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z-PERFECTLY CONTINUOUS FUNCTIONS AND THEIR FUNCTION SPACES

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Abstract. A new class of functions called z-perfectly continuous functions is introduced, which properly includes the class of pseudo perfectly continuous functions [24] but turns out to be independent of continuity. Besides the study of the basic properties of z-perfectly continuous functions and the interplay with topological properties, sufficient conditions are outlined for their function spaces to be closed, respectively compact, in the topology of pointwise convergence.

1. Introduction

Taking account of [24], we can mention some remarkable classes of functions in descending order, each class properly containing the next class: z-perfectly continuous functions, pseudo perfectly continuous functions [24], quasi-perfectly continuous functions [27], δ-perfectly continuous functions [22], perfectly continuous functions due to Noiri [32], strongly continuous functions of Levine [28].

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Naimpally [31] showed the set $S(X,Y)$ of all strongly continuous functions from a locally connected space X into a Hausdorff space Y is closed in Y^X in the topology of pointwise convergence. This result was extended for larger classes of spaces and functions ([21], [22], [24], [27], [36]). One purpose of this paper is to further strengthen these results. Moreover, conditions are given for certain classes of functions to be compact Hausdorff subspaces of Y^X in the topology of pointwise convergence.

The organization of the paper is as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3, we introduce the notion of a 'z-perfectly continuous function' and reflect upon its place in the hierarchy of variants of continuity which already exist in the literature. Moreover, examples are included to ascertain the distinctiveness of the notions so defined from the existing ones in the mathematical literature. Section 4 is devoted to the study of basic properties of z-perfectly continuous functions. The notion of z-partition topology is introduced and sufficient conditions are given for its direct and inverse preservation under mappings. A sum theorem is proved showing that the z-perfect continuity of a certain family of restrictions of a function implies the z-perfect continuity of that function on the whole space. In Section 5 we discuss the interplay between topological properties of X, Y and the z-perfectly continuous functions $f : X \to Y$. Section 6 is devoted to function spaces wherein it is shown that if X is sum connected [16] (e.g. connected or locally connected) and Y is a functionally Hausdorff space, then the function space $P_z(X,Y)$ of all z-perfectly continuous functions from X to Y is closed in Y^X in the topology of pointwise convergence. Moreover, if Y is a compact Hausdorff space, then $P_z(X,Y)$ and several other function spaces are shown to be compact Hausdorff in the topology of pointwise convergence.

2. Preliminaries and basic definitions

In what follows, X and Y are topological spaces. A subset A of X is called ***regular G_δ-set*** [30] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^0$, where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a ***regular F_σ-set***. A point $x \in X$ is called a ***θ-adherent point*** [41] of $A \subset X$ if every closed neigbourhood of x intersects A. Let $\operatorname{cl}_\theta A$ denote the set of all θ-adherent points of A. The set A is called ***θ-closed*** if $A = \operatorname{cl}_\theta A$.

The complement of a θ-closed set is referred to as a ***θ-open set***. A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e., $A = \bar{A}^{\circ}$. The complement of a regular open set is referred to as a ***regular closed*** set. Any union of regular open sets is called ***δ-open*** [41]. The complement of a δ-open set is referred to as a ***δ-closed*** set. A subset A of a space X is said to be ***cl-open*** [35] if for each $x \in A$ there exists a clopen set H such that $x \in H \subset A$; or equivalently A is expressible as a union of clopen sets. The complement of a cl-open set is referred to as a ***cl-closed*** set. Any union of cozero sets is called ***z-open*** [19]. The complement of a z-open set is referred to as a ***z-closed*** set.

2.1. **Definitions.** A function $f : X \to Y$ is said to be

(a) ***perfectly continuous*** ([25], [32]) if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.
(b) ***slightly continuous***[1] [13] if $f^{-1}(V)$ is open in X for every clopen set $V \subset Y$.
(c) ***δ-perfectly continuous*** [22] if for each δ-open set V in Y, $f^{-1}(V)$ is a clopen set in X.
(d) ***almost perfectly continuous*** [36] ($\equiv$ ***regular set connected*** [8]) if $f^{-1}(V)$ is clopen for every regular open set V in Y.
(e) ***quasi perfectly continuous*** [27] if $f^{-1}(V)$ is clopen in X for every θ-open set V in Y.
(f) ***pseudo perfectly continuous*** [24] if $f^{-1}(V)$ is clopen in X for every regular F_σ-set V in Y.

2.2. **Definitions.** A function $f : X \to Y$ is said to be

(a) ***cl-supercontinuous*** [35] ($\equiv$ ***clopen continuous*** [33]) if for each $x \in X$ and each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.
(b) ***z-supercontinuous*** [19] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.
(c) ***z-continuous*** [34] if for each $x \in X$ and for each cozero set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.
(d) ***z-cl-supercontinuos*** [37] if for each $x \in X$ and each cozero set V containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset V$.

[1]Slightly continuous functions have been referred to as cl-continuous in ([20], [26]).

2.3. **Definition.** A function $f : X \to Y$ is said to be ***strongly continuous*** [28] if $f(\bar{A}) \subset f(A)$ for each subset A of X.

2.4. **Definition.** A filter $\mathcal{F}$ is said to ***z-converge*** [19] (***cl-converge*** [35]) to a point x, written as $\mathcal{F} \xrightarrow{z} x(\mathcal{F} \xrightarrow{\text{cl}} x)$ if every cozero (clopen) set containing x contains a member of $\mathcal{F}$.

2.5. **Definition.** A net (x_λ) in X is said to ***z-converge*** [19] (***cl-converge*** [35]) to a point x, written as $x_\lambda \xrightarrow{z} x(x_\lambda \xrightarrow{\text{cl}} x)$ if it is eventually in every cozero (clopen) set containing x.

3. z-perfectly continuous functions

3.1. **Definition.** A subset A of X is called a cozero set if there exists a continuous real valued function f on X such that $A = \{x \in X : f(x) \neq 0\}$.

3.2. **Definition.** A function $f : X \to Y$ from a topological space X into a topological space Y is called ***z-perfectly continuous*** if $f^{-1}(V)$ is clopen in X for every cozero set V in Y.

The following diagram well exhibits the interrelations that exist among pseudo perfect continuity and other variants of continuity that already exist in the literature and are related to the theme of the present paper.

$$\begin{array}{ccccc}
\text{strongly continuous} & & & & \\
\downarrow & & & & \\
\text{perfectly continuous} & \to & \text{cl-supercontinuous} & \to & \text{z-supercontinuous} \\
\downarrow & & & & \\
\delta\text{-perfectly continuous} & \to & \text{almost perfectly continuous} & & \\
\downarrow & & & & \\
\text{quasi perfectly continuous} & \to & \text{pseudo perfectly continuous} & \to & \text{z-perfectly continuous} \\
 & & & & \downarrow \\
 & & & & \text{z-continuous}
\end{array}$$

Examples.

3.1. Let X denote the real line with usual topology and let Y be endowed with the topology $\tau = \{\varnothing, \{1\}, R\}$. Then the identity function $f : X \to Y$ is z-perfectly continuous but not continuous.

3.2. Let X be the real line endowed with usual topology. Then the identity function defined on X is continuous as well as z-supercontinuous but not z-perfectly continuous.

3.3. Let Y be the space of Hewitt's example [12] of an infinite regular Hausdorff space on which every continuous real valued function is constant. Let X denote the same set with indiscrete topology and f denote the identity mapping of X onto Y. Then f is z-perfectly continuous. However, f is not pseudo perfectly continuous since Y contains regular G_δ-set which is not a zero set in Y (see Mack [30]).

4. Basic properties of z-perfectly continuous functions

4.1. **Theorem.** *For a function $f : (X, \tau) \to (Y, \vartheta)$ the following statements are equivalent.*

(a) *The function f is z-perfectly continuous.*
(b) *$f^{-1}(V)$ is cl-open in X for every z-open set $V \subset Y$.*
(c) *$f^{-1}(B)$ is cl-closed in X for every z-closed set $B \subset Y$.*
(d) *$f(\mathcal{F}) \xrightarrow{z} f(x)$ for every filter $\mathcal{F} \xrightarrow{\mathrm{cl}} x$.*
(e) *$f(x_\lambda) \xrightarrow{z} f(x)$ for every net $x_\lambda \xrightarrow{\mathrm{cl}} x$.*

Proof. (a)⇒(b): Let V be a z-open set in Y, then $V = \bigcup_{\alpha\in\Lambda} U_\alpha$, where each U_α is a cozero set. Since f is z-perfectly continuous, $f^{-1}(U_\alpha)$ is clopen for each α. Hence $f^{-1}(V) = \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha)$ is cl-open in X.

(b)⇒(c): Let B be a z-closed set in Y, then $Y\backslash B$ is z-open in Y. By (b) $X\backslash f^{-1}(B) = f^{-1}(Y\backslash B)$ is cl-open in X. Hence $f^{-1}(B)$ is cl-closed in X.

(c)⇒(b): obvious.

(b)⇒(d): Let V be a z-open set in Y containing $f(x)$. Then $V = \bigcup_{\alpha\in\Lambda} V_\alpha$ where each V_α is a cozero set and suppose that $f(x) \in V_\beta$ for some $\beta \in \Lambda$. Then by (b) $f^{-1}(V_\beta)$ is a cl-open set in X containing x. Let U be a clopen set such that $x \in U \subset f^{-1}(V_\beta)$. Since $\mathcal{F} \xrightarrow{\mathrm{cl}} x$, there exists a $G \in \mathcal{F}$ such that $G \in U \subset f^{-1}(V_\beta)$. Hence $f(G) \subset V_\beta$ and so $\mathcal{F} \xrightarrow{z} f(x)$.

(d)⇒(e): Let $x_\lambda \xrightarrow{\mathrm{cl}} x$ and $\mathcal{F}_{x_\lambda}$ be the filter generated by (x_λ). Then each clopen set containing x contains a member of $\mathcal{F}_{x_\lambda}$ and so $\mathcal{F}_{x_\lambda} \xrightarrow{\mathrm{cl}} x$. By (d) $f(\mathcal{F}_{x_\lambda}) \xrightarrow{z} f(x)$. So every cozero set V containing $f(x)$ contains a member of $f(\mathcal{F}_{x_\lambda})$ and hence the net $f(x_\lambda)$ is eventually in V. That is $f(x_\lambda) \xrightarrow{z} f(x)$.

(e)⇒(a): Let V be a cozero set in Y. To show $f^{-1}(V)$ is clopen, assume the contrary and let $x \in f^{-1}(V)$. Thus there is a net (x_α) in X which cl-converges to x and misses $f^{-1}(V)$ frequently. Then the net $(f(x_\alpha))$ misses V frequently, which is a contradiction. ∎

4.2. **Proposition.** *If $f : X \to Y$ is a z-perfectly continuous function and $g : Y \to Z$ is z-continuous, then $g \circ f$ is a z-perfectly continuous function. In particular composition of two z-perfectly continuous functions is z-perfectly continuous.*

4.3. **Corollary.** *If $f : X \to Y$ is a z-perfectly continuous function and $g : Y \to Z$ is a continuous function, then $g \circ f$ is a z-perfectly continuous function.*

4.4. **Proposition.** *Let $f : X \to Y$ be a slightly continuous function and let $g : Y \to Z$ be a z-perfectly continuous function. Then $g \circ f$ is z-perfectly continuous.*

4.5. **Definitions.** A space X is said to be endowed with a

(i) ***z-partition topology*** if every cozero set in X is closed; or equivalently every zero set in X is open.
(ii) ***pseudo partition topology*** [24] if every regular F_σ-set in X is closed; or equivalently every regular G_δ-set in X is open.
(iii) ***partition topology*** [39] if every open set in X is closed.
(iv) δ***-partition topology*** [22] if every δ-open set in X is closed
(v) ***almost partition topology*** [36] ($\equiv$***extremally disconnected topology***) if every regular open set in X is closed.
(vi) ***quasi partition topology*** [27] if every θ-open set in X is closed.

The following implications are immediate from definitions.

$$\begin{array}{ccc}
\text{partition topology} & & \\
\Downarrow & & \\
\delta\text{-partition topology} & \Rightarrow & \text{almost partition topology} \\
 & & (\equiv \text{extremally disconnected topology}) \\
\Downarrow & & \\
\text{quasi partition topology} & \Rightarrow & \text{pseudo partition topology} \\
 & & \Downarrow \\
 & & \text{z-partition topology}
\end{array}$$

However, none of the above implications is reversible as shown in ([22], [24], [36]) and the following example.

4.6. **Example.** Consider the Hewitt's example [12] of an infinite, regular, Hausdorff space X on which every continuous real valued function is constant. Then X has z-partition topology but it is not endowed with pseudo partition topology, since X contains a regular G_δ-set which is not a zero set.

4.7. **Theorem.** *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. If g is z-perfectly continuous, then so is f and the space X is endowed with a z-partition topology. Further, if X has a z-partition topology and f is z-perfectly continuous, then g is z-cl-supercontinuous.*

Proof. Suppose that the graph function $g : X \to X \times Y$ is z-perfectly continuous. Consider the projection map $p_y : X \times Y \to Y$. Since it is continuous, it is z-continuous. Hence in view of Proposition 4.2 the function $f = p_y \circ g$ is z-perfectly continuous. To prove that the space X possesses a z-partition topology, let U be a cozero set in X. Then $U \times Y$ is a cozero set in $X \times Y$. Since g is z-perfectly continuous, $g^{-1}(U \times Y) = U$ is clopen in X and so the topology of X is a z-partition topology.

Finally, suppose that X has z-partition topology and f is a z-perfectly continuous function. Let $U \times V$ be a basic cozero set in $X \times Y$. Then $g^{-1}(U \times V) = U \cap f^{-1}(V)$ is a clopen set in X and so g is z-cl-supercontinuous. ∎

4.8. **Theorem.** *Let $f : X \to Y$ be a z-perfectly continuous surjection which maps clopen sets to closed (open) sets. Then Y is endowed with a z-partition topology. Moreover, if f is a bijection which maps cozero (zero) sets to cozero (zero) sets, then X is also equipped with a z-partition topology.*

Proof. Suppose f maps clopen sets to closed (open) sets. Let V be a cozero (zero) sets in Y. In view of z-perfect continuity of f, $f^{-1}(V)$ is a clopen set in X. Again, since f is a surjection which maps clopen sets to closed (open) sets, the set $f(f^{-1}(V)) = V$ is closed (open) in Y and hence clopen in Y. Thus Y is endowed with a z-partition topology.

To prove the last part of the theorem assume that f is a bijection which maps cozero (zero) sets to cozero (zero) sets and let U be a cozero (zero) set in X. Then $f(U)$ is a cozero (zero) set in Y. Since f is a z-perfectly continuous bijection, $f^{-1}(f(U)) = U$ is a clopen set in X and so X is endowed with a z-partition topology. ∎

4.9. **Proposition.** *If $f : X \to Y$ is a surjection which maps clopen sets to open sets and $g : Y \to Z$ is a function such that $g \circ f$ is z-perfectly continuous, then g is a z-continuous function. Moreover, if f maps clopen sets to clopen sets, then g is a z-perfectly continuous function.*

Proof. Let V be a cozero set in Z. Since $g \circ f$ is z-perfectly continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is a clopen set in X. Again, since f is a surjection which maps clopen sets to open sets, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open in Y and so g is a z-continuous function. The last assertion is immediate, since in this case $g^{-1}(V)$ is a clopen set in Y.

4.10. **Remark.** *A space X is endowed with a z-partition topology if and only if every z-continuous function $f : X \to Y$ is z-perfectly continuous. Necessity is obvious in view of definitions. To prove sufficiency, we prove its contrapositive. Let V be a cozero set in X which is not clopen. Then the identity mapping defined on X is z-continuous but not z-perfectly continuous.*

4.11. **Proposition.** *If $f : X \to Y$ is a z-perfectly continuous function and $g : Y \to Z$ is a z-supercontinuous function, then their composition is cl-supercontinuous.*

Proof. Let V be an open set in Z. In view of z-supercontinuity of g, $g^{-1}(V)$ is a z-open set in Y and so $g^{-1}(V) = \bigcup\limits_{\alpha} V_\alpha$, where each V_α is a cozero set. Since f is z-perfectly continuous, each $f^{-1}(V_\alpha)$ is a clopen set. Hence $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}\left(\bigcup\limits_{\alpha} V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha)$ is cl-open. So $g \circ f$ is cl-supercontinuous. ∎

4.12. **Proposition.** *If $f : X \to Y$ is a z-perfectly continuous function and $g : Y \to Z$ is almost z-supercontinuous, then their composition $g \circ f$ is almost cl-supercontinuous.*

4.13. **Proposition.** *If $f : X \to Y$ is a z-perfectly continuous function and $g : Y \to Z$ is quasi z-supercontinuous, then their composition $g \circ f$ is quasi cl-supercontinuous.*

4.14. **Theorem.** *Let $f : X \to Y$ be a function and let $\boldsymbol{Q} = \{X_\alpha : \alpha \in \Lambda\}$ be a locally finite clopen cover of X. For each $\alpha \in \Lambda$, let $f_\alpha = f|_{X_\alpha} : X_\alpha \to Y$ denote the restriction map. Then f is z-perfectly continuous if and only if each f_α is z-perfectly continuous.*

Proof. Necessity is immediate in view of the fact that z-perfect continuity is preserved under the restriction of domain. To prove sufficiency, let V be a cozero set in Y. Then $f^{-1}(V) = \bigcup\limits_{\alpha\in\Lambda} (f|X_\alpha)^{-1}(V) = \bigcup\limits_{\alpha\in\Lambda} (f^{-1}(V) \bigcap X_\alpha)$. Since each $f^{-1}(V) \bigcap X_\alpha$ is clopen in X_α and hence in X. Thus $f^{-1}(V)$ is open being the union of clopen sets. Moreover, since the collection $\boldsymbol{Q}$ is locally finite, the collection $\{f^{-1}(V)\bigcap X_\alpha : \alpha \in \Lambda\}$ is a locally finite collection of clopen sets. Since the union of a locally finite collection of closed sets is closed, $f^{-1}(V)$ is also closed and hence clopen. ∎

4.15. **Definition** ([3], [6]). A subset S of a space X is said to be z-***embedded*** in X if every zero set in S is the intersection of a zero set in X with S; or equivalently every cozero set in S is the intersection of a cozero set in X with S.

4.16. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous function. If $f(X)$ is z-embedded in Y, then $f : X \to f(X)$ is z perfectly continuous.*

Proof. Let V_1 be a cozero set in $f(X)$. Since $f(X)$ is z-embedded in Y, there exists a cozero set V in Y such that $V_1 = V \bigcap f(X)$. In view of z-perfect continuity of f, $f^{-1}(V)$ is clopen in X. Now $f^{-1}(V_1) = f^{-1}(V \cap f(X)) = f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(V)$ and hence the result. ∎

4.17. **Definition** ([2]). *A topological space X is called an **Alexandroff space** if any intersection of open sets in X is open in X, or equivalently any union of closed sets in X is closed in X.*

Alexandroff spaces have been referred to as ***saturated spaces*** by Lorrain in [29].

4.18. **Theorem.** *For each $\alpha \in \Lambda$, let $f_\alpha : X \to X_\alpha$ be a function and let $f : X \to \prod\limits_{\alpha\in\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. If f is z-perfectly continuous, then each f_α is z-perfectly continuous. Further, if X is an Alexandroff space and each f_α is z-perfectly continuous, then f is z-perfectly continuous.*

Proof. Let f be z-perfectly continuous. Now for each α, $f_\alpha = \Pi_\alpha \circ f$, where Π_α denotes the projection map. Since each projection map Π_α is continuous and hence z-continuous, in view of Proposition 4.2 it follows that each f_α is z-perfectly continuous.

Conversely, suppose that X is an Alexandroff space and each f_α is a z-perfectly continuous function. Since X is Alexandroff, to show that the function f is z-perfectly continuous, it is sufficient to show that $f^{-1}(S)$ is clopen for every subbasic cozero set S in the product space $\prod\limits_{\alpha\in\Lambda} X_\alpha$. Let $U_\beta \times \prod\limits_{\alpha\in\Lambda} X_\alpha$ be a subbasic cozero set in $\prod\limits_{\alpha\in\Lambda} X_\alpha$, where U_β is a cozero set in X_β. Then $f^{-1}(U_\beta \times \prod_{\alpha \neq \beta} X_\alpha) = f^{-1}(\Pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$ is clopen in X and so f is z-perfectly continuous. ∎

5. Interplay between topological properties and z-perfectly continuous functions

5.1. **Definition.** A topological space X is called ***functionally Hausdorff*** if any two distinct points of X can be separated by disjoint cozero sets, equivalently, if X every pair of distinct points, x and y in X, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

5.2. **Definition** ([24])**.** A space X is called $D_\delta T_0$***-space*** if for each pair of distinct points x, y in X, there is a regular F_σ-set U containing one of the points x and y but not the other.

The following theorem is related to a class of spaces, important in the theories studying the p-adic topologies and the Stone duality for Boolean algebras, namely spaces having large inductive dimension zero or ultranormal spaces [38]. These are precisely the spaces in which each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets.

5.3. **Theorem.** *Let $f : X \to Y$ be a closed, z-perfectly continuous injection into a normal space Y. Then X is an ultranormal space.*

Proof. Let A and B be any two disjoint closed sets in X. Since the function f is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of Y. Again, since Y is normal, by Urysohn's Lemma there exists a continuous function $\varphi : Y \to [0,1]$ such that $\varphi(f(A)) = 0$ and $\varphi(f(B)) = 1$. Then $V = \varphi^{-1}([0,1/2))$ and $W = \varphi^{-1}((1/2,1])$ are disjoint cozero sets in Y containing $f(A)$ and $f(B)$, respectively. Since f is z-perfectly continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint clopen sets containing A and B, respectively and so X is an ultranormal space. ∎

5.4. **Definition.** A space X is said to be ***quasicompact*** [10] (***mildly compact*** [38]) if every cover of X by cozero (clopen) sets has a finite subcover.

5.5. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous function from a mildly compact space X onto a space Y. Then Y is quasicompact.*

Proof. Let $\mathbf{\Omega} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of Y by cozero sets. Since f is z-perfectly continuous, the collection $\boldsymbol{\beta} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a clopen cover of X. Since X is mildly compact, let $\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})\}$ be a finite subcollection of $\boldsymbol{\beta}$ which covers X. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite subcollection of $\mathbf{\Omega}$ which covers Y. Hence Y is quasicompact. ∎

5.6. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous function from a space X onto a space Y. If (i) f is an open bijection; or (ii) f is a closed surjection, then any pair of disjoint cozero sets in Y are clopen in Y.*

Proof. Let A and B be disjoint cozero-subsets of Y. Since f is z-perfectly continuous $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint clopen subsets of X.

(i) In case f is an open bijection, $f(f^{-1}(A)) = A$ and $f(f^{-1}(B)) = B$ are disjoint open sets and hence clopen sets in Y.

(ii) In case f is a closed surjection, the sets $A = Y - f(X - f^{-1}(A))$ and $B = Y - f(X - f^{-1}(B))$ are disjoint clopen sets in Y. ∎

5.7. **Proposition.** *Let $f, g : X \to Y$ be z-perfectly continuous functions from a space X into a functionally Hausdorff space Y. Then the set $A = \{x : f(x) = g(x)\}$ is cl-closed in X.*

Proof. Let $x \in X \setminus A$. Then $f(x) \neq g(x)$, and so by hypothesis on Y, there are disjoint cozero sets U and V containing $f(x)$ and $g(x)$, respectively. Since f and g are z-perfectly continuous, the sets $f^{-1}(U)$ and $g^{-1}(V)$ are clopen and containing the point x. Let $G = f^{-1}(U) \cap g^{-1}(V)$. Then G is a clopen set containing x and $G \cap A = \varnothing$. Thus A is cl-closed in X. ∎

5.8. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous function from a space X into a functionally Hausdorff space Y. Then the set $A = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is cl-closed in $X \times X$.*

Proof. Let $(x_1, x_2) \in (X \times X) \setminus A$. Then $f(x_1) \neq f(x_2)$. Since Y is functionally Hausdorff, there exist disjoint cozero sets U and V containing $f(x_1)$ and $f(x_2)$, respectively. Since f is z-perfectly continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint clopen sets in X containing x_1 and x_2, respectively. Let $G = f^{-1}(U) \times f^{-1}(V)$. Then G is a clopen subset of $X \times X$ containing (x_1, x_2) and $G \cap A = \varnothing$. Thus A is cl-closed in $X \times X$. ∎

5.9. **Definition.** A space X is said to be

(i) ***z-hyperconnected*** if there exists no proper zero set in X or equivalently there exists no proper cozero set in X ($\equiv X$ is the only cozero set in X)
(ii) ***pseudo hyperconnected*** [24] if there exists no proper regular G_δ-set in X or equivalently there exists no proper regular F_σ-set in X ($\equiv X$ is the only regular F_σ-set in X)
(iii) ***hyperconnected*** ([1], [39]) if every nonempty open subset of X is dense in X ($\equiv X$ is the only regular open set in X).
(iv) ***quasi hyperconnected*** [18] if there exists no proper θ-open set in X or equivalently there exists no proper θ-closed set in X ($\equiv X$ is the only θ-open set in X).

The following implications are immediate from definitions.

$$\begin{array}{ccccc} \text{hyperconnected} & \Rightarrow & \text{quasi hyperconnected} & \Rightarrow & \text{pseudo hyperconnected} \\ & & & & \Downarrow \\ & & & & \text{z-hyperconnected} \end{array}$$

5.10. **Example.** Hewitt's example [12] of an infinite regular Hausdorff space on which every continuous real valued function is constant is z-hyperconnected but not pseudo hyperconnected.

5.11. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous surjection from a connected space X onto Y. Then Y is z-hyperconnected.*

Proof. Suppose Y is not z-hyperconnected and let V be a non empty cozero set in Y. Since f is z-perfectly continuous, $f^{-1}(V)$ is a non empty proper clopen subset of X contradicting the fact that X is connected. ∎

5.12. **Remark.** There exists no z-perfectly continuous surjection from a connected space onto a non z-hyperconnected space.

5.13. **Definition.** The graph $G(f)$ of a function $f : X \to Y$ is said to be

(i) ***clopen z-closed*** if for each $(x, y) \notin G(f)$ there exists a clopen set U of x and a cozero set V containing y such that $(U \times V) \cap G(f) = \varnothing$.
(ii) ***clopen D_δ-closed*** [24] if for each $(x, y) \notin G(f)$ there exists a clopen set U of x and a regular F_σ-set V containing y such that $(U \times V) \cap G(f) = \varnothing$.
(iii) ***clopen θ-closed*** [18] if for each $(x, y) \notin G(f)$ there exists a clopen set U of x and a θ-open set V containing y such that $(U \times V) \cap G(f) = \varnothing$.

5.14. **Proposition.** *Let $f : X \to Y$ be a z-perfectly continuous function into a functionally Hausdorff space Y. Then the graph $G(f)$ of f is clopen z-closed in $X \times Y$.*

Proof. Suppose $(x, y) \notin G(f)$. Then $f(x) \neq y$. Since Y is functionally Hausdorff there exist disjoint cozero sets V and W containing $f(x)$ and y, respectively. Since f is z-perfectly continuous, $f^{-1}(V)$ is a clopen set containing x. Clearly $(f^{-1}(V) \times W) \cap G(f) = \varnothing$ and so the graph $G(f)$ of f is clopen z-closed in $X \times Y$. ∎

5.15. **Corollary.** *If $f : X \to Y$ is a z-perfectly continuous function into a functionally Hausdorff space Y, then the graph $G(f)$ of f is clopen D_δ-closed (clopen θ-closed) in $X \times Y$.*

6. Function spaces and z-perfectly continuous functions

It is of considerable importance and from applications viewpoint to formulate conditions on the spaces X, Y and subsets of $C(X, Y)$ or Y^X to be closed/compact in the topology of pointwise convergence. Results of this type and Ascoli type theorems abound in the literature (see [4], [14]).

Herein we strengthen the results of ([18], [21], [22], [24], [27], [28], [36]) to show that if X is a sum connected space and Y is functionally Hausdorff, then $S(X, Y) = P(X, Y) = L(X, Y) = P_\Delta(X, Y) = P_\delta(X, Y) = P_q(X, Y) = P_p(X, Y) = P_z(X, Y)$ denote the function spaces of all strongly continuous, perfectly continuous, cl-supercontinuous, δ-perfectly continuous, almost perfectly continuous, quasi perfectly continuous, pseudo perfectly continuous functions from X into Y, respectively with the topology of pointwise convergence and are closed in Y^X in the topology of pointwise convergence.

6.1. **Theorem.** *Let $f : X \to Y$ be a z-perfectly continuous function into a functionally Hausdorff space Y. Then f is constant on each connected subset of X. In particular, if X is connected, then f is constant on X and hence strongly continuous.*

Proof. The method of proof by contradiction is used. Let C be the connected subset of X such that $f(C)$ is not a singleton. Let $f(x), f(y) \in f(C)$, $f(x) \neq f(y)$. Since Y is functionally Hausdorff, there exist disjoint cozero sets U and V containing $f(x)$ and $f(y)$, respectively. Since f is a z-perfectly continuous, $f^{-1}(U) \cap C$ and $f^{-1}(V) \cap C$ are non empty proper clopen subsets of C, contradicting the fact that C is connected. The last part of the theorem is immediate, since every constant function is strongly continuous. ∎

6.2. **Remark.** The hypothesis of 'functionally Hausdorff space' in Theorem 6.1 cannot be omitted. For let X be the real line with usual topology and let Y denote the real line endowed with cofinite topology. Let f denote the identity mapping from X onto Y. Then f is a nonconstant z-perfectly continuous function.

6.3. **Corollary.** *Let $f : X \to Y$ be a z-perfectly continuous function from a sum connected space X into a functionally Hausdorff space Y. Then f is constant on each component of X and hence strongly continuous.*

Proof. Clearly, in view of Theorem 6.1 f is constant on each component of X. Since X is a sum connected space, each component of X is clopen in X. Hence it follows that any union of components of X and the complement of this union are complementary clopen sets in X. Thus f is constant on each component on X. Therefore, for every subset A of Y, $f^{-1}(A)$ and $X \backslash f^{-1}(A)$ are complementary clopen sets in X being the union of component of X. So f is strongly continuous. ∎

Next, we quote the following result from [24].

6.4. **Theorem** ([24, Theorem 6.7])**.** *Let $f : X \to Y$ be a function from a sum connected space X into a $D_\delta T_0$-space Y. Then the following statements are equivalent.*

(a) *f is strongly continuous.*
(b) *f is perfectly continuous.*
(c) *f is cl-supercontinuous.*
(d) *f is δ-perfectly continuous.*
(e) *f is almost perfectly continuous.*
(f) *f is quasi perfectly continuous.*
(g) *f is pseudo perfectly continuous.*

6.5. **Theorem.** *Let $f : X \to Y$ be a function from a sum connected space X into a functionally Hausdorff space Y. Then the following statements are equivalent.*

(a) *f is strongly continuous.*
(b) *f is perfectly continuous.*
(c) *f is cl-supercontinuous.*
(d) *f is δ-perfectly continuous.*
(e) *f is almost perfectly continuous.*
(f) *f is quasi perfectly continuous.*
(g) *f is pseudo perfectly continuous.*
(h) *f is z-perfectly continuous.*

Proof. Since every functionally Hausdorff space is $D_\delta T_0$-space, the equivalence of the assertions (a)-(g) is a consequence of Theorem 6.4. The implications (a)⇒(b) ⇒(d) ⇒(f) ⇒(g) ⇒(h) are trivial and the implication (h)⇒(a) is immediate in view of Corollary 6.3. ∎

6.6. **Theorem.** *Let X be a sum connected space and let Y be a functionally Hausdorff space. Then $S(X,Y) = P(X,Y) = L(X,Y) = P_\Delta(X,Y) = P_\delta(X,Y) = P_q(X,Y) = P_p(X,Y) = P_z(X,Y)$ is closed in Y^X in the topology of pointwise convergence.*

Proof. It is immediate from Theorem 6.5 that the above eight classes of functions are identical and its closedness in Y^X in the topology of pointwise convergence follows either from [22, Theorem 5.4] or [36, Theorem 4.6] or [27, Theorem 5.7]. ∎

The above results are important from applications view point since in particular it follows that if X is sum connected (e.g. connected or locally connected) and Y is functionally Hausdorff, then the pointwise limit of a sequence $\{f_n : X \to Y : n \in N\}$ of z-perfectly continuous functions is z-perfectly continuous.

We conclude this section with the following result which seems to be of considerable significance from applications view point.

6.7. **Theorem.** *If X is a sum connected space and Y is a compact Hausdorff space, then the spaces $S(X,Y) = P(X,Y) = L(X,Y) = P_\Delta(X,Y) = P_\delta(X,Y) = P_q(X,Y) = P_p(X,Y) = P_z(X,Y)$ are compact Hausdorff subspaces of Y^X in the topology of pointwise convergence.*

7. Change of topology

The technique of change of topology of a space is prevalent all through mathematics and is of considerable significance and widely used in topology, functional analysis and several other branches of mathematics. For example, to see the applications of the technique in topology see ([11], [15], [17], [23], [42]).

Here we discuss the behaviour of a z-perfectly continuous function if its domain or range or both domain and range are retopologized in an appropriate way. This method also suggests the alternative proofs of certain results of preceding sections.

Let (X, τ) be a topological space and let B_z denote the collection of all cozero subsets of (X, τ). Since the intersection of two cozero sets is a cozero set, the collection B_z is a base for a topology τ_z on X which is coarser than τ (see [19]). The topology τ_z has been referred to by several authors in the literature. See for example, Aull [5], D'Aristotle [7] and Stephenson [40].

Further, $B_{\rm cl}$ denotes the collection of all clopen subsets of (X, τ). Since the intersection of two clopen sets is a clopen set, the collection $B_{\rm cl}$ is a base for a topology $\tau_{\rm cl}$ on X. Clearly $\tau_{\rm cl} \subset \tau$ (see [9], [35]). The space (X, τ) is zero-dimensional if and only if $\tau_{\rm cl} = \tau$.

For interrelations and interplay among various other coarser topologies obtained in this way for a given topology we refer the interested reader to [23]. Finally, we conclude with the following result.

7.1. **Theorem.** *For a function $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.*

(i) *$f : (X, \tau) \to (Y, \sigma)$ is a z-perfectly continuous function*
(ii) *$f : (X, \tau) \to (Y, \sigma_z)$ is cl-supercontinuous*
(iii) *$f : (X, \tau_{cl}) \to (Y, \sigma)$ is z-continuous*
(iv) *$f : (X, \tau_{cl}) \to (Y, \sigma_z)$ is continuous.*

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