [26]) if it has a base of regular F_σ-sets. A point $x \in X$ is called a θ-adherent point [38] of A if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-adherent points of A. The set A is called θ-closed if $A = cl_\theta A$. The complement of a θ-closed set is referred to as a θ-open set. A subset A of a space X is said to be regular open if it is the interior of its closure, i.e., if $A = \overline{A}^o$. The complement of a regular open set is referred to as regular closed. A point $x \in$X is called a $u\theta$-adherent point ([23] [24]]) of a set A if every θ-open set containing x intersects A. Let $A_{u\theta}$ denote the set of all $u\theta$-adherent points of the set A. It turns out that $A_{u\theta}$ is the smallest θ-closed set containing A. Moreover, it is shown in [[24], Lemma 5.2] that the correspondence $A \to A_{u\theta}$ is Kuratowski closure operator.

2.1. **Definitions.** A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be
(a) ***upper (lower) perfectly continuous (respectively almost perfectly continuous, respectively quasi perfectly continuous, respectively δ - perfectly continuous)*** [19] if $\varphi_-^{-1}(U)$ $(\varphi_+^{-1}(U))$ is clopen in X for every open (respectively regular open, respectively θ-open, respectively δ-open) subset U of Y;
(b) ***upper quasi z-supercontinuous (D_δ-supercontinuous)*** ([22] [29]) if for each $x \in X$ and each θ-open set V containing $\varphi(x)$, there exists a cozero set (regular F_σ-set) U containing x such that $\varphi(U) \subset V$;
(c) ***lower quasi z-supercontinuous (D_δ-supercontinuous)*** ([22] [29]) if for each $x \in X$ and each θ-open set V with $\varphi(x) \cap V \neq \phi$, there exists a cozero set (regular F_σ-set) U containing x such that $\varphi(z) \cap V \neq \phi$ for each $z \in U$; and
(d) ***upper (lower) quasi θ-continuous (faintly continuous)*** if for every θ-open set $V \subset Y$, $\varphi_-^{-1}(V)$($\varphi_+^{-1}(V)$) is θ-open (open) in X.

[1] [10] L. Górniecowicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1999.

2.2. **Definitions.** A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be
(a) ***upper (almost) cl-supercontinuous*** ([21] [20] [27]) (respectively ***z-supercontinuous*** ([2] [30]), respectively D_δ ***- supercontinuous*** [3], respectively ***strongly*** θ***-continuous*** [32]) at $x \in X$ if for each open (regular open) set V with $\varphi(x) \subset V$, there exists a clopen set (respectively cozero set, respectively regular F_σ-set, respectively θ-open set) U containing x such that $\varphi(U) \subset V$;

(b) ***lower (almost) cl-supercontinuous*** ([21] [20] [27]) (respectively ***z - supercontinuous*** ([2] [30]), respectively D_δ ***- supercontinuous*** [3], respectively ***strongly*** θ***-continuous*** [32]) at $x \in X$ if for each open (regular open) set V with $\varphi(x) \cap V \neq \phi$, there exists a clopen set (respectively cozero set, respectively regular F_σ-set, respectively θ-open set) U containing x such that $\varphi(z) \cap V \neq \phi$ for each $z \in$U;

(c) ***upper supercontinuous (***δ***-continuous)*** [1] if for each $x \in X$ and each open (regular open) set V containing $\varphi(x)$, there exists a regular open set U containing x such that $\varphi(U) \subset V$; and

(d) ***lower supercontinuous (***δ***-continuous)*** [1] if for each $x \in X$ and each open (regular open) set V with $\varphi(x) \cap V \neq \phi$, there exists a regular open set U containing x such that $\varphi(z) \cap V \neq \phi$ for each $z \in U$.

2.3. **Definitions.** We say that a multifunction $\varphi : X \multimap Y$ is
(a) ***upper quasi cl-supercontinuous*** if for each $x \in X$ and each θ-open set V containing $\varphi(x)$ there exists a clopen set U containing x such that $\varphi(U) \subset V$; and
(b) ***lower quasi cl-supercontinuous*** if for each $x \in X$ and each θ-open set V with $\varphi(x) \cap V \neq \phi$, there exists a clopen set U containing x such that $\varphi(z) \cap V \neq \phi$ for each $z \in U$.
The following diagrams well illustrates the interrelations that exist among variants of continuity of multifunctions defined in Definitions 2.1, 2.2, and 2.3.

However, none of the above implications is reversible as is well illustrated by the examples in the sequel and the examples in ([1], [2], [3], [21], [19], [20]).

2.4. **Example.** Let $X = \{x, y, z\}$ and let $\Im_X$={ϕ, X, $\{x\}$, $\{x, y\}$}. Let $Y = \{a, b, c\}$ and let $\Im_Y$={ϕ, Y, $\{a\}$, $\{b\}$, $\{a, b\}$}. Define a

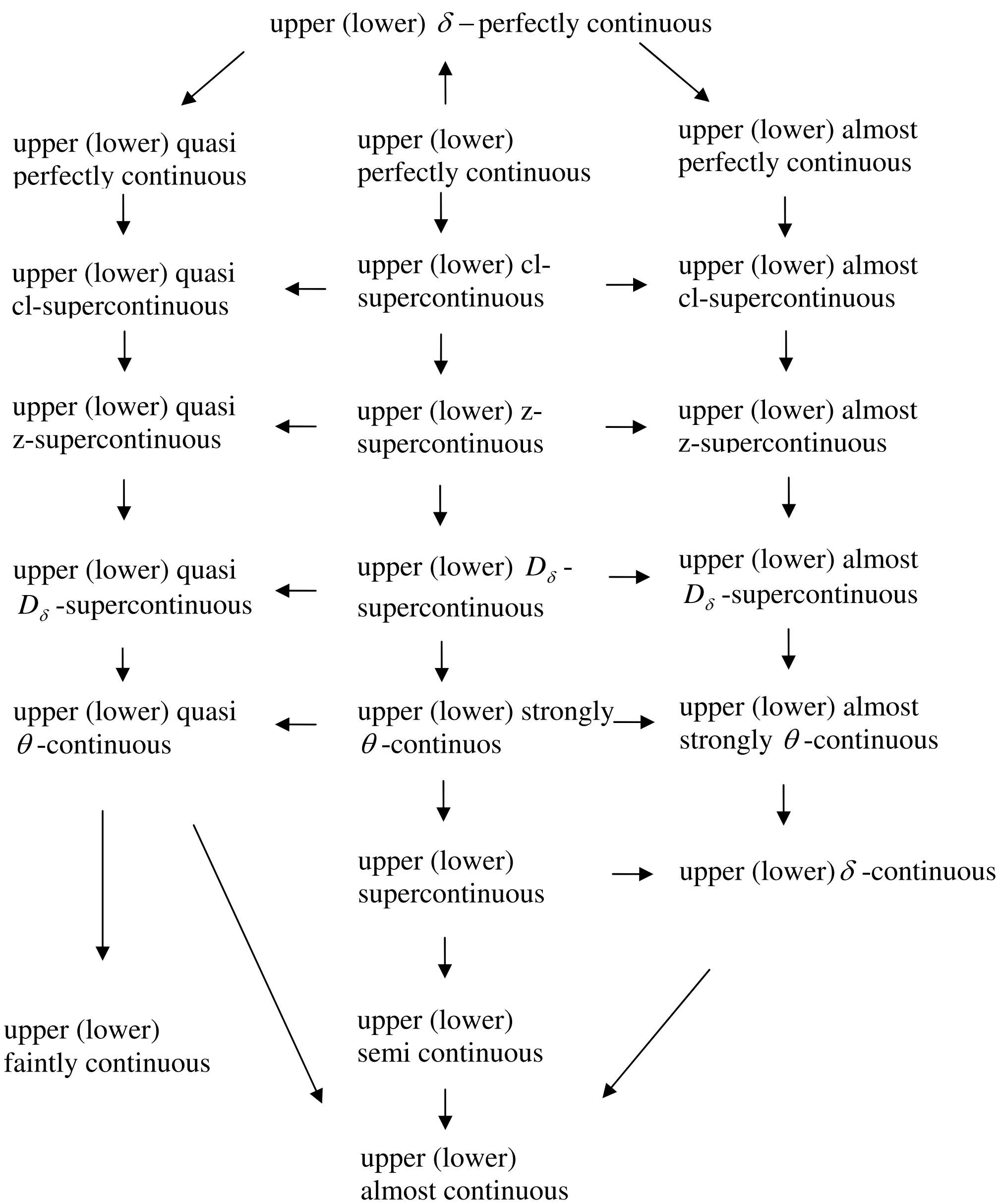

FIGURE 1

multifunction $\varphi : X \multimap Y$ by $\varphi(x) = \{a\}$, $\varphi(y) = \{b, c\}$, $\varphi(z) = \{a, c\}$. Then the multifunction φ is upper quasi perfectly continuous but not upper δ-perfectly continuous. This example also shows that lower quasi perfect continuity need not imply lower δ-perfect continuity.

$$
\begin{tikzcd}
& \text{lower } \delta\text{-perfectly continuous} \arrow[dl] \arrow[dr] & \\
\text{lower almost perfectly continuous} \arrow[d] & & \text{lower quasi perfectly continuous} \arrow[d] \\
\text{lower almost cl-supercontinuous} \arrow[rr] \arrow[d] & & \text{lower quasi cl-supercontinuous} \arrow[d] \\
\text{lower almost z-supercontinuous} \arrow[rr] \arrow[d] & & \text{lower quasi z-supercontinuous} \arrow[d] \\
\text{lower almost } D_\delta\text{- supercontinuous} \arrow[rr] \arrow[d] & & \text{lower quasi } D_\delta\text{-supercontinuous} \arrow[d] \\
\text{lower almost strongly } \theta\text{-continuous} \arrow[rr] \arrow[d] & & \text{lower quasi } \theta\text{-continuous} \arrow[d] \\
\text{lower } \delta\text{-continuous} \arrow[rr] \arrow[d] & & \text{lower faintly continuous} \\
\text{lower almost continuous} & &
\end{tikzcd}
$$

FIGURE 2

2.5. **Example.** In Example 2.4 the multifunction $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous, and hence upper quasi z-supercontinuous, as well as upper quasi D_δ-supercontinuous, as well as upper quasi θ-continuous, but not upper strongly θ-continuous and hence neither upper D_δ-supercontinuous, nor upper z-supercontinuous, nor upper cl-supercontinuous multifunction. This example also shows that multifunction φ is lower quasi cl-supercontinuous, and hence lower quasi z-supercontinuous, as well as lower quasi D_δ-supercontinuous, as well as lower quasi θ-continuous, but not lower almost strongly θ-continuous hence neither lower almost D_δ-supercontinuous, nor lower almost z-superconti-nuous, nor lower almost cl-supercontinuous.

2.6. **Example.** Let X be the real line endowed with usual topology and let $Y = [-1, 1]$ as a subspace of X. Then the multifunction φ :

$X \multimap Y$ defined by $\varphi(x) =$ [-1,1] if $x = 0$ and $\varphi(x) = \{0\}$ if $x \neq 0$ is upper quasi z-supercontinuous but not upper quasi cl-supercontinuous.

2.7. **Example.** Let the spaces X and Y be same as in Example 2.6. Then the multifunction $\varphi : X \multimap Y$ defined by $\varphi(x) = \{0\}$ if $x = 0$ and $\varphi(x) = [-1, 1]$ if $x \neq 0$ is lower quasi z-supercontinuous but not lower quasi cl-supercontinuous.

2.8. **Example.** Consider the space X defined on page 504 of E. Hewitt [12] which is a D_δ-completely regular space but not completely regular(see [26]). A multifunction $\varphi : X \multimap Y$ defined by $\varphi(x) = \{x\}$ is upper (lower) quasi D_δ-supercontinuous but not upper (lower) quasi z-supercontinuous.

2.9. **Example.** Let $X = Y$ denote the mountain chain space due to Heldermann[11]. Then X is a regular space which is not a D_δ-completely regular space[26]. The identity multifunction φ from X into Y is upper (lower) quasi strongly θ-continuous but not upper (lower) quasi D_δ-supercontinuous.

2.10. **Example.** Let $X = \{a, b, c\}$ and $\Im_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $Y = \{x, y, z\}$ and $\Im_Y = \{\phi, Y, \{x\}, \{y, z\}\}$. Define a multifunction $\varphi : X \multimap Y$ by $\varphi(a) = \{x\}$, $\varphi(b) = \{x, y\}$, $\varphi(c) = \{y\}$. Then the multifunction φ is upper faintly continuous but not upper quasi θ-continuous.

2.11. **Example.** Let N be the set of natural numbers with topology τ in which every odd integer is open and a neighbourhood of even integer contains its predecessor as well as its successor. Let Y denote the one point compactification of the space (N,τ). Let X be the same set as Y endowed with the topology $\tau_1 = \{\phi, X, X - N\}$. Define a multifunction $\varphi : X \multimap Y$ by $\varphi(x) = X - N$ if $x \in N$, $\varphi(x) = N$ if $x \in X - N$ Then the multifunction φ is lower faintly continuous but not lower quasi θ-continuous.

3. Properties of upper quasi cl-supercontinuous multifunctions

3.1. **Theorem.** For a multifunction $\varphi : X \multimap Y$ the following statements are equivalent.

(a) φ is upper quasi cl-supercontinuous.
(b) $\varphi_-^{-1}(V)$ is cl-open in X for each θ-open set $V \subset Y$.
(c) $\varphi_+^{-1}(B)$ is cl-closed in X for each θ-closed set $B \subset Y$.

(d) $(\varphi_+^{-1}(B))_{cl} \subset \varphi_+^{-1}(B_{u\theta})$ for every set $B \subset Y$.

Proof. $(a) \Rightarrow (b)$. Let V be a θ-open subset of Y. To show that $\varphi_-^{-1}(V)$ is cl-open in X, let $x \in \varphi_-^{-1}(V)$. Then $\varphi(x) \subset V$. Since φ is upper quasi cl-supercontinuous, there exists a clopen set H containing x such that $\varphi(H) \subset V$. Hence $x \in H \subset \varphi_-^{-1}(V)$ and so $\varphi_-^{-1}(V)$ is a cl-open set in X being a union of clopen sets.
$(b) \Rightarrow (c)$. Let B be a θ-closed subset of Y. Then $Y - B$ is a θ-open subset of Y. In view of (b), $\varphi_-^{-1}(Y - B)$ is cl-open set in X. Since $\varphi_-^{-1}(Y - B) = X - \varphi_+^{-1}(B)$, $\varphi_+^{-1}(B)$ is a cl-closed set in X.
$(c) \Rightarrow (d)$. Since $B_{u\theta}$ is θ-closed, $\varphi_+^{-1}(B_{u\theta})$ is a cl-closed set containing $\varphi_+^{-1}(B)$ and so $(\varphi_+^{-1}(B))_{cl} \subset \varphi_+^{-1}(B_{u\theta})$.
$(d) \Rightarrow (a)$. Let $x \in X$ and let V be a θ-open set in Y such that $\varphi(x) \subset V$. Then $\varphi(x) \cap (Y - V) = \phi$ and $Y - V$ is θ-closed. So $(Y - V)_{u\theta} = Y - V$, and hence $(\varphi_+^{-1}(Y - V))_{cl} \subset \varphi_+^{-1}(Y - V) = X - \varphi_-^{-1}(V)$. Since $\varphi_+^{-1}(Y - V)$ is cl-closed, its complement $\varphi_-^{-1}(V)$ is a cl-open set containing x. So there is a clopen set U containing x and contained in $\varphi_-^{-1}(V)$, whence $\varphi(U) \subset V$. Thus φ is upper quasi cl-supercontinuous.

3.2. **Theorem.** If $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous and $\psi : Y \multimap Z$ is upper quasi θ-continuous, then $\psi o \varphi$ is upper quasi cl-supercontinuous. In particular, the composition of two upper quasi cl-supercontinuous multifunctions is upper quasi cl-supercontinuous.

Proof. Let W be a θ-open set in Z. Since ψ is upper quasi θ-continuous, $\psi_-^{-1}(W)$ is a θ-open set in Y. Again, since φ is upper quasi cl-supercontinuous, $\varphi_-^{-1}(\psi_-^{-1}(W)) = (\psi o \varphi)_-^{-1}(W)$ is a cl-open set in X and so $\psi o \varphi$ is upper quasi cl-supercontinuous.

3.3. **Theorem.** If $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ are upper quasi cl-supercontinuous, then the multifunction $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is upper quasi cl-supercontinuous.

Proof. Let B be a θ-open subset of Y. Since φ and ψ are upper quasi cl-supercontinuous, $\varphi_-^{-1}(B)$ and $\psi_-^{-1}(B)$ are cl-open sets in X. Since $(\varphi \cup \psi)_-^{-1}(B) = \varphi_-^{-1}(B) \cap \psi_-^{-1}(B)$ and since finite intersection of cl-open sets is cl-open, $(\varphi \cup \psi)_-^{-1}(B)$ is cl-open in X. Thus $\varphi \cup \psi$ is upper quasi cl-supercontinuous.

In contrast in general the intersection of two upper quasi cl - supercon-

-tinuous multifunctions need not be upper quasi cl-supercontinuous.

However, in the forthcoming theorem we formulate a sufficient condition for the intersection of two multifunctions to be upper quasi cl-supercontinuous.

3.4. **Definition.** The graph Γ_φ of a multifunction $\varphi : X \multimap Y$ is said to be quasi cl-closed with respect to X if for each $(x, y) \notin \Gamma_\varphi$ there exists a clopen set U containing x and a θ-open set V containing y such that $\Gamma_\varphi \cap (U \times V) = \phi$.

3.5. **Theorem.** Let $\varphi, \psi : X \multimap Y$ be upper quasi cl-supercontinuous multifunctions from a space X into a Hausdorff space Y such that $\varphi(x)$ is compact for each $x \in X$ and the graph Γ_ψ of ψ is quasi cl-closed with respect to X. Then the multifunction $\varphi \cap \psi$ defined by $(\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x)$ for each $x \in X$, is upper quasi cl-supercontinuous.

Proof. Let $x_0 \in X$ and let V be a θ-open set containing $\varphi(x_0) \cap \psi(x_0)$. It suffices to find a clopen set U containing x_0 such that $(\varphi \cap \psi)(U) \subset V$. If $V \supset \varphi(x_0)$, it follows from upper quasi cl-supercontinuity of φ. If not, then consider the set $K = \varphi(x_0) \setminus V$ which is compact. Now for each $y \in K$, $y \notin \psi(x_0)$. This implies that $(x_0, y) \notin \Gamma_\psi$. Since the graph of ψ is quasi cl-closed with respect to X, there exists a clopen set U_y containing x_0 and a θ-open set V_y containing y such that $\Gamma_\psi \cap (U_y \times V_y) = \phi$. Therefore, for each $x \in U_y$, $\psi(x) \cap V_y = \phi$. Since K is compact, there exist finitely many $y_1, y_2, ..., y_n$ in K such that $K \subset \cup_{i=1}^n V_{y_i}$. Let $W = \cup_{i=1}^n V_{y_i}$. Then $V \cup W$ is a θ-open set containing $\varphi(x_0)$. Since φ is upper quasi cl-supercontinuous, there exists a clopen set U_0 containing x_0 such that $\varphi(U_0) \subset V \cup W$. Let $U = U_0 \cap (\cap_{i=1}^n U_{y_i})$. Then U is a clopen set containing x_0. Hence for each $z \in U$, $\varphi(z) \subset V \cup W$ and $\psi(z) \cap W = \phi$. Therefore, $\varphi(z) \cap \psi(z) \subset V$ for each $z \in U$. This proves that $\varphi \cap \psi$ is upper quasi cl-supercontinuous at x_0.

3.6. **Corollary.** Let $\psi : X \multimap Y$ be a multifunction from a space X into a compact Hausdorff space Y such that the graph Γ_ψ of ψ is quasi cl-closed with respect to X. Then ψ is upper quasi cl-supercontinuous.

3.7. **Definition.** A topological space X is said to be quasi zero dimensional [17] if for each $x \in X$ and each θ-open set U containing x, there exists a clopen set H containing x such that $x \in H \subset U$.

3.8. **Theorem.** Let $\varphi : X \multimap Y$ be any multifunction. The multifunction $g : X \multimap X \times Y$ defined by $g(x) = \{(x, y) \in X \times Y | y \in \varphi(x)\}$ for each $x \in X$, is called the graph multifunction. If g is upper quasi cl-supercontinuous, then φ is upper quasi cl-supercontinuous and the space X is quasi zero dimensional.

Proof. Suppose that g is upper quasi cl-supercontinuous. In view of Theorem3.2, the multifunction $\varphi = p_y o g$ is upper quasi cl - supercontinuous, where $p_y : X \times Y \to Y$ denotes the projection mapping. To show that X is quasi zero dimensional, let U be a θ-open set in X and let $x \in U$. Then $U \times Y$ is a θ-open set in $X \times Y$ and $g(x) \subset U \times Y$. Since g is upper quasi cl-supercontinuous, there exists a clopen set W containing x such that $g(W) \subset U \times Y$ and so $W \subset g_-^{-1}(U \times Y) = U$. Hence $x \in W \subset U$ and thus X is quasi zero dimensional.

We do not know whether the converse of Theorem 3.8 is true. The forthcoming theorem elaborates upon the behaviour of quasi cl-supercontinuity of multifunctions under restriction, and shrinking / expansion of range.

3.9. **Definition.** A subset S of a space X is said to be θ-embedded in X ([23])if every θ-closed subset in the subspace topology of S is the intersection with S of a θ-closed set in X, or equivalently every θ-open set in S is the intersection with S of a θ-open set in X.

3.10. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction from a topological space X into a topological space Y. Then the following statements are true.

(a) if φ is upper quasi cl-supercontinuous and $\varphi(X)$ is θ-embedded in Y, then the multifunction $\varphi : X \multimap \varphi(X)$ is upper quasi cl-supercontinuous.
(b) if φ is upper quasi cl-supercontinuous and Y is a subspace of Z, then the multifunction $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper quasi cl-supercontinuous.
(c) if φ is upper quasi cl-supercontinuous and $A \subset X$, then the restriction $\varphi_{|A} : A \multimap Y$ is upper quasi cl-supercontinuous. Further, if $\varphi(A)$ is θ-embedded in Y, then $\varphi_{|A} : A \multimap \varphi(A)$ is also upper quasi cl-supercontinuous.

Proof. (a) Let V_1 be a θ-open set in $\varphi(X)$. Since $\varphi(X)$ is θ-embedded in Y, there exists a θ-open set V in Y such that $V_1 = V \cap \varphi(X)$. Again, since, $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous, $\varphi_-^{-1}(V)$ is cl-open in X. Now $\varphi_-^{-1}(V_1) = \varphi_-^{-1}(V \cap \varphi(X)) = \varphi_-^{-1}(V) \cap \varphi_-^{-1}(\varphi(X)) = \varphi_-^{-1}(V) \cap X = \varphi_-^{-1}(V)$ and so $\varphi : X \multimap \varphi(X)$ is upper

quasi cl-supercontinuous.
(b) Let W be a θ-open set in Z. Then $W \cap Y$ is a θ-open set in Y. Since φ is upper quasi cl-supercontinuous, $\varphi_-^{-1}(W \cap Y)$ is cl-open in X. Now since $\psi_-^{-1}(W) = \psi_-^{-1}(W \cap Y) = \varphi_-^{-1}(W \cap Y)$, it follows that ψ is upper quasi cl-supercontinuous.
(c) Let V be a θ-open set in Y. Then $(\varphi_{|A})_-^{-1}(V) = \varphi_-^{-1}(V) \cap A$. Since φ is upper quasi cl-supercontinuous, $\varphi_-^{-1}(V)$ is cl-open in X. Consequently $\varphi_-^{-1}(V) \cap A$ is cl-open in A and so $\varphi_{|A}$ is upper quasi cl-supercontinuous.

3.11. **Theorem.** Let $\varphi : X \multimap Y$ be any multifunction. If $\{U_\alpha : \alpha \in \Delta\}$ is a cl-open cover of X and if for each α, the restriction $\varphi_\alpha = \varphi_{|U_\alpha} : U_\alpha \multimap Y$ is upper quasi cl-supercontinuous, then $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous.

Proof. Let W be an θ-open set in Y. Since $\varphi_\alpha = \varphi_{|U_\alpha} : U_\alpha \multimap Y$ is upper quasi cl-supercontinuous, $(\varphi_\alpha)_-^{-1}(W)$ is a cl-open set in U_α and consequently cl-open in X. Since $\varphi_-^{-1}(W) = \cup_{\alpha\in\Delta}(\varphi_\alpha)_-^{-1}(W)$ and since the union of cl-open set is cl-open, $\varphi_-^{-1}(W)$ is cl-open set in X. In view of Theorem 3.1, $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous.

4. Properties of lower quasi cl-supercontinuous multifunctions

4.1. **Theorem.** For a multifunction $\varphi : X \multimap Y$, the following statements are equivalent.
(a) φ is lower quasi cl-supercontinuous.
(b) $\varphi_+^{-1}(V)$ is cl-open for each θ-open set $V \subset Y$.
(c) $\varphi_-^{-1}(V)$ is cl-closed for each θ-closed set $B \subset Y$.
(d) $(\varphi_-^{-1}(B))_{cl} \subset \varphi_-^{-1}(B_{u\theta})$ for every subset B of Y.

Proof. $(a) \Rightarrow (b)$ Let V be a θ-open subset of Y. To show that $\varphi_+^{-1}(V)$ is cl-open in X, let $x \in \varphi_+^{-1}(V)$. Then $\varphi(x) \cap V \neq \phi$. Since φ is lower quasi cl-supercontinuous, there exists a clopen set H containing x such that $\varphi(z) \cap V \neq \phi$ for each $z \in H$. Hence $x \in H \subset \varphi_+^{-1}(V)$ and so $\varphi_+^{-1}(V)$ is a cl-open set in X.
$(b) \Rightarrow (c)$ Let B be a θ-closed subset of Y. Then $Y - B$ is a θ-open subset of Y. In view of (b), $\varphi_+^{-1}(Y - B)$ is a cl-open set in X. Since $\varphi_+^{-1}(Y - B) = X - \varphi_-^{-1}(B)$, $\varphi_-^{-1}(B)$ is a cl-closed set in X.
$(c) \Rightarrow (d)$ Since $B_{u\theta}$ is θ-closed, $\varphi_-^{-1}(B_{u\theta})$ is a cl-closed set containing $\varphi_-^{-1}(B)$ and so $(\varphi_-^{-1}(B))_{cl} \subset \varphi_-^{-1}(B_{u\theta})$.
$(d) \Rightarrow (a)$ Let $x \in X$ and let V be a θ-open set in Y such that $\varphi(x) \cap V \neq \phi$. Then $Y - V$ is a θ-closed set and so $(Y - V)_{u\theta} = Y - V$.

Hence $(\varphi_-^{-1}(Y-V))_{cl} \subset \varphi_-^{-1}(Y-V) = X - \varphi_+^{-1}(V)$. Since $\varphi_-^{-1}(Y-V)$ is cl-closed, its complement $\varphi_+^{-1}(V)$ is a cl-open set containing x. So there is a clopen set U containing x and contained in $\varphi_+^{-1}(V)$, whence $\varphi(z) \cap V \neq \phi$ for each $z \in U$. Thus φ is lower quasi cl-supercontinuous.

4.2. **Theorem.** If $\varphi : X \multimap Y$ is lower quasi cl-supercontinuous and $\varphi : Y \multimap Z$ is lower quasi θ-continuous, then the multifunction $\psi o \varphi$ is lower quasi cl-supercontinuous. In particular, the composition of two quasi lower cl-supercontinuous multifunctions is quasi lower cl-supercontinuous.

Proof. Let W be a θ-open set in Z. Since ψ is lower quasi θ-continuous, $\psi_+^{-1}(W)$ is a θ-open set in Y. Again, since φ is lower quasi cl-supercontinuous, $\varphi_+^{-1}(\psi_+^{-1}(W)) = (\psi o \varphi)_+^{-1}(W)$ is a cl-open set in X and so the multifunction $\psi o \varphi : X \multimap Z$ is lower quasi cl-supercontinuous.

4.3. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction from a topological space X into a topological space Y. The following statements are equivalent.

(a) φ is lower quasi cl-supercontinuous.
(b) $\varphi(A_{cl}) \subset (\varphi(A))_{u\theta}$ for every set $A \subset X$.
(c) $(\varphi_-^{-1}(B))_{cl} = \varphi_-^{-1}(B_{u\theta})$ for every set $B \subset Y$.

Proof. $(a) \Rightarrow (b)$ Let A be subset of X. Then $(\varphi(A))_{u\theta}$ is a θ-closed subset of Y. By Theorem 4.1 $\varphi_-^{-1}((\varphi(A))_{u\theta})$ is a cl-closed set in X. Since $A \subset \varphi_-^{-1}((\varphi(A))_{u\theta})$, $A_{cl} \subset \varphi_-^{-1}((\varphi(A))_{u\theta})$ and so $\varphi(A_{cl}) \subset \varphi(\varphi_-^{-1}((\varphi(A))_{u\theta})) \subset (\varphi(A))_{u\theta}$.
$(b) \Rightarrow (c)$ Let $B \subset Y$. Using (b) $\varphi((\varphi_-^{-1}(B))_{cl}) \subset (\varphi(\varphi_-^{-1}(B)))_{u\theta} \subset B_{u\theta}$. So it follows that $(\varphi_-^{-1}(B))_{cl} \subset \varphi_-^{-1}(B_{u\theta})$.
$(c) \Rightarrow (a)$ Let F be any θ-closed set in Y. Then by (c) $(\varphi_-^{-1}(F))_{cl} \subset \varphi_-^{-1}(F_{u\theta}) = \varphi_-^{-1}(F)$. Again, since $\varphi_-^{-1}(F) \subset (\varphi_-^{-1}(F))_{cl}$, $\varphi_-^{-1}(F) = (\varphi_-^{-1}(F))_{cl}$ which in its turn implies that $\varphi_-^{-1}(F)$ is cl-closed and so in view of Theorem 4.1 φ is lower quasi cl-supercontinuous.

4.4. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction from a topological space X into a topological space Y. Then the following statements are true.

(a) if φ is lower quasi cl-supercontinuous and $\varphi(X)$ is θ-embedded in Y, then the multifunction $\varphi : X \multimap \varphi(X)$ is lower quasi cl-supercontinuous.

(b) if φ is lower quasi cl-supercontinuous and Y is a subspace of Z then the multifunction $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is lower quasi cl-supercontinuous.

(c) if φ is lower quasi cl-supercontinuous and $A \subset X$, then the restriction $\varphi_{|A} : A \multimap Y$ is lower quasi cl-supercontinuous. Further, if $\varphi(A)$ is θ-embedded in Y, then $\varphi_{|A} : A \multimap \varphi(A)$ is also lower quasi cl-supercontinuous.

Proof. (a) Let V_1 be a θ-open set in $\varphi(X)$. Since $\varphi(X)$ is θ-embedded in Y, there exists a θ-open set V in Y such that $V_1 = V \cap \varphi(X)$. Again, since, $\varphi : X \multimap Y$ is lower quasi cl-supercontinuous, $\varphi_+^{-1}(V)$ is cl-open in X. Now $\varphi_+^{-1}(V_1) = \varphi_+^{-1}(V \cap \varphi(X)) = \varphi_+^{-1}(V) \cap \varphi_+^{-1}(\varphi(X)) = \varphi_+^{-1}(V) \cap X = \varphi_+^{-1}(V)$ and so $\varphi : X \multimap \varphi(X)$ is lower quasi cl-supercontinuous.
(b) Let W be a θ-open set in Z. Then $W \cap Y$ is a θ-open set in Y. Since φ is lower quasi cl-supercontinuous, $\varphi_+^{-1}(W \cap Y)$ is cl-open in X. Now Since $\psi_+^{-1}(W) = \psi_+^{-1}(W \cap Y) = \varphi_+^{-1}(W \cap Y)$, it follows that ψ is lower quasi cl-supercontinuous.
(c) Let V be a θ-open set in Y. Then $(\varphi_{|A})_+^{-1}(V) = \varphi_+^{-1}(V) \cap A$. Since φ is lower quasi cl-supercontinuous, $\varphi_+^{-1}(V)$ is cl-open in X. Consequently $\varphi_+^{-1}(V) \cap A$ is cl-open in A and so $\varphi_{|A}$ is lower quasi cl-supercontinuous.

5. Change of Topology

The technique of change of topology of a space is prevalent all through mathematics and is of considerable significance and widely used in topology, functional analysis and several other branches of mathematics. For example, weak and weak* topology of a Banach space, weak and strong operator topologies on $\mathbb{B}(H)$ the space of operators on a Hilber space H, hull kernel topology and multitude of other topologies on $Id(A)$ the space of all closed two sided ideals of a Banach algebra A([5] [6] [40]). Moreover to taste the flavour of applications of the technique in topology see([9] [15] [16] [28] [44]).
In this section we restrict ourselves to study the behaviour of an upper(lower) quasi cl-supercontinuous multifunction if its domain and/or range are retopologized in an appropriate way. This suggests alternative proofs of certain results of preceding sections. Moreover, in addition it suggests new results.

(1) Let (X, τ) be a topological space and let β denote the collection of all clopen subsets of (X, τ). Since the intersection of two clopen sets is a clopen set, the collection β is a base

for a topology τ^* on X. Clearly $\tau^* \subset \tau$ and any topological property which is preserved under continuous bijections is transferred from (X, τ) to (X, τ^*). Moreover, the space (X, τ) is zero dimensional if and only if $\tau = \tau^*$. The topology τ^* has been extensively referred to in the mathematical literature (see [8] [28] [38] [41]).

(2) Let (Y, σ) be a topological space, and let σ_θ denote the collection of all θ-open subsets of (Y, σ). Since the finite intersection and arbitrary union of θ-open sets is θ-open (see [42]), the collection σ_θ is a topology for Y considered in ([31] [7]). Clearly, $\sigma_\theta \subset \sigma$ and any topological property which is preserved by continuous bijections is transferred from (Y, σ) to (Y, σ_θ). Moreover, the space (Y, σ) is a regular space if and only if $\sigma = \sigma_\theta$. Throughout the section, the symbol σ_θ will have the same meaning as in the above paragraph.

5.1. **Theorem.** For a multifunction $\varphi : (X, \tau) \multimap (Y, \sigma)$, the following statements are equivalent

(a) $\varphi : (X, \tau) \multimap (Y, \sigma)$ is upper(lower) quasi cl-supercontinuous.
(b) $\varphi : (X, \tau) \multimap (Y, \sigma_\theta)$ is upper(lower) cl-supercontinuous.
(c) $\varphi : (X, \tau^*) \multimap (Y, \sigma)$ is upper(lower) faintly continuous.
(d) $\varphi : (X, \tau^*) \multimap (Y, \sigma_\theta)$ is upper(lower) semi continuous.

Proof. $(a) \Rightarrow (b)$. Let V be a open set in (Y, σ_θ). Then V is θ-open in (Y, σ). By (a) $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is cl-open in (X, τ). So φ is upper(lower) cl-supercontinuous.
$(b) \Rightarrow (c)$. Let V be a θ-open set in (Y, σ). By (b) $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is cl-open in (X, τ). Since every cl-open set is a union of clopen sets, hence $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is open in (X, τ^*).
$(c) \Rightarrow (d)$. Let V be an open set in (Y, σ_θ). Then V is θ-open in (Y, σ). By (c) $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is open in (X, τ^*). So φ is upper(lower) semi continuous.
$(d) \Rightarrow (a)$. Let V be a θ-open set in (Y, σ). Then V is open in (Y, σ_θ). By (d) $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is open in (X, τ^*). So $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ being union of clopen sets is cl-open in (X, τ).

Now Theorem 5.1 can be used to provide alternative proofs of certain results of preceding sections. To illustrate our viewpoint, we present an alternative proof of Theorem 3.1 using Theorem 5.1. First for the clarity and convenience of the reader, we rephrase Theorem 3.1:

5.2. **Theorem.** For a multifunction $\varphi : (X, \tau) \multimap (Y, \nu)$ the following statements are equivalent.

(a) φ is upper quasi cl-supercontinuous.
(b) $\varphi_-^{-1}(V)$ is cl-open in (X, τ) for each θ-open set V in (Y, ν).
(c) $\varphi_+^{-1}(B)$ is cl-closed in (X, τ) for each θ-closed set B in (Y, ν).
(d) $(\varphi_+^{-1}(B))_{cl} \subset \varphi_+^{-1}(B_{u\theta})$ for every set B in (Y, ν).

Proof. $(a) \Rightarrow (b)$. Suppose $\varphi : (X, \tau) \multimap (Y, \nu)$ is upper quasi cl-supercontinuous. By Theorem 5.1(4) $\varphi : (X, \tau^*) \multimap (Y, \nu_\theta)$ is upper semicontinuous. Let V be a θ-open set in (Y, ν), then V is open in (Y, ν_θ). In view of upper semicontinuity of the map $\varphi : (X, \tau^*) \multimap (Y, \nu_\theta)$, $\varphi_-^{-1}(V)$ is open in (X, τ^*) and so $\varphi_-^{-1}(V)$ is cl-open in (X, τ).
$(b) \Rightarrow (c)$. Let B be θ-closed set in (Y, ν). Then $Y - B$ is a θ-open set in (Y, ν). By Theorem 5.1(3) $\varphi_-^{-1}(Y - B) = X - \varphi_+^{-1}(B)$ is open set in (X, τ^*) and so $\varphi_+^{-1}(B)$ is closed in (X, τ^*) and hence cl-closed in (X, τ).
$(c) \Rightarrow (d)$. Let B be any set in Y. Then $B_{u\theta}$ is a θ-closed set in (Y, ν) and so $B_{u\theta}$ is closed in (Y, ν_θ). By Theorem 5.1(4) $\varphi_+^{-1}(B_{u\theta})$ is a closed set in (X, τ^*) containing $\varphi_+^{-1}(B)$ and so $(\varphi_+^{-1}(B))_{cl} \subset \varphi_+^{-1}(B_{u\theta})$.
$(d) \Rightarrow (a)$. Let $x \in X$ and let V be a θ-open set in (Y, ν) containing $\varphi(x)$. Then V is an open set in (Y, ν_θ) containing $\varphi(x)$. By Theorem 5.1(4), there exists a basic open set U in (X, τ^*) containing x and $\varphi(U) \subset V$. Clearly U is a clopen set containing x. Hence the multifunction $\varphi : X \multimap Y$ is upper quasi cl-supercontinuous.

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Dr. J.K. Kohli,
Department of Mathematics,
Hindu college, University of Delhi,
Delhi-110007, INDIA.
Email: jk_kohli@yahoo.com

Dr. Chaman Prakash Arya,
ISSA/DRDO,
Metcalfe House,
Delhi 110054, INDIA.
Email:carya28@gmail.com

Dr. D. Singh,
Department of Mathematics,

Sri Aurobindo College, University of Delhi.
New Delhi 110017, INDIA.
Email:dstopology@rediffmail.com