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COMMON $(E.A)$ - PROPERTY AND ALTERING DISTANCE IN METRIC SPACES

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Abstract. In this paper a general fixed point theorem for mappings satisfying an implicit relation is proved for two pairs of mappings with a common $(E.A)$ - property, which generalize the main results from [1]. In the last part of the paper, as applications we obtain some fixed point results for mappings satisfying contractive condition of integral type, for almost contractive mappings and for (ψ, φ) - weakly contractive mappings.

1. Introduction and Preliminaries

Let (X, d) be a metric space and S, T be two self mappings of X. In [24], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

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The concept was frequently used to prove the existence theorems in common fixed point theory.

The study of common fixed points of noncompatible mappings is also interesting, work along these lines has been initiated by Pant in [29], [30], [31].

Aamri and El - Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.1 ([1])**.** Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy $(E.A)$ - property if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Remark 1.2. *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or non existent. Therefore, two noncompatible self mappings of a metric space (X, d) satisfy property $(E.A)$. Liu et al.* [28] *extend Definition 1.1 for two pairs of mappings.*

Definition 1.3 ([28])**.** Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy the common $(E.A)$ - property if there exists two sequences (x_n) and (y_n) in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

Definition 1.4 ([25])**.** Two self mappings S and T of a metric space (X, d) are said to be weakly compatible if $Su = Tu$ implies $STu = TSu$.

Remark 1.5. *The notion of weak compatibility is equivalent with the notion of pointwise R - weakly commuting introduced and studied in* [32], [33].

Remark 1.6. *It is known* [34], [35] *that the notion of weakly compatible mappings and mappings satisfying $(E.A)$ - property are independent.*

Definition 1.7. Let S and T be two self mappings of a metric space (X, d). A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and the point $w = Sx = Tx$ is said to be a point of coincidence of S and T.

The set of coincidence points of S and T is denoted by $C(S, T)$.

Lemma 1.8 ([2])**.** *Let f and g be weakly compatible self mappings on a nonempty set X. If f and g have an unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 1.9 ([10])**.** A pair of maps (S, T) is said to be occasionally weakly compatible (owc) if there exists $x \in C(S, T)$ such that $STx = TSx$.

Every pair of weakly compatible mappings is owc, but the converse is not true [10].

Remark 1.10. *It is proved in* [12] *that the notions of owc mappings and* $(E.A)$ *- property are independent.*

In [8] a general fixed point for two pairs of self mappings of a metric space is proved.

2. Implicit contractive mappings and fixed points

In [36], [37] the study of fixed points for implicit contractive mappings was introduced. Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi – metric spaces, compact metric spaces, paracompact metric spaces, reflexive spaces, probabilistic metric spaces, convex metric spaces, intuitionistic metric spaces, in two or three metric spaces, for single valued functions, hybrid pairs of mappings and multivalued mappings. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type and in fuzzy metric spaces. The method unified different types of contractive and expansive conditions.

With this method the proofs of fixed point theorems are more simple.

The study of fixed points for mappings satisfying an implicit relation and with $(E.A)$ - property is initiated in [8], [35], [38], [41].

Definition 2.1. Let $\mathfrak{F}_{EA}$ be the set of all lower semi - continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : $F(t, 0, t, 0, 0, t) > 0$, $\forall t > 0$,

(F_2) : $F(t, 0, 0, t, t, 0) > 0$, $\forall t > 0$,

(F_3) : $F(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

Examples of functions satisfying the conditions $(F_1) - (F_3)$ are in [8], [35], [38], [41].

The following theorem is proved in [8].

Theorem 2.2. *Let f, g, I and J be self mappings of a metric space (X, d) such that:*

(2.1) the pairs (f, I) and (g, J) satisfy the common $(E.A)$ - property,

(2.2) $I(X)$ and $J(X)$ are closed subsets of X,

(2.3)

$$F(d(fx, gy), d(Ix, Jy), d(Ix, fx), d(Jy, gy), d(Ix, gy), d(Jy, fx)\} \leq 0$$

for all $x, y \in X$, where $F \in \mathfrak{F}_{EA}$. Then

a) f and I have a coincidence point,

b) g and J have a coincidence point.

Moreover, if the pairs (f, I) and (g, J) are weakly compatible then f, g, I and J have an unique common fixed point.

Definition 2.3. An altering distance is a function $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(ψ_1) : ψ is increasing and continuous,

(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving an altering distance have been studied in [40], [41], [45], [46] and in other papers.

In this paper a generalization of Theorem 2.2 using altering distance is obtained. In the last part of this paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings and for (ψ, φ) - weakly contractive mappings, are obtained.

3. Main results

Theorem 3.1. *Let f, g, I and J be self mappings of a metric space (X, d) satisfying the following inequality*

$$F(\psi(d(fx,gy)),\psi(d(Ix,Jy)),\psi(d(Ix,fx)),\psi(d(Jy,gy)),\psi(d(Ix,gy)),\psi(d(Jy,fx)))\leq 0 \tag{3.1}$$

for all $x, y \in X$, where F satisfy property (F_3) and ψ is an altering distance. If there exist $u, v \in X$ such that $fu = Iu$ and $gv = Jv$, then there exists $t \in X$ such that t is the unique point of coincidence of f and I, as well is the unique point of coincidence of g and J.

Proof. First we prove that $fu = gv$. Suppose that $fu \neq gv$. Then by (3.1) we get

$$F(\psi(d(fu,gv)),\psi(d(fu,gv)),0,0,\psi(d(fu,gv)),\psi(d(fu,gv)))\leq 0\ ,$$

a contradiction of (F_3). Hence $fu = gv$ which implies $fu = Iu = gv = Jv = t$ and t is a common point of coincidence for (f, I) and (g, J). Suppose that there exists $z = fw = Iw$, $z \neq t$ other point of coincidence for f and I. Then by (3.1) we obtain

$$F(\psi(d(fw,gv)),\psi(d(fw,gv)),0,0,\psi(d(fw,gv)),\psi(d(fw,gv)))\leq 0$$

a contradiction of (F_3). Hence $z = fw = Iw = fu = Iu = t$ and t is the unique point of coincidence for f and I. Similarly, t is the unique point of coincidence for g and J. □

Theorem 3.2. *Let f, g, I and J be self mappings of a metric space (X, d) such that:*

(3.2) the pairs (f, I) and (g, J) satisfy the common $(E.A)-$ property,

(3.3) $I(X)$ and $J(X)$ are closed subsets of X,

(3.4) f, g, I and J satisfy inequality (3.1), for all $x, y \in X$, where $F \in \mathfrak{F}_{EA}$ and ψ is an altering distance. Then

a) f and I have a coincidence point,

b) g and J have a coincidence point.

Moreover, if the pairs (f, I) and (g, J) are weakly compatible, then f, g, I and J have an unique common fixed point.

Proof. Since the pairs (f, I) and (g, J) satisfy the common $(E.A)-$ property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ix_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t,$$

for some $t \in X$.

Since $I(X)$ is closed in X, there exists a point $u \in X$ such that $t = Iu$. We prove that $fu = Iu$. Suppose that $fu \neq Iu$, then by (3.1) we get

$$F(\psi(d(fu, gy_n)), \psi(d(Iu, Jy_n)), \psi(d(Iu, fu)),$$
$$\psi(d(Jy_n, gy_n)), \psi(d(Iu, gy_n)), \psi(d(Jy_n, fu))) \leq 0.$$

Letting n tend to infinity we obtain

$$F(\psi(d(fu, Iu)), 0, \psi(d(fu, Iu)), 0, 0, \psi(d(fu, Iu))) \leq 0,$$

a contradiction of (F_1). Hence $Iu = fu$ and $C(f, I) \neq \emptyset$. Since $J(X)$ is closed, there exists $v \in X$ such that $t = Jv$. We prove that $gv = Jv$. Suppose that $gv \neq Jv$. Then by (3.1) we have successively

$$F(\psi(d(fu, gv)), \psi(d(Iu, Jv)), \psi(d(Iu, fu)),$$
$$\psi(d(Jv, gv)), \psi(d(Iu, gv)), \psi(d(Jv, fu))) \leq 0,$$

$$F(\psi(d(Jv, gv)), 0, 0, \psi(d(Jv, gv)), \psi(d(Jv, gv)), 0) \leq 0,$$

a contradiction of (F_2). Hence $Jv = gv$ and $C(g, J) \neq \emptyset$. Since $t = Iu = fu = gv = Jv$, by Theorem 3.1, t is the unique point of coincidence for (f, I) and (g, J).

Moreover, if (f, I) and (g, J) are weakly compatible, by Lemma 1.8, t is the unique common fixed point of f, g, I and J. $\square$

Remark 3.3. *If $\psi(t) = t$, by Theorem 3.2 we obtain Theorem 2.2.*

4. Applications

4.1. **Fixed points for mappings satisfying conditions of integral type.** In [20], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

Theorem 4.1 ([20])**.** *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt \tag{4.1}$$

where $h(t) : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has an unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Theorem 4.1 has been generalized in several papers, e.g. it has been extended to a pair of compatible mappings in [27].

Theorem 4.2 ([27])**.** *Let f, g be compatible self mappings of a complete metric space (X, d), with g - continuous, satisfying the following conditions:*

(1) $f(X) \subset g(X)$,

(2) $\int_0^{d(fx,gy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt$,

for some $c \in (0, 1)$, whenever $x, y \in X$ and $h(t) : [0, \infty) \to [0, \infty)$ satisfies the assumptions from Theorem 4.1. Then f and g have an unique common fixed point.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [9], [39], [40], [44] and in other papers.

Lemma 4.3 ([40])**.** *Let $h(t) : [0, \infty) \to [0, \infty)$ as in Theorem 4.1. Then, $\psi(t) = \int_0^t h(x)dx$ is an altering distance.*

Theorem 4.4. *Let f, g, I and J be self mappings of a metric space (X, d) such that:*

(4.1.2) *the pairs (f, I) and (g, J) satisfies the common $(E.A)$ - property,*

(4.1.3) *$I(X)$ and $J(X)$ are closed subsets of X,*

(4.1.4)

$$F\Big(\int_0^{d(fx,gy)} h(t)dt, \int_0^{d(Ix,Jy)} h(t)dt, \int_0^{d(Ix,fx)} h(t)dt,$$
$$\int_0^{d(Jy,gy)} h(t)dt, \int_0^{d(Ix,gy)} h(t)dt, \int_0^{d(Jy,fx)} h(t)dt\Big) \leq 0$$

for all $x, y \in X$, $F \in \mathfrak{F}_{EA}$ *and* $h(t)$ *is as in Theorem 4.1.*

Then

a) f *and* I *have a coincidence point,*

b) g *and* J *have a coincidence point.*

Moreover, if (f, I) *and* (g, J) *are weakly compatible, then* f, g, I *and* J *have an unique common fixed point.*

Proof. By Lemma 4.3, $\psi(t) = \int_0^t h(x)dx$ is an altering distance. By (**??**) we obtain

$$F\left(\psi(d(fx, Jy)), \psi(d(Ix, Jy)), \psi(d(Ix, fx)),\right.$$
$$\left.\psi(d(Jy, gy)), \psi(d(Ix, gy)), \psi(d(Jy, fx))\right) \leq 0,$$

which is the inequality (3.1). Hence, the conditions of Theorem 3.2 are satisfied.

Theorem 4.4 it follows by Theorem 3.2. □

4.2. **Fixed points for almost contractive mappings.**

Definition 4.5. Let (X, d) be a metric space. A mapping $T : (X, d) \to (X, d)$ is called weak contractive [16], [17] or almost contractive [18] if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

The following theorems are proved in [19].

Theorem 4.6. *Let* (X, d) *be a metric space and* $T, S : (X, d) \to (X, d)$ *be the mappings for which there exists* $a \in (0, 1)$ *and* $L \geq 0$ *such that:*

$$d(Tx, Ty) \leq ad(Sx, Sy) + Ld(Sy, Tx),$$

for all $x, y \in X$.

If $T(X) \subset S(X)$ *and* $S(X)$ *is a complete subspace of* X*, then* S *and* T *have an unique point of coincidence. Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have an unique common fixed point.*

Theorem 4.7. *Let (X,d) be a metric space and $T,S:(X,d)\to(X,d)$ be two mappings for which there exists $a\in(0,1)$ and $L\geq 0$ such that:*

$$d(Tx,Ty)\leq$$
$$\leq ad(Sx,Sy)+L\min\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Tx,Sy)\},$$

for all $x,y\in X$.

If $T(X)\subset S(X)$ and $S(X)$ is a complete subspace of X, then T and S have an unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Some generalizations of Theorems 4.6, 4.7 are obtained in [11]. The following theorem is proved in [11].

Theorem 4.8. *Let (X,d) be a metric space and $f,T:(X,d)\to(X,d)$ such that $T(X)\subset f(X)$. Assume that there exists $\delta\in(0,1)$ and $L\geq 0$ such that:*

$$d(Tx,Ty)\leq\delta m(x,y)+$$
$$+L\min\{d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\},$$

for all $x,y\in X$, where

$$m(x,y)=\max\left\{d(fx,Ty),\right.$$
$$\left.\frac{d(fx,Tx)+d(fy,Ty)}{2},\frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}.$$

If $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have an unique point of coincidence.

Moreover, if f and T are weakly compatible, then f and T have an unique common fixed point.

The following function $F(t_1,...,t_6):\mathbb{R}_+^6\to\mathbb{R}$ satisfies property $(F_1),(F_2),(F_3)$.

Example 4.9. $F(t_1,...,t_6)=t_1-\delta\max\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\}-$
$-L\min\{t_3,t_4,t_5,t_6\}$, *where $\delta\in(0,1)$ and $L\geq 0$.*

Example 4.10. $F(t_1,...,t_6)=t_1-at_2-$
$-L\min\{t_3,t_4,t_5,t_6\}$, *where $a\in(0,1)$ and $L\geq 0$.*

Example 4.11. $F(t_1,...,t_6) = t_1 - k\max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\} -$
$- L\min\{t_3, t_4, t_5, t_6\}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.12. $F(t_1,...,t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\} -$
$- L\min\{t_3, t_4, t_5, t_6\}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.13. $F(t_1,...,t_6) = t_1 - k\max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\} -$
$- L\min\{t_3, t_4, \sqrt{t_4t_6}, \sqrt{t_5t_6}\}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.14. $F(t_1,...,t_6) = t_1 - c\max\{t_2, t_3, \sqrt{t_4t_6}, \sqrt{t_5t_6}\} -$
$- L\min\{t_3, t_4, t_5, t_6\}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.15. $F(t_1,...,t_6) = t_1 - \max\{t_2, k(t_3+t_4), k(t_5+t_6)\} -$
$- L\min\{t_3, t_4, t_5, t_6\}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.16. $F(t_1,...,t_6) = t_1 - \max\{t_2, \alpha t_3, \alpha t_4, \frac{\alpha(t_5+t_6)}{2}\} -$
$- L\min\{t_3, t_4, t_5, t_6\}$, *where* $\alpha \in (0,1)$ *and* $L \geq 0$.

Example 4.17. $F(t_1,...,t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\} -$
$- L(\sqrt{t_2t_3} + \sqrt{t_2t_4} + \sqrt{t_3t_5})$, *where* $k \in (0,1)$ *and* $L \geq 0$.

Example 4.18. $F(t_1,...,t_6) = t_1 - k\max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\} -$
$- L\frac{\sqrt{t_4t_6} + \sqrt{t_3t_4}}{1 + \sqrt{t_2t_3} + \sqrt{t_3t_5}}$, *where* $k \in (0,1)$ *and* $L \geq 0$.

By Theorem 3.2 and Example 4.9 we obtain

Theorem 4.19. *Let* f, g, I *and* J *be self mappings of a metric space* (X, d) *such that*

(4.2.1) *the pairs* (f, I) *and* (g, J) *satisfy the common* $(E.A)$ *- property,*

(4.2.2) $I(X)$ *and* $J(X)$ *are closed subsets of* X*,*

(4.2.3)

$$\psi(d(fx,gy)) \leq \delta \quad \max\{\psi(d(Ix,Jy)), \frac{\psi(d(Ix,fx))+\psi(d(Jy,gy))}{2}, \frac{\psi(d(Ix,gy))+\psi(d(Jy,fx))}{2}\}- -L\min\{\psi(d(Ix,fx)),\psi(d(Jy,gy)) -\psi(d(Ix,gy)),\psi(d(Jy,fx))\},$$

where $\delta \in (0,1)$, $L \geq 0$ and $\psi(t)$ is an altering distance.

Then,

a) f and I have a point of coincidence,

b) g and J have a point of coincidence.

Moreover, if the pairs (f,I) and (g,J) are weakly compatible then f,g,I and J have an unique common fixed point.

If $\psi(t)=t$, then by Theorem 4.19 we obtain

Theorem 4.20. *Let f,g,I and J be self mappings of a metric space (X,d) satisfying the conditions (4.2.1) and (4.2.2) of Theorem 4.19 and*

$$d(fx,gy) \leq k\max\{d(Ix,Ty), \frac{d(Ix,fx)+d(Jy,gy)}{2}, \frac{d(Ix,gy)+d(Jy,fx)}{2}\}- -L\min\{d(Ix,fx),d(Jy,gy),d(Ix,gy),d(Jy,fx)\},$$

for all $x,y \in X$, where $k \in [0,1)$ and $L \geq 0$. Then,

a) I and f have a point of coincidence,

b) J and g have a point of coincidence.

Moreover, if the pairs (f,I) and (g,J) are weakly compatible then f,g,I and J have an unique common fixed point.

Similarly, by Theorem 3.2 and Examples 4.10 - 4.18 we obtain new particular results.

By Theorem 4.4 and Example 4.9 we obtain

Theorem 4.21. *Let f,g,I and J be self mappings of a metric space (X,d) satisfying the conditions (4.2.1) and (4.2.2) of Theorem 4.19* *and*

$$\int_0^{d(fx,gy)} h(t)dt \leq k \max\{\int_0^{d(Ix,Ty)} h(t)dt, \frac{\int_0^{d(Ix,fx)} h(t)dt + \int_0^{d(Jy,gy)} h(t)dt}{2}, \frac{\int_0^{d(Ix,gy)} h(t)dt + \int_0^{d(Jy,fx)} h(t)dt}{2}\} - -L \min\{\int_0^{d(Ix,fx)} h(t)dt, \int_0^{d(Jy,gy)} h(t)dt, \int_0^{d(Ix,gy)} h(t)dt, \int_0^{d(Jy,fx)} h(t)dt\},$$

where $k \in [0,1)$ and $L \geq 0$, for all $x, y \in X$ and h is as in Theorem 4.1. Then,

a) I and f have a point of coincidence,

b) J and g have a point of coincidence.

Moreover, if (f, I) and (g, J) are weakly compatible then f, g, I and J have an unique common fixed point.

Similarly, by Theorem 4.4 and Examples 4.10 - 4.18 we obtain new particular results.

4.3. **Fixed point of** $(\psi, \varphi)-$ **weakly contractive mappings.** In 1997, Alber and Guerre - Delabrierre [6] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self mapping on a Hilbert space. Rhoades [43] extended this concept to metric spaces. In [14] the authors studied the existence of fixed points for a pair of $(\psi, \varphi)-$ weakly contractive mappings.

New results are obtained in [4], [21], [22], [15], [42] and in other papers. Quite recently in [13] and [5] the study of common fixed points of $(\psi, \varphi)-$ weakly contraction, with common $(E.A)$ - property is initiated.

Definition 4.22. Let Ψ be the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

a) ψ is continuous,

b) $\psi(t) = 0$ and $\psi(t) > 0$, $\forall t > 0$.

Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:

a) ϕ is lower semi - continuous,

b) $\phi(t) = 0$ and $\phi(t) > 0, \forall t > 0$.

In [5] the following theorem is proved.

Theorem 4.23 (Theorem 2.1 [5]). *Let (X, d) be a metric space f, g, I and J be self mappings of X, $\psi \in \Psi$ and $\phi \in \Phi$. We suppose that:*

(4.3.1) the pairs (f, I) and (g, J) satisfy the common $(E.A)$ - property,

(4.3.2) $I(X)$ and $J(X)$ are closed subsets of X,

(4.3.3)

$$\psi(d(fx, gy)) \leq \psi(M(x, y) - \phi(M(x, y))),$$

where $M(x, y) = \max\{d(Ix, Jy), d(fx, Ix), d(gy, Jy), \frac{1}{2}[d(Ix, gy) + d(Jy, fx)]\}$.

Then,

a) the pair (f, I) has a point of coincidence,

b) the pair (g, J) has a point of coincidence.

Moreover, if the pairs (f, I) and (g, J) are weakly compatible then f, g, I and J have an unique common fixed point.

Proof. Let **w1** be $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, \dfrac{1}{2}[t_5 + t_6]\}) + \phi(\max\{t_2, t_3, t_4, \dfrac{1}{2}[t_5 + t_6]\})$. Then

(F_1) : $F(t, 0, t, 0, 0, t) = \phi(t) > 0, \forall t > 0$,

(F_2) : $F(t, 0, 0, t, t, 0) = \phi(t) > 0, \forall t > 0$,

(F_3) : $F(t, t, 0, 0, t, t) > 0, \forall t > 0$.

Hence, $F \in \mathfrak{F}_{EA}$ and the proof it follows from Theorem 2.2. □

In [13] the following theorem is proved.

Theorem 4.24 (Theorem 3.2 [13]). *Let f, g, I and J be self mappings of a metric space (X, d) such that*

(a_1) (f, I) and (g, J) satisfy the common $(E.A)$ - property,

(a_2) (f, I) and (g, J) are occasionally weakly compatible,

(a_3)

$$\psi(d(fx, gy)) \leq \psi(M_1(x, y)) - \phi(M_1(x, y)), \forall x, y \in X,$$

where

$M_1(x,y) = \max\{d(Ix,Jy), d(Ix,fx), d(Jy,gy), d(Ix,gy), d(Jy,fx)\}$.

If $I(X)$ and $J(X)$ are closed in X, then f, g, I and J have an unique fixed point.

Proof. Let **w2** be
$F(t_1,...,t_6) = t_1 - \psi(\max\{t_2,t_3,t_4,t_5,t_6\}) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$.
As in the proof of Theorem 4.23, $F \in \mathfrak{F}_{EA}$ and the proof it follows by Theorem 2.2. □

The following functions $F \in \mathfrak{F}_{EA}$.

Example w3. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

Example w4. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{t_2,t_3,t_4,t_5,t_6\}) + \phi(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\})$.

Example w5. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}) + \phi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_5t_6}\})$.

Example w6. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_5t_6}\}) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

Example w7. $F(t_1,...,t_6) = \psi(t_1) - \psi\left(\frac{\sqrt{t_3t_6}+\sqrt{t_4t_5}+\sqrt{t_2t_6}}{1+\sqrt{t_3t_4}+\sqrt{t_2t_3}+\sqrt{t_4t_6}}\right) + \phi(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\})$.

Example w8. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{\sqrt{t_2t_5} + b\sqrt{t_2t_6} + \sqrt{t_3t_6} + \sqrt{t_4t_5}\}) + \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$, where $a,b \geq 0$ and $a+b=1$.

Example w9. $F(t_1,...,t_6) = \psi(t_1) - \psi(a\max\{t_2,t_3,t_4\} + b\max\{t_5,t_6\}) + \phi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_5t_6}\})$, where $a,b \geq 0$ and $a+b=1$.

Example w10. $F(t_1,...,t_6) = \psi(t_1) - \psi(\max\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}) + \phi(a\max\{t_2,t_3,t_4\} + b\max\{t_5,t_6\})$, where $a,b \geq 0$ and $a+b=1$.

Remark 4.25. *1. By Example w3 - w10 and Theorem 2.2 we obtain new particular results.*

2. Similar results we obtain combining the results from this paragraph with the results from paragraph 4.2.

By the following examples we obtain new particular results:

Example w.A.1. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + \phi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + L\min\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $L \geq 0$.

Example w.A.2. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\}) + L\min\{t_2, t_3, t_4, t_5, t_6\}$, where $L \geq 0$.

Example w.A.3. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\}) + L\min\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $L \geq 0$.

Example w.A.4. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}) + \phi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + L\min\{t_2, t_3, t_4, t_5, t_6\}$, where $L \geq 0$.

Example w.A.5. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + \phi(\max\{\sqrt{t_3 t_6}, \sqrt{t_2 t_5}, \sqrt{t_5 t_6}\}) + L\min\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $L \geq 0$.

Example w.A.6. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + \phi(\max\{\sqrt{t_3 t_6}, \sqrt{t_2 t_5}, \sqrt{t_5 t_6}\}) + L\min\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $L \geq 0$.

Example w.A.7. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\frac{\sqrt{t_3 t_6} + \sqrt{t_4 t_5} + \sqrt{t_2 t_6}}{1 + \sqrt{t_3 t_4} + \sqrt{t_2 t_3} + \sqrt{t_4 t_6}}\right) + \phi(\max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}) + L\min\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $L \geq 0$.

Example w.A.8. $F(t_1, ..., t_6) = \psi(t_1) - \psi(a\sqrt{t_2 t_5} + b\sqrt{t_2 t_6} + \sqrt{t_3 t_6} + \sqrt{t_4 t_5}) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\}) + L\min\{t_3, t_4, t_5, t_6\}$, where $a, b \geq 0$, $a + b = 1$ and $L \geq 0$.

Example w.A.9. $F(t_1, ..., t_6) = \psi(t_1) - \psi(a\max\{t_2, t_3, t_4\} + b\max\{t_5, t_6\}\} + \phi(\max\{\sqrt{t_3 t_6}, \sqrt{t_2 t_5}, \sqrt{t_5 t_6}\}) + L\min\{t_3, t_4, t_5, t_6\}$, where $a, b \geq 0$, $a + b = 1$ and $L \geq 0$.

Example w.A.10. $F(t_1,...,t_6) = \psi(t_1) - \psi(a\max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\} + b\max\{t_5, t_6\}\} + L\min\{t_3, t_4, t_5, t_6\}$, where $a, b \geq 0$, $a + b = 1$ and $L \geq 0$.

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