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ON UPPER AND LOWER m-ITERATE CONTINUOUS MULTIFUNCTIONS

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Abstract. We introduce the notion of mIT-structures determined by operators mInt and mCl on an m-space (X, m_X). By using mIT-structures, we introduce and investigate new classes of multifunctions $F : (X, m_X) \to (Y, \sigma)$ called upper/lower mIT-continuous. As special cases of upper/lower mIT-continuity, we obtain upper/lower γ-M continuity [34] and upper/lower δ-M-precontinuity [35].

1. Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and b-open sets play an important role in the researches of generalizations of continuity of functions and multifunctions. By using these sets, several authors introduced and studied various types of non-continuous functions and multifunctions.

In [26] and [28], the present authors introduced and studied the notions of minimal structures, m-spaces, m-continuity and M-continuity. Quite recently, in [16], [17], [18], [19], and [20], Min and Kim introduced the notions of m-semi-open sets, m-preopen sets, m-α-open sets, m-β-open sets which generalize the notions of semi-open sets, preopen sets, α-open sets, and β-open sets, respectively and also the notions of m-semi-continuity, m-precontinuity, m-α-continuity, m-β-continuity which generalize the notions of of semi-continuity, precontinuity, α-continuity, β-continuity and m-continuity.

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In [6], [7], [31] and [33], the notions of m-semi-open sets, m-preopen sets, m-α-open sets and m-β-open sets are also introduced and studied. Recently, in [25] the notion of iterate m-continuous functions are introduced and studied. Also, in [34] and [35], the notions of γ-M continuity and δ-M-precontinuity for multifunctions are initiated and studied.

In the present paper, we introduce the notions of upper / lower mIT-continuous multifunctions which generalize the notions of upper / lower γ-M continuity and δ-M-precontinuity. By using upper/lower M-continuity [22], we obtain many properties of upper / lower mIT - continuous multifunctions which provide properties of upper / lower γ-M continuity and δ-M-precontinuity. And also, the notions of upper/lower mIT-continuous multifunctions provide the definitions of the new notions: upper/lower m-semi-continuity, upper / lower m-precontinuity, upper/lower m-α-continuity, and upper/lower m - β-continuity. The properties of these multifunctions are obtained from those of upper/lower mIT-continuity as the special cases.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A point $x \in X$ is called a δ-cluster (resp. θ-cluster) point of a subset A if $\mathrm{Int}(\mathrm{Cl}((U)) \cap A \neq \emptyset$ (resp. $\mathrm{Cl}((U) \cap A \neq \emptyset)$ for every open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the δ-closure (rep. θ-closure) of A and is denoted by $\mathrm{Cl}\delta(A)$ (resp. $\mathrm{Cl}\theta(A)$). If $A = \mathrm{Cl}\delta(A)$ (resp. $A = \mathrm{Cl}\theta(A)$), then A is said to be δ-closed (resp. θ-closed) [32]. The complement of a δ-closed (resp. θ-closed) set is said to be δ-open (resp. θ-open). The union of all δ-open (resp. θ-open) sets contained in A is called the δ-interior (resp. θ-interior) of A and is denoted by $\mathrm{Int}\delta(A)$ (resp. $\mathrm{Int}\theta(A)$).

We recall some generalized open sets in topological spaces.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [21] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [12] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [14] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *b-open* [4] or *γ-open* [9] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(5) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all α-open (resp. semi-open, preopen, γ-open, β-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\gamma(X)$, $\beta(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [15] (resp. *semi-closed* [8], *preclosed* [14], *γ-closed* [9], *β-closed* [1]) if the complement of A is α-open (resp. semi-open, preopen, γ-open, β-open).

Definition 2.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, b-closed, β-closed) sets of X containing A is called the *α-closure* [15] (resp. *semi-closure* [8], *preclosure* [10], *γ-closure* [9], *β-closure* [2]) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\gamma\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$).

Definition 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, γ-open, β-open) sets of X contained in A is called the *α-interior* [15] (resp. *semi-interior* [8], *preinterior* [10], *γ-interior* [9], *β-interior* [2]) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\gamma\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$).

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *semi-continuous* [12] (resp. *precontinuous* [14], *α-continuous* [15], *γ - continuous* [9], *β-continuous* [1]) at $x \in X$ if for each open set V containing $f(x)$, there exists a semi-open (resp. preopen, α-open, γ-open, β-open) set U of X containing x such that $f(U) \subset V$. The function f is said to be *semi-continuous* (resp. *precontinuous* , *α-continuous*, *γ-continuous*, *β-continuous*) if it has this property at each point $x \in X$.

Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

3. Minimal structures and upper/lower m-continuity

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [26], [27] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be m_X-*open* (briefly m-open) and the complement of an m_X-open set is said to be m_X-*closed* (briefly m-closed).

Remark 3.1. Let (X, τ) be a topological space. The families τ, $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\gamma(X)$ and $\beta(X)$ are all minimal structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-*closure* of A and the m_X-*interior* of A are defined in [13] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\gamma(X)$, $\beta(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\gamma\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$) .

Lemma 3.1. (Maki et al. [13]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) If $(X \setminus A) \in m_X$, then $\mathrm{mCl}(A) = A$ *and if $A \in m_X$, then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2. (Popa and Noiri [26]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [13] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.3. If (X, τ) is a topological space, then $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\gamma(X)$ and $\beta(X)$ have property $\mathcal{B}$.

Lemma 3.3. (Popa and Noiri [30]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is* m_X*-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is* m_X*-closed.*

Definition 3.4. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m-continuous* at $x \in X$ [28] if for each open set $V \in \sigma$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is m-continuous if it has this property at each $x \in X$.

Remark 3.4. Let (X, τ) be a topological space. If $f : (X, m_X) \to (Y, \sigma)$ is m-continuous and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\gamma(X)$ and $\beta(X)$), then we obtain Definition 2.5.

Definition 3.5. Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper m-continuous* at $x \in X$ [29] if for each open set $V \in \sigma$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower m-continuous* at $x \in X$ [29] if for each open set $V \in \sigma$ meeting $F(x)$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower m-continuous* if it has this property at each $x \in X$.

Theorem 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ *for every open set V of Y;*
(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ *for every closed set K of Y;*
(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for every subset B of Y;*
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ *for every subset B of Y.*

Proof. The proof follows from Theorem 3.1 of [22] and Lemma 4.1 of [29].

Corollary 3.1. *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^+(V)$ is m-open for every open set V of Y;
(3) $F^-(K)$ is m-closed for every closed set K of Y.

Proof. The proof follows from Theorem 3.1 and Lemma 3.3.

Theorem 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-continuous;
(2) $F^{-}(V) = \mathrm{mInt}(F^{-}(V))$ for every open set V of Y;
(3) $F^{+}(K) = \mathrm{mCl}(F^{+}(K))$ for every closed set K of Y;
(4) $\mathrm{mCl}(F^{+}(B)) \subset F^{+}(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F^{-}(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^{-}(B))$ for every subset B of Y;
(6) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X.

Proof. The proof follows from Theorem 3.2 of [22] and Lemma 4.1 of [29].

Corollary 3.2. *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^{-}(V)$ is m-open for every open set V of Y;
(3) $F^{+}(K)$ is m-closed for every closed set K of Y.

Proof. The proof follows from Theorem 3.2 and Lemma 3.3.

For a function $F : (X, m_X) \to (Y, \sigma)$, we define $D^{+}_{m}(F)$ and $D^{-}_{m}(F)$ as follows:

$$D^{+}_{m}(F) = \{x \in X : F \text{ is not upper } m\text{-continuous at } x\},$$
$$D^{-}_{m}(F) = \{x \in X : F \text{ is not lower } m\text{-continuous at } x\}$$

Theorem 3.3. (Noiri and Popa [23]). *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^{+}_{m}(F) &= \textstyle\bigcup_{G\in\sigma}\{F^{+}(G) \setminus \mathrm{mInt}(F^{+}(G))\} \\ &= \textstyle\bigcup_{B\in \mathcal{P}(Y)} \{F^{+}(\mathrm{Int}(B)) \setminus \mathrm{mInt}(F^{+}(B))\} \\ &= \textstyle\bigcup_{B\in \mathcal{P}(Y)} \{\mathrm{mCl}(F^{-}(B)) \setminus F^{-}(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{K\in \mathcal{F}} \{\mathrm{mCl}(F^{-}(K)) \setminus F^{-}(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Theorem 3.4. (Noiri and Popa [23]). *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^{-}_{m}(F) &= \textstyle\bigcup_{G\in\sigma}\{F^{-}(G) \setminus \mathrm{mInt}(F^{-}(G))\} \\ &= \textstyle\bigcup_{B\in \mathcal{P}(Y)} \{F^{-}(\mathrm{Int}(B)) \setminus \mathrm{mInt}(F^{-}(B))\} \\ &= \textstyle\bigcup_{B\in \mathcal{P}(Y)} \{\mathrm{mCl}(F^{+}(B)) \setminus F^{+}(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{A\in \mathcal{P}(X)} \{\mathrm{mCl}(A) \setminus F^{+}(\mathrm{Cl}(F(A)))\} \\ &= \textstyle\bigcup_{K\in \mathcal{F}} \{\mathrm{mCl}(F^{+}(K)) \setminus F^{+}(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

4. Iterate m-structures and iterate m-continuity for multifunctions

Definition 4.1. Let (X, m_X) be an m-space. A subset A of X is said to be

(1) *m-α-open* [17] if $A \subset \mathrm{mInt}(\mathrm{mCl}(\mathrm{mInt}(A)))$,
(2) *m-semi-open* [16] if $A \subset \mathrm{mCl}(\mathrm{mInt}(A))$,
(3) *m-preopen* [19] if $A \subset \mathrm{mInt}(\mathrm{mCl}(A))$,
(4) *m-β-open* [6], [33] if $A \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}(A)))$,
(5) *m-γ-open* [34] or *m-b-open* [25] if $A \subset \mathrm{mInt}(\mathrm{mCl}(A)) \cup \mathrm{mCl}(\mathrm{mInt}(A))$,
(6) *m-regular open* (resp. *m-regular closed*) if $A = \mathrm{mInt}(\mathrm{mCl}(A))$ (resp. $A = \mathrm{mCl}(\mathrm{mInt}(A))$.

The family of all m-α-open (resp. m-semi-open, m-preopen, m-β-open, m-γ-open) sets in (X, m_X) is denoted by $\mathrm{m}\alpha(X)$ (resp. $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\beta(X)$, $\mathrm{m}\gamma(X)$).

Remark 4.1. Similar definitions of m-semi-open sets, m-preopen sets, m-α-open sets, m-β-open sets are provided in [7], [31] and [33].

Let A be a subset of an m-space (X, m_X). The union of m-regular open sets of X contained in A is called *m_X-δ-interior* of A and is denoted by m_X-$\delta\mathrm{Int}(A)$. A subset A is said to be m_X-δ-open if $A = m_X$-$\delta\mathrm{Int}(A)$. The complement of an m_X-δ-open set is said to be m_X-δ-closed. The intersection of all m_X-δ-closed sets of X containing A is called the m_X-δ-closure of A and is denoted by m_X-$\delta\mathrm{Cl}(A)$. A subset A of (X, m_X) is said to be m_X-δ-preopen [35] if $A \subset \mathrm{mInt}(m_X\text{-}\delta\mathrm{Cl}(A))$. The complement of an m_X-δ-preopen set is said to be m_X-δ-preclosed. The family of all m_X-δ-open (resp. m_X-δ-preopen) sets of X is denoted by $\mathrm{m}\delta(X)$ (resp. $\mathrm{m}\delta\mathrm{PO}(X)$).

Let (X, m_X) be an m-space. Then $\mathrm{m}\alpha(X)$, $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\beta(X)$, $\mathrm{m}\gamma(X)$, $\mathrm{m}\delta(X)$ and $\mathrm{m}\delta\mathrm{PO}(X)$ are all minimal structures on X and are determined by iterating operators mInt, mCl, m_X-δInt and m_X-δCl. Hence, they are called *m-iterate structures* [25] and are denoted by $\mathrm{mIT}(X)$ (briefly mIT).

Remark 4.2. (1) It easily follows from Lemma 3.1(3)(4) that $\mathrm{m}\alpha(X)$, $\mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\beta(X)$ and $\mathrm{mBO}(X)$ are minimal structures with property $\mathcal{B}$. They are also shown in Theorem 3.5 of [16], Theorem 3.4 of [19], Theorem 3.4 of [17], Proposition 3.5 of [34].

(2) Let (X, m_X) be an m-space and $\mathrm{mIT}(X)$ an iterate structure on X. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X)$), $\mathrm{m}\gamma(X)$,

mδPO(X)), then we obtain the following definitions provided in [16] (resp. [19], [17], [20], [34], [35]):

mITCl(A) = msCl(A) (resp. mpCl(A), mαCl(A), mβCl(A), $m\gamma$Cl(A), mδpCl(A)),

mITInt(A) = msInt(A) (resp. mpInt(A), mαInt(A), mβInt(A), mγInt(A), mδpInt(A)).

Remark 4.3. (1) By Lemmas 3.1 and 3.3, we obtain Theorems 3.7 and 3.8 of [19], Theorems 3.8 and 3.9 of [17], Remark 3.10 of [34].

(2) By Lemma 3.2, we obtain Lemma 3.9 of [19] and Theorem 3.10 of [17].

Definition 4.2. Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper γ-M-continuous* [34] (resp. *upper δ-M-precontinuous* [35]) at $x \in X$ if for each open set $V \in \sigma$ containing $F(x)$, there exists $U \in m\gamma(X)$ (resp. mδPO(X)) containing x such that $F(U) \subset V$,

(2) *lower γ-M-continuous* [34] (resp. *lower δ-M-precontinuous* [35]) at $x \in X$ if for each open set $V \in \sigma$ meeting $F(x)$, there exists $U \in m\gamma(X)$ (resp. mδPO(X)) containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower γ-M-continuous* (resp. *upper/lower δ-M-precontinuous*) if it has this property at each $x \in X$.

Remark 4.4. By Definition 4.2 and Remark 4.2, it follows that a multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper/lower γ-M-continuous (resp. upper/lower δ-M-precontinuous) at x (on X) if a multifunction $F : (X, \mathrm{m}\gamma(X)) \to (Y, \sigma)$ (resp. $F : (X, \mathrm{m}\delta\mathrm{PO}(X)) \to (Y, \sigma)$) is upper/lower m-continuous at x (on X).

Definition 4.3. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *upper/lower mIT-continuous* at $x \in X$ (on X) if $F : (X, mIT(X)) \to (Y, \sigma)$ is upper/lower m-continuous at $x \in X$ (on X).

Remark 4.5. Let (X, m_X) be a minimal space. If mIT(X) = mSO(X) (resp. mPO(X), m$\alpha(X)$, m$\beta(X)$, m$\gamma(X)$, mδPO(X)) and $F : (X, m_X) \to (Y, \sigma)$ is mIT-continuous, then F is upper/lower m-semicontinuous (resp. upper/lower m-precontinuous, upper/lower m-α-continuous, upper/lower m-β-continuous, upper/lower γ-M-continuous [34], upper/lower δ-M-precontinuous [35]).

Since mIT(X) has property $\mathcal{B}$, by Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 we obtain the following theorems.

Theorem 4.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper mIT-continuous;
(2) $F^+(V)$ is mIT-open for every open set V of Y;
(3) $F^-(K)$ is mIT-closed for every closed set K of Y;
(4) $\mathrm{mITCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for every subset B of Y;*
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mITInt}(F^+(B))$ *for every subset B of Y.*

Remark 4.6. (1) If $\mathrm{mIT}(X) = \mathrm{m}\gamma(X)$ (resp. $\mathrm{m}\delta\mathrm{PO}(X)$), then by Theorem 4.1 we obtain the results established in Theorem 4.5 (1), (2), (3) and (4) of [34] (resp. Theorem 3.5 (1), (2), (3), (4) and (7) of [35]),
(2) If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X)$), then we obtain characterizations of upper m-semicontinuous (resp. upper m-precontinuous, upper m-α-continuous, upper m-β-continuous) multifunctions.

Theorem 4.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower mIT-continuous;
(2) $F^-(V)$ is mIT-open for every open set V of Y;
(3) $F^+(K)$ is mIT-closed for every closed set K of Y;
(4) $\mathrm{mITCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y;*
(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mITInt}(F^-(B))$ *for every subset B of Y;*
(6) $F(\mathrm{mITCl}(A)) \subset \mathrm{Cl}(F(A))$ *for every subset A of X.*

Remark 4.7. (1) If $\mathrm{mIT}(X) = \mathrm{m}\gamma(X)$ (resp. $\mathrm{m}\delta\mathrm{PO}(X)$), then by Theorem 4.2 we obtain the results established in Theorem 4.6 (1), (2), (3), (4) and (5) of [34] (resp. Theorem 3.6 (1), (2), (3), (6) and (7) of [35]),
(2) By Theorem 4.2 (6), we obtain a new characterization for lower γ-M-continuous (resp. lower δ-M-continuous) multifunctions.
(3) If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X)$), then we obtain characterizations of lower m-semicontinuous (resp. lower m-precontinuous, lower m-α-continuous, lower m-β-continuous) multifunctions.

For a function $F : (X, m_X) \to (Y, \sigma)$, we define $D^+_{mIT}(F)$ and $D^-_{mIT}(F)$ as follows:

$$D^+_{mIT}(F) = \{x \in X : F \text{ is not upper } mIT\text{-continuous at } x\},$$
$$D^-_{mIT}(F) = \{x \in X : F \text{ is not lower } mIT\text{-continuous at } x\}$$

By Theorems 3.3 and 3.4, we obtain the following theorems:

Theorem 4.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^+_{mIT}(F) &= \bigcup_{G\in\sigma}\{F^+(G) \setminus \mathrm{mITInt}(F^+(G))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) \setminus \mathrm{mITInt}(F^+(B))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{\mathrm{mITCl}(F^-(B)) \setminus F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{K\in\ \mathcal{F}} \{\mathrm{mITCl}(F^-(K)) \setminus F^-(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Theorem 4.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^-_{mIT}(F) &= \bigcup_{G\in\sigma}\{F^-(G) \setminus \mathrm{mITInt}(F^-(G))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) \setminus \mathrm{mITInt}(F^-(B))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{\mathrm{mITCl}(F^+(B)) \setminus F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\ \mathcal{P}(X)} \{\mathrm{mITCl}(A) \setminus F^+(\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{K\in\ \mathcal{F}} \{\mathrm{mITCl}(F^+(K)) \setminus F^+(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, we define $D^+_{m\gamma}(F)$ and $D^-_{m\gamma}(F)$ as follows:

$$D^+_{m\gamma}(F) = \{x \in X : F \text{ is not upper } m\gamma\text{-continuous at } x\},$$
$$D^-_{m\gamma}(F) = \{x \in X : F \text{ is not lower } m\gamma\text{-continuous at } x\}$$

By Theorems 4.3 and 4.4, we obtain the following corollaries:

Corollary 4.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^+_{m\gamma}(F) &= \bigcup_{G\in\sigma}\{F^+(G) \setminus \mathrm{m\gamma Int}(F^+(G))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) \setminus \mathrm{m\gamma Int}(F^+(B))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{\mathrm{m\gamma Cl}(F^-(B)) \setminus F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{K\in\ \mathcal{F}} \{\mathrm{m\gamma Cl}(F^-(K)) \setminus F^-(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Corollary 4.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following equalities hold:*

$$\begin{aligned} D^-_{m\gamma}(F) &= \bigcup_{G\in\sigma}\{F^-(G) \setminus \mathrm{m\gamma Int}(F^-(G))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) \setminus \mathrm{m\gamma Int}(F^-(B))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)} \{\mathrm{m\gamma Cl}(F^+(B)) \setminus F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\ \mathcal{P}(X)} \{\mathrm{m\gamma Cl}(A) \setminus F^+(\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{K\in\ \mathcal{F}} \{\mathrm{m\gamma Cl}(F^+(K)) \setminus F^+(K)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Remark 4.8. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X))$, or $\mathrm{m}\delta\mathrm{PO}(X)$, then we obtain the similar results.

5. Some properties of mIT-continuous functions

Lemma 5.1. (Noiri and Popa [24]). *Let (Y, σ) be a regular space and m_X have property $\mathcal{B}$. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^-(\mathrm{Cl}\theta(B))$ is m-closed for every subset B of Y;
(3) $F^-(K)$ is m-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is m-open for every θ-open set V of Y.

Lemma 5.2. (Noiri and Popa [24]). *Let (Y, σ) be a regular space and m_X have property $\mathcal{B}$. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^+(\mathrm{Cl}\theta(B))$ is m-closed for every subset B of Y;
(3) $F^+(K)$ is m-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is m-open for every θ-open set V of Y.

By Definition 4.3, Lemmas 5.1 and 5.2 and Remark 4.2(1), we obtain the following theorems:

Theorem 5.1. *Let (Y, σ) be a regular space. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper mIT-continuous;
(2) $F^-(\mathrm{Cl}\theta(B))$ is mIT-closed for every subset B of Y;
(3) $F^-(K)$ is mIT-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is mIT-open for every θ-open set V of Y.

Theorem 5.2. *Let (Y, σ) be a regular space. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower mIT-continuous;
(2) $F^+(\mathrm{Cl}\theta(B))$ is mIT-closed for every subset B of Y;
(3) $F^+(K)$ is mIT-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is mIT-open for every θ-open set V of Y.

Corollary 5.1. *Let (Y, σ) be a regular space. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper γ-M-continuous;
(2) $F^-(\mathrm{Cl}\theta(B))$ is γ-M-closed for every subset B of Y;
(3) $F^-(K)$ is γ-M-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is γ-M-open for every θ-open set V of Y.

Corollary 5.2. *Let (Y, σ) be a regular space. Then, for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower γ-M-continuous;
(2) $F^+(\mathrm{Cl}\theta(B))$ is γ-M-closed for every subset B of Y;
(3) $F^+(K)$ is γ-M-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is γ-M-open for every θ-open set V of Y.

Remark 5.1. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X)$ or $\mathrm{m}\delta\mathrm{PO}(X)$, then we obtain the similar corollaries with Corollaries 5.1 and 5.2.

Definition 5.1. A subset A of a topological space (Y, σ) is said to be

(1) *α-regular* [11] if for each $a \in A$ and each open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(2) *α-paracompact* [36] if every X-open cover of A has an X-open refinement which covers A and is locally finite for each point of X.

For a multifunction $F : X \to (Y, \sigma)$, by $\mathrm{Cl}(F) : X \to (Y, \sigma)$ [5] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we denote $\mathrm{sCl}(F)$, $\mathrm{pCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma\mathrm{Cl}(F)$, $\delta\mathrm{pCl}(F)$.

Lemma 5.3. *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$, then $G^+(V) = F^+(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$, $\delta pCl(F)$.*

Proof. The proof for $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or $\beta\mathrm{Cl}(F)$ follows from Lemma 3.6 of [24]. The proof for $\gamma\mathrm{Cl}(F)$ or $\delta\mathrm{pCl}(F)$ is similar with the proof of Lemma 3.6 of [24].

Lemma 5.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold: $G^-(V) = F^-(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Proof. The proof for $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or $\beta\mathrm{Cl}(F)$ follows from Lemma 3.7 of [24]. The proof for $\gamma\mathrm{Cl}(F)$ or $\delta\mathrm{pCl}(F)$ is similar with the proof of Lemma 3.7 of [24].

Theorem 5.3. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper m-continuous if and only if G is upper m-continuous, where G $= \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Proof. Let V be an open set of Y and F is upper m-continuous. By Lemma 5.3 and Theorem 3.1, $G^+(V) = F^+(V) = \mathrm{mInt}(F^+(V) = m\mathrm{Int}(G^+(V)$. By Theorem 3.1 G is upper m-continuous.

Conversely, let V be an open set of Y and G is upper m-continuous. By Lemma 5.3 and Theorem 3.1, $F^+(V) = G^+(V) = \mathrm{mInt}(G^+(V) = m\mathrm{Int}(F^+(V)$. By Theorem 3.1 F is upper m-continuous.

Theorem 5.4. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. Then F is lower m-continuous if and only if G is lower m-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Proof. Let V be an open set of Y and F is lower m-continuous. By Lemma 5.4 and Theorem 3.2, $G^-(V) = F^-(V) = \mathrm{mInt}(F^-(V) = m\mathrm{Int}(G^-(V)$. By Theorem 3.2 G is lower m-continuous.

Conversely, let V be an open set of Y and G is lower m-continuous. By Lemma 5.4 and Theorem 3.2, $F^-(V) = G^-(V) = \mathrm{mInt}(G^-(V) = m\mathrm{Int}(F^-(V)$. By Theorem 3.2 F is lower m-continuous.

By Definition 4.3, Theorems 5.3 and 5.4, we obtain the following theorems:

Theorem 5.5. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper mIT-continuous if and only if G is upper mIT-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Theorem 5.6. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. Then F is lower mIT-continuous if and only if G is lower mIT-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Corollary 5.3. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper γ-M-continuous if and only if G is upper γ-M-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Corollary 5.4. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. Then F is lower γ-M-continuous if and only if G is lower γ-M-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\gamma Cl(F)$ or $\delta pCl(F)$.*

Remark 5.2. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X))$ or $\mathrm{m}\delta\mathrm{PO}(X)$, then we obtain the similar corollaries with Corollaries 5.3 and 5.4.

Definition 5.2. Let (X, m_X) be an m-space and A a subset of X. The *m-frontier* of A [27], $\mathrm{mFr}(A)$, is defined as follows: $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X \setminus A)$.

Theorem 5.7. *The set of all points $x \in X$ at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper (resp. lower) m-continuous is* *identical with the union of the m-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Proof. Let x be a point of X at which F is not upper m-continuous. Then there exists an open sets V of Y such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2, $x \in \mathrm{mCl}(X \backslash (F^+(V))$. On the other hand, $x \in F^+(V) \subset \mathrm{Cl}(F^+(V)$ and hence $x \in \mathrm{mFr}(F^+(V)$.

Conversely, suppose that F is upper m-continuous. For any $x \in X$ and any open sets of Y containing $F(x)$, we have $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction. Hence F is not upper m-continuous. In case F is lower m-continuous the proof is similar.

Definition 5.3. Let (X, m_X) be an m-space and A a subset of X. The *mIT-frontier* of A, $\mathrm{mITFr}(A)$, is defined as follows: $\mathrm{mITFr}(A) = \mathrm{mITCl}(A) \cap \mathrm{mITCl}(X \setminus A)$.

Theorem 5.8. *The set of all points $x \in X$ at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper (resp. lower) mIT-continuous is identical with the union of the mIT-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Proof. The proof follows from Definition 4.3 and Theorem 5.7.

Remark 5.3. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $\mathrm{m}\alpha(X)$, $\mathrm{m}\beta(X)$, $m\gamma(X)$ or $\mathrm{m}\delta\mathrm{PO}(X)$, then we obtain the similar results.

Definition 5.4. An m-space (X, m_X) is said to be *m-compact* [22], [26] if every cover of X by m-open sets has a finite subcover.

A subset K of an m-space (X, m_X) is said to be *m-compact* [22], [26] if every cover of K by m-open sets has a finite subcover.

Definition 5.5. An m-space (X, m_X) is said to be *mIT-compact* if every cover of X by mIT-open sets has a finite subcover.

A subset K of an m-space (X, m_X) is said to be *mIT-compact* if every cover of K by mIT-open sets has a finite subcover.

Remark 5.4. If $\mathrm{mIT}(X) = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then by Definition 5.5 we obtain the definition of m-semicompact spaces [18] (resp. m-precompact spaces [20]).

Lemma 5.5. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper m-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and K is an m-compact set of X, then $F(K)$ is compact.*

Proof. The proof follows from Theorem 4.1 of [22].

Corollary 5.5. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper m-continuous surjective multifunction such that $F(x)$ is compact for each $x \in X$ and (X, m_X) is m-compact, then (Y, σ) is compact.*

Lemma 5.6. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper mIT-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and K is an mIT-compact set of X, then $F(K)$ is compact.*

Proof. The proof follows from Theorem 4.1 of [22].

Corollary 5.6. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper mIT-continuous surjective multifunction such that $F(x)$ is compact for each $x \in X$ and (X, m_X) is mIT-compact, then (Y, σ) is compact.*

Corollary 5.7. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper γ-M-continuous surjective multifunction such that $F(x)$ is compact for each $x \in X$ and (X, m_X) is γ-M-compact, then (Y, σ) is compact.*

Definition 5.6. An m-space (X, m_X) is said to be *m-connected* [28] if X cannot be written as the union of two nonempty disjoint m-open sets.

Definition 5.7. An m-space (X, m_X) is said to be *mIT-connected* if an m-space $(X, mIT(X))$ is m-connected. Hence an m-space $(X, mIT(X))$ is m-connected if X cannot be written as the union of two nonempty disjoint mIT-open sets.

Lemma 5.7. (Noiri and Popa [22]). *If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower m-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is connected.*

Theorem 5.9. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower mIT-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is mIT-connected, where m_X has property $\mathcal{B}$, then (Y, σ) is connected.*

Corollary 5.8. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower γ-M-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is γ-M-connected, then (Y, σ) is connected.*

Remark 5.5. If mIT(X) = mSO(X), mPO(X), m$\alpha(X)$, m$\beta(X)$ or mδPO(X), then we obtain the similar result with Corollary 5.8.

6. Separation axioms and mIT-continuous multifunctions

Definition 6.1. A subset A of (X, m_X) is said to be *m-dense* in X if $\mathrm{mCl}(A) = X$.

Theorem 6.1. *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$ and (Y, σ) be a Hausdorff space.*
If the following four conditions are satisfied,
(1) a multifunction $G : (X, m_X^1) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F : (X, m_X^2) \to (Y, \sigma)$ is upper m-continuous,
(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,
(4) $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of (X, m_X^2),
then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Proof. Let $A = \{ x \in X : F(x) \cap G(x) \neq \emptyset \}$. Suppose that $x \in X \setminus A$. Then we have $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are compact sets of a Hausdorff space Y, there exist open sets V and W of Y such that $F(x) \subset V, G(x) \subset W$ and $V \cap W = \emptyset$. Since G is upper m-continuous, there exists $U_1 \in m_X^1$ containing x such that $G(U_1) \subset W$. Since F is upper m-continuous, there exists $U_2 \in m_X^2$ containing x such that $F(U_2) \subset V$. Now, set $U = U_1 \cap U_2$, then we have $U \in m_X^2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 we have $x \in X \setminus \mathrm{m}_X^2\mathrm{Cl}(A)$ and hence $A =_X^2 \mathrm{Cl}(A)$. On the other hand, $F(x) \cap G(x) \neq \emptyset$ on D and hence $D \subset A$. Since D is m-dense in (X, m_X^2), we have $X = \mathrm{m}_X^2\mathrm{Cl}(D) \subset \mathrm{m}_X^2\mathrm{Cl}(A) = A$. Therefore, we obtain $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

Definition 6.2. A subset A of (X, m_X) is said to be *mIT-dense* in X if $\mathrm{mITCl}(A) = X$.

Theorem 6.2. *Let (X, m_X) be an m-space with two m-iterate structures $mIT^1(X)$ and $mIT^2(X)$ such that $U \cap V \in mIT^2(X)$ whenever $U \in mIT^1(X)$ and $V \in mIT^2(X)$ and (Y, σ) be a Hausdorff space.*
If the following four conditions are satisfied,
(1) a multifunction $G : (X, mIT^1(X)) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F : (X, mIT^2(X)) \to (Y, \sigma)$ is upper m-continuous,
(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,
(4) $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of $(X, mIT^2(X))$,
then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Corollary 6.1. *Let (X, m_X) be an m-space with two $m\gamma$-iterate structures $m\gamma^1(X)$ and $m\gamma^2(X)$ such that $U \cap V \in m\gamma^2(X)$ whenever*

$U \in m\gamma^1(X)$ and $V \in m\gamma^2(X)$ and (Y, σ) be a Hausdorff space.
If the following four conditions are satisfied,
(1) a multifunction $G : (X, m\gamma^1(X)) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F : (X, m\gamma^2(X)) \to (Y, \sigma)$ is upper m-continuous,
(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,
(4) $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of $(X, m\gamma^2(X))$,
then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Remark 6.1. If mIT(X) = mSO(X), mPO(X), m$\alpha(X)$, m$\beta(X)$ or mδPO(X), then we obtain the similar result with Corollary 6.1.

Definition 6.3. An m-space (X, m_X) is said to be *m-T_2* [28] if for each distinct points $x, y \in X$ there exist $U, V \in m_X$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Definition 6.4. An m-space (X, m_X) is said to be *mIT-T_2* if an m-space $(X, mIT(X))$ is m-T_2. Hence an m-space (X, m_X) is mIT-T_2 if for each distinct points $x, y \in X$ there exist $U, V \in mIT(X)$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Remark 6.2. Let (X, m_X) be an m-space. If mIT(X) = mSO(X) (resp. mPO(X)), then we obtain the definition of m-semi-T_2 spaces [18] (resp. m-pre-T_2 spaces [20]).

Definition 6.5. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Theorem 6.3. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper m-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then X is m-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = \emptyset$ since F is injective. Since $F(x)$ is compact for each $x \in X$ and Y is Haudorff, there exist an open set V_i such that $F(x_i) \subset V_i$ for $i = 1, 2$ and $V_1 \cap V_2 = \emptyset$. Since F is upper m-continuous, there exists $U_i \in m_X$ containing x_i such that $F(U_i) \subset V_i$ for $i = 1, 2$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$ and hence X is m-T_2.

Theorem 6.4. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper mIT-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then (X, m_X) is mIT-T_2.*

Corollary 6.2. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper γ-M-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then (X, m_X) is $m\gamma(X)$-T_2.*

Remark 6.3. Let (X, m_X) be an m-space. If $mIT(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $m\alpha(X)$, $m\beta(X)$ or $m\delta\mathrm{PO}(X)$, we obtain the similar result with Corollary 6.2.

Theorem 6.5. *Let X be a nonempty set with two minimal structures m^1_X and m^2_X such that $U \cap V \in m^2_X$ whenever $U \in m^1_X$ and $V \in m^2_X$ and (Y, σ) be a Hausdorff space. If the following four conditions are satisfied:*

(1) a multifunction $F_1 : (X, m^1_X) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F_2 : (X, m^2_X) \to (Y, \sigma)$ is upper m-continuous,
(3) $F_1(x)$ and $F_2(x)$ are compact sets of (Y, σ) for each $x \in X$,
(4) $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$,

then a multifunction $F : (X, m^2_X) \to (Y, \sigma)$, defined by formula $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper m-continuous.

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $A = F_1(x) \setminus V$ and $B = F_2(x) \setminus V$ are disjoint compact sets. Hence, there exist open sets V_1 and V_2 such that $A \subset V_1, B \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since F_1 is upper m-continuous, there exists $U_1 \in m^1_X$ containing x such that $F_1(U_1) \subset V_1 \cup V$. Since F_2 is upper m-continuous, there exists $U_2 \in m^2_X$ containing x such that $F_2(U_2) \subset V_2 \cup V$. Set $U = U_1 \cap U_2$, then $U \in m^2_X$ containing x. If $y \in F(x_0)$ for any $x_0 \in U$, then $y \in (V_1 \cup V) \cap (V_2 \cup V) = (V_1 \cap V_2) \cup V = V$ because $V_1 \cap V_2 = \emptyset$. Hence, we have $y \in V$ and hence $F(U) \subset V$. Therefore, F is upper m-continuous.

Theorem 6.6. *Let (X, m_X) an m-space with two m-iterate structures $mIT^1(X)$ and $mIT^2(X)$ such that $U \cap V \in mIT^2(X)$ whenever $U \in mIT^1(X)$ and $V \in mIT^2(X)$ and (Y, σ) be a Hausdorff space. If the following four conditions are satisfied:*

(1) a multifunction $F_1 : (X, mIT^1(X)) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F_2 : (X, mIT^2(X)) \to (Y, \sigma)$ is upper m-continuous,
(3) $F_1(x)$ and $F_2(x)$ are compact sets of (Y, σ) for each $x \in X$,
(4) $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$,

then a multifunction $F : (X, mIT^2(X)) \to (Y, \sigma)$, defined by $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper m-continuous.

Corollary 6.3. *Let (X, m_X) an m-space with two $m\gamma$-structures $m\gamma^1(X)$ and $m\gamma^2(X)$ such that $U \cap V \in m\gamma^2(X)$ whenever $U \in m\gamma^1(X)$ and $V \in m\gamma^2(X)$ and (Y, σ) be a Hausdorff space. If the following four conditions are satisfied:*

(1) a multifunction $F_1 : (X, m\gamma^1(X)) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $F_2 : (X, m\gamma^2(X)) \to (Y, \sigma)$ is upper m-continuous,

(3) $F_1(x)$ and $F_2(x)$ are compact sets of (Y, σ) for each $x \in X$,
(4) $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$,
then a multifunction $F : (X, m\gamma^2(X)) \to (Y, \sigma)$, defined by $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper m-continuous.

Remark 6.4. Let (X, m_X) be an m-space. If $mIT(X) = \mathrm{mSO}(X)$, $\mathrm{mPO}(X)$, $m\alpha(X)$, $m\beta(X)$ or $m\delta\mathrm{PO}(X)$, then we obtain the similar result with Corollary 6.3.

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