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A GENERAL FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE HYBRID MAPPINGS

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Abstract. In the present paper a general fixed point theorem for two pairs of occasionally weakly compatible mappings is proved. This theorem generalizes some results by metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, generalized metric spaces, G - metric spaces.

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1. Introduction

Definition 1.1. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a metric on X if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is a metric space.

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Definition 1.2. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a symmetric on X if for each $x, y \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$.

The pair (X, d) is a symmetric space.

There exists a vast literature concerning fixed points in symmetric spaces.

Definition 1.3. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a quasi - metric on X [51] if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is a quasi - metric space.

Some fixed point theorems in quasi - metric spaces are proved in [11], [22], [23], [29], [47], [48] and in other papers.

Definition 1.4. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a b - metric on X [12] if there exists $s \geq 1$ such that for all $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is a b - metric space.

Some fixed point theorems in b - metric spaces are proved in [12], [13], [14], [15], [37], [49] and in other papers.

Definition 1.5. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a generalized metric on X [9] if for each $x, y, z, w \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$.

The pair (X, d) is a generalized metric space.

Some fixed point theorems in generalized metric spaces are proved in [9], [16], [17], [18], [21], [28] and in other papers.

Remark 1.1. *In Definitions 1.1 - 1.5 the condition 1) is single common condition. In [26] and [42] some fixed point theorems in symmetric spaces for mappings without the condition of symmetry and triangle inequality are proved. Also in [7] some fixed point theorems satisfying only condition 1) are proved.*

Definition 1.6. *Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}_+$ is a minimal condition metric (briefly mc - metric) on X if $m(x, y) = 0$ if and only if $x = y$.*

The pair (X, m) is a minimal condition metric space (briefly mc - metric space).

Remark 1.2. *1) By Definition 1.6 it follows that every metric, symmetric, quasi - metric, b - metric, generalized metric is a mc - metric.*

2) The metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, generalized metric spaces are all mc - metric spaces.

In the following we denote $M(A, B) = \inf\{m(a, b) : a \in A, b \in B\}$.

2. Preliminaries

Let A and S be self mappings of a metric space (X, d). Jungck [24] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$. We denote by $C(A, S)$ the set of all coincidence points of A and S.

In [38], Pant defined A and S to be pointwise R - weakly commuting if for each $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \leq Rd(Ax, Sx)$. It is proved in [39] that pointwise R - weakly commuting is equivalent with the commuting at coincidence points.

Definition 2.1. *A and S is said to be weakly compatible* [25] *if $ASu = SAu$ for $u \in C(A, S)$.*

Remark 2.1. *A and S are pointwise R - weakly commuting if and only if A and S are weakly compatible.*

Definition 2.2. *A and S are said to be occasionally weakly compatible (briefly owc)* [6] *if $ASu = SAu$ for some $u \in C(A, S)$.*

Remark 2.2. *If $C(A, S) \neq \varnothing$ and A and S are weakly compatible, then A and S are owc, but the converse is not true (Example* [6]*).*

Some fixed point theorems for owc mappings are proved in [4], [26], [42], [45] and in other papers.

Let X be a nonempty set and $f : X \to X$ and $F : X \to 2^X$. A point $x \in X$ is a coincidence point of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.

Definition 2.3. *The pair (f, F) is occasionally weakly compatible* [1], [2] *if $fFu \subset Ffu$ for some $u \in C(f, F)$.*

Some fixed point theorems for owc hybrid mappings are proved in [1], [2], [5], [8] and in other papers.

The study of fixed points in metric spaces satisfying an implicit relation is initiated in [40], [41].

In the present paper a general fixed point theorem for owc hybrid mappings satisfying an implicit relation is proved. As application, two general fixed point theorems for mappings satisfying contractive conditions of integral type and for hybrid mappings in G - metric spaces are obtained.

3. Implicit relations

Definition 3.1. *Let ϕ_d be the set of all real functions $\phi(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:*

(ϕ_1): *ϕ is nonincreasing in variables t_2, t_5, t_6,*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.*

Example 3.1. *$\phi(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, where $k \in (0, 1)$.*

(ϕ_1): *Obviously.*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) = t(1 - k)$, $\forall t > 0$.*

Example 3.2. *$\phi(t_1, ..., t_6) = t_1 - h \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $h \in (0, 1)$.*

(ϕ_1): *Obviously.*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) = t(1 - h)$, $\forall t > 0$.*

Example 3.3. *$\phi(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in (0, 1)$.*

(ϕ_1): *Obviously.*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) = t(1 - k)$, $\forall t > 0$.*

Example 3.4. *$\phi(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4) - c \max\{t_2, t_5, t_6\}$, where $a, b, c \geq 0$ and $a + c < 1$.*

(ϕ_1): *Obviously.*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) = t(1 - (a + c))$, $\forall t > 0$.*

Example 3.5. *$\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c \min\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + c < 1$.*

(ϕ_1): *Obviously.*

(ϕ_2): *$\phi(t, t, 0, 0, t, t) = t(1 - (a + c))$, $\forall t > 0$.*

Example 3.6. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\sqrt{t_5 t_6}$, *where* $a, b, c \geq 0$ *and* $a + c < 1$.

(ϕ_1): *Obviously.*

(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - (a + c))$, $\forall t > 0$.

Example 3.7. $\phi(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1-\alpha)(at_5 + bt_6)$, *where* $0 < \alpha < 1$, $a, b \geq 0$ *and* $a + b < 1$.

(ϕ_1): *Obviously.*

(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - \alpha)(1 - (a + b))$, $\forall t > 0$.

Example 3.8. $\phi(t_1, ..., t_6) = t_1^2 - at_2^2 - b\frac{\min\{t_5^2, t_6^2\}}{1+t_3+t_4}$, *where* $a, b \geq 0$ *and* $a + b < 1$.

(ϕ_1): *Obviously.*

(ϕ_2): $\phi(t, t, 0, 0, t, t) = t^2(1 - (a + b))$, $\forall t > 0$.

Example 3.9. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\frac{t_5+t_6}{1+t_3+t_4}$, *where* $a, b \geq 0$ *and* $a + 2b < 1$.

(ϕ_1): *Obviously.*

(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - (a + 2b))$, $\forall t > 0$.

Example 3.10. $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0, 1)$, $a, b \geq 0$ *and* $\max\{c, a + b\} < 1$.

(ϕ_1): *Obviously.*

(ϕ_2): $\phi(t, t, 0, 0, t, t) = t(1 - \max\{c, a + b\})$, $\forall t > 0$.

4. Main results

Theorem 4.1. *Let* f, h *be self mappings of a mc - metric space* (X, m) *and* F, H *be maps of* X *into* 2^X *such that the pairs* (f, F) *and* (h, H) *are owc. If*

$$\begin{gathered}\phi(m(fx, hy), M(Fx, Hy), M(fx, Fx), \\ M(hy, Hy), M(fx, Hy), M(Fx, hy)) \leq 0\end{gathered} \tag{4.1}$$

for all $x, y \in X$ *for which* $fx \neq hy$ *and* $\phi \in \phi_d$, *then* f, h, F *and* H *have a unique common fixed point.*

Proof. Since (f, F) and (h, H) are owc, there exists $x, y \in X$ such that $fx \in Fx$, $hy \in Hy$ and $fFx \subset Ffx$ and $hHy \subset Hhy$. First we prove that $fx = hy$. Suppose that $fx \neq hy$. Then $0 \neq m(fx, hy) \geq M(fx, Hy)$. By (4.1) and (ϕ_1) we have

$$\phi(m(fx, hy), m(fx, hy), 0, 0, m(fx, hy), m(fx, hy)) \leq 0,$$

a contradiction of (ϕ_2). Hence $fx = hy$.

Next we prove that $fx = f^2x$. Suppose that $fx \neq f^2x$. Since $f^2x \in fFx \subset Ffx$, then $0 \neq m(f^2x, fx) \geq M(Ffx, fx) = M(Ffx, hy) \geq M(Ffx, Hy)$. By (4.1) and ($\phi_1$) we have successively

$$\phi(m(f^2x, hy), M(Ffx, Hy), 0, 0, M(f^2x, Hy), M(Ffx, hy)) \leq 0,$$

$$\phi(m(f^2x, hy), m(f^2x, hy), 0, 0, m(f^2x, hy), m(f^2x, hy)) \leq 0,$$

a contradiction of (ϕ_2). Hence $fx = f^2x$ and fx is a fixed point of f. Similarly, $h^2y = hy$. Therefore, $fx = f^2x = hy = h^2y = hfx$ and fx is a fixed point of h. On the other hand, $fx = f^2x \in fFx \subset Ffx$, hence fx is a fixed point of F. Similarly, $fx = f^2x = hy = h^2y \in hHy \subset Hhy = Hfx$ and fx is a fixed point of H. Therefore, $w = fx$ is a common fixed point of f, h, F and H.

Suppose that $w' \neq w$ is an other common fixed point of f, h, F and H. Then, by (4.1) and (ϕ_1) we have successively

$$\phi(M(fw, hw'), M(Fw, Hw'), 0, 0, M(fw, Hw'), M(Fw, hw')) \leq 0,$$

$$\phi(m(w, w'), m(w, w'), 0, 0, m(w, w'), m(w, w')) \leq 0,$$

a contradiction of (ϕ_2). Hence w is the unique common fixed point of f, h, F and H. □

Corollary 4.1. *Let f, h be self mappings in a metric (symmetric, quasi - metric, b - metric, generalized metric) space and $F, H : X \to 2^X$ such that the pairs (f, F) and (h, H) are owc. If the inequality (4.1) holds for all $x, y \in X$ with $fx \neq hy$ and $\phi \in \phi_d$, then f, h, F and H have a unique common fixed point.*

If $f = h$ and $F = H$ then by Theorem 4.1 we obtain

Theorem 4.2. *Let f be a self mapping of a mc - metric space (X, m) and F be a map of X into 2^X such that the pair (f, F) is owc. If*

$$\begin{gathered}\phi(m(fx, fy), M(Fx, Fy), M(fx, Fx), \\ M(fy, Fy), M(fx, Fy), M(Fx, fy)) \leq 0\end{gathered} \tag{4.2}$$

for all $x, y \in X$ for which $fx \neq fy$ and $\phi \in \phi_d$, then f and F have a unique common fixed point.

If f, h, F and H are single valued mappings, then by Theorem 4.1 we obtain

Theorem 4.3. *Let f, h, F and H be self mappings of a mc - metric space (X, m) such that the pairs (f, F) and (h, H) are owc. If the inequality*

$$\begin{gathered}\phi(m(fx, hy), m(Fx, Hy), m(fx, Fx), \\ m(hy, Hy), m(fx, Hy), m(Fx, hy)) \leq 0\end{gathered} \tag{4.3}$$

for all $x, y \in X$ for which $fx \neq hy$, then f, h, F and H have a unique common fixed point.

Corollary 4.2. *Let f, h, F and H be self mappings of a metric (symmetric, quasi - metric, b - metric, generalized metric) space such that the pairs (f, F) and (h, H) are owc. If the inequality (4.3) holds for all $x, y \in X$ for which $fx \neq hy$, then f, h, F and H have a unique common fixed point.*

By Theorem 4.1 we and Examples 3.1 - 3.10 we obtain

Corollary 4.3. *Let f, h be self mappings of a mc - metric space (X, m) and F, H be maps of X into 2^X. If one of the following inequalities holds for all $x, y \in X$ for which $fx \neq hy$:*

1)

$$\begin{gathered} m(fx, hy) \leq k \max\{M(Fx, Hy), M(fx, Fx), \\ M(hy, Hy), M(fx, Hy), M(Fx, hy)\}, \end{gathered}$$

where $k \in (0, 1)$,

2)

$$\begin{gathered} m(fx, hy) \leq k \max\{M(Fx, Hy), M(fx, Fx), \\ M(hy, Hy), \tfrac{1}{2}[M(fx, Hy) + M(Fx, hy)]\}, \end{gathered}$$

where $k \in (0, 1)$,

3)

$$\begin{gathered} m(fx, hy) \leq k \max\{M(Fx, Hy), \\ \tfrac{1}{2}[M(fx, Fx) + M(hy, Hy)], \\ \tfrac{1}{2}[M(fx, Hy) + M(Fx, hy)]\}, \end{gathered}$$

where $k \in (0, 1)$,

4)

$$\begin{gathered} m(fx, hy) \leq aM(Fx, Hy) + b \max\{M(fx, Fx), M(hy, Hy)\}, \\ c \max\{M(Fx, Hy), M(fx, Hy), M(Fx, hy)\}, \end{gathered}$$

where $a, b, c \geq 0$ and $a + c < 1$,

5)

$$\begin{gathered} m(fx, hy) \leq aM(Fx, Hy) + b[M(fx, Fx) + M(hy, Hy)] \\ +c \min\{M(fx, Hy), M(Fx, hy)\}, \end{gathered}$$

where $a, b, c \geq 0$ and $a + c < 1$,

6)

$$\begin{gathered} m(fx, hy) \leq aM(Fx, Hy) + b[M(fx, Fx) + M(hy, Hy)], \\ +c\sqrt{M(fx, Hy) \cdot M(Fx, hy)}, \end{gathered}$$

where $a, b, c \geq 0$ and $a + c < 1$,

7)

$$m(fx,hy) \le \alpha \max\{M(Fx,Hy), M(fx,Fx), M(hy,Hy)\}$$
$$+(1-\alpha)[aM(fx,Hy)+bM(Fx,Hy)],$$

where $\alpha \in (0,1)$, $a,b \ge 0$ *and* $a+b<1$,

8)

$$m^2(fx,hy) \le aM^2(Fx,Hy) + b\frac{\min\{M^2(fx,Hy), M^2(Fx,hy)\}}{1+M(fx,Fx)+M(hy,Hy)},$$

where $a,b \ge 0$ *and* $a+b<1$,

9)

$$m(fx,hy) \le aM(Fx,Hy) + b\frac{M(fx,Hy)+M(Fx,hy)}{1+M(fx,Fx)+M(hy,Hy)},$$

where $a,b \ge 0$ *and* $a+2b<1$,

10)

$$m(fx,hy) \le \max\{cM(Fx,Hy), cM(fx,Fx),$$
$$cM(hy,Hy), aM(fx,Hy)+bM(Fx,hy)\},$$

where $c \in (0,1)$, $a,b \ge 0$ *and* $\max\{c, a+b\} < 1$,

and if (f,F) *and* (h,H) *are owc, then* f,h,F *and* H *have a unique common fixed point.*

Remark 4.1. *By Theorems 4.2, 4.3 and Examples 3.1 - 3.10 we obtain new corollaries.*

5. Applications

a) Fixed points in G - metric spaces

In [19], [20] Dhage introduced a new class of generalized metric spaces named D - metric spaces. Mustafa and Sims [32], [33] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric spaces, named G - metric spaces. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [30], [33], [34], [35], [36], [43] and other papers.

Definition 5.1. *Let* X *be a nonempty set and* $G: X^3 \to \mathbb{R}_+$ *be a function satisfying the following properties:*

(G_1) : $G(x,y,z)=0$ *if* $x=y=z$,

(G_2) : $0<G(x,x,y)$ *for all* $x,y \in X$ *with* $y \ne x$,

(G_3) : $G(x,y,y) \le G(x,y,z)$ *for all* $x,y,z \in X$ *and* $z \ne y$,

(G_4) : $G(x,y,z) = G(y,z,x) = G(z,x,y) = \ldots$ *(symmetry in all three variables),*

(G_5) : $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

The function G *is called a* G *- metric on* X *and the pair* (X,G) *is called a* G *- metric space,* [32], [33].

Remark 5.1. *If* $G(x,y,z) = 0$*, then* $x = y = z$ [33].

Lemma 5.1. *Let* (X,G) *be a* G *- metric space and* $q(x,y) = G(x,y,y)$*. Then* $q(x,y)$ *is a quasi - metric on* X*.*

Proof. 1) By (G_1) and Remark 5.1 $q(x,y) = 0$ if and only if $x = y$.
2) By (G_5) we have

$$q(x,y) = G(x,y,y) \le G(x,z,z) + G(z,y,y) = q(x,z) + q(z,y).$$

Hence $q(x,y)$ is a quasi - metric on X. □

Theorem 5.1. *Let* f,h,F *and* H *be self mappings of a* G *- metric space* (X,G) *such that* (f,F) *and* (h,H) *are owc. If*

$$\begin{gathered}\phi(G(fx,hy,hy), G(Fx,Hy,Hy), G(fx,Fx,Fx),\\ G(hy,Hy,Hy), G(fx,Hy,Hy), G(Fx,hy,hy)) \le 0,\end{gathered} \tag{5.1}$$

for all $x,y \in X$ *for which* $fx \neq hy$ *and* $\phi \in \phi_d$*. Then* f,h,F *and* H *have a unique common fixed point.*

Proof. As in Lemma 5.1, $q(x,y) = G(x,y,y)$ is a quasi - metric on X. Then,

$$\begin{gathered}G(fx,hy,hy) = q(fx,hy), G(Fx,Hy,Hy) = q(Fx,Hy),\\ G(fx,Fx,Fx) = q(fx,Fx), G(hy,Hy,Hy) = q(hy,Hy),\\ G(fx,Hy,Hy) = q(fx,Hy), G(Fx,hy,hy) = q(Fx,hy).\end{gathered}$$

Then in (X,q) by (5.1) we have

$$\begin{gathered}\phi(q(fx,hy), q(Fx,Hy), q(fx,Fx),\\ q(hy,Hy), q(fx,Hy), q(Fx,hy)) \le 0,\end{gathered} \tag{5.2}$$

which is the inequality (4.3) for $m(x,y) = q(x,y)$.

Hence, the conditions of Theorem 4.3 are satisfied and f,h,F and H have a unique common fixed point. □

By Theorem 5.1 and Examples 3.1 - 3.10 we obtain

Corollary 5.1. *Let* f,h,F *and* H *be self mappings of a* G *- metric space* (X,G) *such that* (f,F) *and* (h,H) *are owc. If one of the following inequalities holds for all* $x,y \in X$ *with* $fx \neq hy$*:*

1)

$$G(fx,hy,hy) \leq k\max\{G(Fx,Hy,Hy),G(fx,Fx,Fx),\\ G(hy,Hy,Hy),G(fx,Hy,Hy),G(Fx,hy,hy)\},$$

where $k \in (0,1)$*,*

2)

$$G(fx,hy,hy) \leq k\max\{G(Fx,Hy,Hy),G(fx,Fx,Fx),\\ G(hy,Hy,Hy),\tfrac{1}{2}[G(fx,Hy,Hy)+G(Fx,hy,hy)]\},$$

where $k \in (0,1)$*,*

3)

$$G(fx,hy,hy) \leq k\max\{G(Fx,Hy,Hy),\\ \tfrac{1}{2}[G(fx,Fx,Fx)+G(hy,Hy,Hy)],\\ \tfrac{1}{2}[G(fx,Hy,Hy)+G(Fx,hy,hy)]\},$$

where $k \in (0,1)$*,*

4)

$$G(fx,hy,hy) \leq aG(Fx,Hy,Hy)+b[G(fx,Fx,Fx)+G(hy,Hy,Hy)]\\ +c\max\{G(Fx,Hy,Hy),G(fx,Hy,Hy),G(Fx,hy,hy)\},$$

where $a,b,c \geq 0$ *and* $a+c<1$*,*

5)

$$G(fx,hy,hy) \leq aG(Fx,Hy,Hy)\\ +b[G(fx,Fx,Fx)+G(hy,Hy,Hy)]\\ +c\min\{G(fx,Hy,Hy),G(Fx,hy,hy)\},$$

where $a,b,c \geq 0$ *and* $a+c<1$*,*

6)

$$G(fx,hy,hy) \leq aG(Fx,Hy,Hy)\\ +b[G(fx,Fx,Fx)+G(hy,Hy,Hy)]\\ +c\sqrt{G(fx,Hy,Hy)\cdot G(Fx,hy,hy)},$$

where $a,b,c \geq 0$ *and* $a+c<1$*,*

7)

$$G(fx,hy,hy) \leq \alpha\max\{G(Fx,Hy,Hy),\\ G(fx,Fx,Fx),G(hy,Hy,Hy)\}\\ +(1-\alpha)[aG(fx,Hy,Hy)+bG(Fx,hy,hy)],$$

where $\alpha \in (0,1)$*,* $a,b \geq 0$ *and* $a+b<1$*,*

8)

$$[G(fx,hy,hy)]^2 \leq aG^2(Fx,Hy,Hy)\\ +b\frac{\min\{G^2(fx,Hy,Hy),G^2(Fx,hy,hy)\}}{1+G(fx,Fx,Fx)+G(hy,Hy,Hy)},$$

where $a,b \geq 0$ *and* $a+b<1$*,*

9)

$$G(fx,hy,hy) \leq aG(Fx,Hy,Hy) + b\frac{G(fx,Hy,Hy)+G(Fx,hy,hy)}{1+G(fx,Fx,Fx)+G(hy,Hy,Hy)},$$

where $a,b \geq 0$ *and* $a+2b<1$,

10)

$$G(fx,hy,hy) \leq \max\{cG(Fx,Hy,Hy), cG(fx,Fx,Fx), cG(hy,Hy,Hy), aG(fx,Hy,Hy)+bG(Fx,hy,hy)\},$$

where $c \in (0,1)$, $a,b \geq 0$ *and* $\max\{c, a+b\} < 1$,

then f,h,F *and* H *have a unique common fixed point.*

b) Fixed point results for mappings satisfying a contractive condition of integral type

In [10], Branciari established the following fixed point theorem, which opened the way to the study of mappings satisfying a contractive condition of integral type.

Theorem 5.2. *Let* (X,d) *be a complete metric space,* $c \in (0,1)$ *and* $f:(X,d)\to(X,d)$ *be a mapping such that for all* $x,y \in X$

$$\int_0^{d(fx,fy)} h(t)dt \leq c\int_0^{d(x,y)} h(t)dt$$

where $h:[0,\infty)\to[0,\infty)$ *is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,\infty)$, *such that, for* $\varepsilon>0$, $\int_0^{\varepsilon} h(t)dt>0$. *Then* f *has a unique fixed point* z *such that for each* $x\in X$, $\lim_{n\to\infty} f^n x = z$.

Some fixed point theorems for compatible, weakly compatible and owc mappings satisfying a contractive condition of integral type are proved in [3], [27], [42], [44], [45], [46], [50] and in other papers.

Let (X,d) be a metric space and $s(x,y)=\int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 5.2. In [31] and [44] is proved that $s(x,y)$ is a symmetric on X and the study of fixed points for mappings satisfying contractive conditions of integral type is reduced to the study of fixed points in symmetric spaces.

Let (X,d) be a metric space and $f,g:X\to X$ and $F,G:X\to 2^X$, and (X,s) is the symmetric space determined by $s(x,y)$. Then

$$\begin{aligned} &s(fx,gy)=\textstyle\int_0^{d(fx,gy)} h(t)dt, S(Fx,Gy)=\int_0^{D(Fx,Gy)} h(t)dt,\\ &S(fx,Fx)=\textstyle\int_0^{D(fx,Fx)} h(t)dt, S(gy,Gy)=\int_0^{D(gy,Gy)} h(t)dt,\\ &S(fx,Gy)=\textstyle\int_0^{D(fx,Gy)} h(t)dt, S(Fx,gy)=\int_0^{D(Fx,gy)} h(t)dt.\end{aligned} \tag{5.3}$$

where $h(t)$ is as in Theorem 5.2.

Theorem 5.3. *Let (X,d) be a metric space, $f,g : X \to X$, $F,G : X \to 2^X$ satisfying*

$$\begin{array}{c} \phi(\int_0^{d(fx,gy)} h(t)dt, \int_0^{D(Fx,Gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt, \\ \int_0^{D(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(Fx,gy)} h(t)dt) \leq 0, \end{array} \tag{5.4}$$

for all $x,y \in X$ with $fx \neq gy$ and $\phi \in \phi_d$. If the pairs (f,F) and (g,G) are owc, then f,g,F and G have a unique common fixed point.

Proof. By (5.3) and (5.4) we obtain

$$\phi(s(fx,gy), S(Fx,Gy), S(fx,Fx), S(gy,Gy), S(fx,Gy), S(Fx,gy)) \leq 0,$$

for all $x,y \in X$ with $fx \neq gy$ and $\phi \in \phi_d$.

Hence the conditions of Theorem 4.1 are satisfied for $m(x,y) = s(x,y)$ in the symmetric space (X,s). By Theorem 4.1, f,g,F and G have a unique common fixed point. $\square$

By Theorem 5.3 and Examples 3.1 - 3.10 we obtain

Corollary 5.2. *Let $f,g : X \to X$ and $F,G : X \to 2^X$ such that (f,F) and (g,G) are owc. If one of the following conditions holds for all $x,y \in X$ with $fx \neq gy$:*

1)

$$\begin{array}{c} \int_0^{d(fx,gy)} h(t)dt \leq k \max\{\int_0^{D(Fx,Gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt, \\ \int_0^{D(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(Fx,gy)} h(t)dt\}, \end{array}$$

where $k \in (0,1)$,

2)

$$\begin{array}{c} \int_0^{d(fx,gy)} h(t)dt \leq k \max\{\int_0^{D(Fx,Gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt, \int_0^{D(gy,Gy)} h(t)dt, \\ \frac{1}{2}[\int_0^{D(fx,Fy)} h(t)dt + \int_0^{D(Fx,gy)} h(t)dt]\}, \end{array}$$

where $k \in (0,1)$,

3)

$$\begin{array}{c} \int_0^{d(fx,gy)} h(t)dt \leq k \max\{\int_0^{D(Fx,Gy)} h(t)dt, \\ \frac{1}{2}[\int_0^{D(fx,Fx)} h(t)dt + \int_0^{D(gy,Gy)} h(t)dt], \\ \frac{1}{2}[\int_0^{D(fx,Gy)} h(t)dt + \int_0^{D(Fx,gy)} h(t)dt]\}, \end{array}$$

where $k \in (0,1)$,

4)

$$\begin{array}{c} \int_0^{d(fx,gy)} h(t)dt \leq a \int_0^{D(Fx,Gy)} h(t)dt \\ +b \max\{\int_0^{D(fx,Fx)} h(t)dt, \int_0^{D(gy,Gy)} h(t)dt\} \\ +c \max\{\int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(Fx,gy)} h(t)dt\}, \end{array}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

5)

$$\begin{gathered}\int_0^{d(fx,gy)} h(t)dt \leq a \int_0^{D(Fx,Gy)} h(t)dt \\ +b[\int_0^{D(fx,Fx)} h(t)dt + \int_0^{D(gy,Gy)} h(t)dt] \\ +c \min\{\int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(Fx,gy)} h(t)dt\},\end{gathered}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

6)

$$\begin{gathered}\int_0^{d(fx,gy)} h(t)dt \leq a \int_0^{D(Fx,Gy)} h(t)dt \\ +b[\int_0^{D(fx,Fx)} h(t)dt + \int_0^{D(gy,Gy)} h(t)dt] \\ +c\sqrt{\int_0^{D(fx,Gy)} h(t)dt \cdot \int_0^{D(Fx,gy)} h(t)dt},\end{gathered}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

7)

$$\begin{gathered}\int_0^{d(fx,gy)} h(t)dt \leq \alpha \max\{\int_0^{D(Fx,Gy)} h(t)dt, \\ \int_0^{D(fx,Fx)} h(t)dt, \int_0^{D(gy,Gy)} h(t)dt\} \\ +(1-\alpha)[a \int_0^{D(fx,Gy)} h(t)dt + b \int_0^{D(Fx,gy)} h(t)dt],\end{gathered}$$

where $\alpha \in (0, 1)$, $a, b \geq 0$ *and* $a + b < 1$,

8)

$$\begin{gathered}\left[\int_0^{d(fx,gy)} h(t)dt\right]^2 \leq a \left[\int_0^{D(Fx,Gy)} h(t)dt\right]^2 \\ +b \min \frac{\left\{\left[\int_0^{D(fx,Gy)} h(t)dt\right]^2, \left[\int_0^{D(Fx,gy)} h(t)dt\right]^2\right\}}{1 + \int_0^{D(fx,Fx)} h(t)dt + \int_0^{D(gy,Gy)} h(t)dt},\end{gathered}$$

where $a, b \geq 0$ *and* $a + b < 1$,

9)

$$\begin{gathered}\int_0^{d(fx,gy)} h(t)dt \leq a \int_0^{d(Fx,Gy)} h(t)dt \\ +b\frac{\int_0^{D(fx,Gy)} h(t)dt + \int_0^{D(Fx,gy)} h(t)dt}{1 + \int_0^{D(fx,Fx)} h(t)dt + \int_0^{D(gy,Gy)} h(t)dt},\end{gathered}$$

where $a, b \geq 0$ *and* $a + 2b < 1$,

10)

$$\begin{gathered}\int_0^{d(fx,gy)} h(t)dt \leq \max\{c \int_0^{D(Fx,Gy)} h(t)dt, c \int_0^{D(fx,Fx)} h(t)dt, \\ c \int_0^{D(gy,Gy)} h(t)dt, a \int_0^{D(fx,G)} h(t)dt + b \int_0^{D(Fx,gy)} h(t)dt\},\end{gathered}$$

where $c \in (0, 1)$, $a, b \geq 0$ *and* $\max\{c, a + b\} < 1$,

then f, g, F *and* G *have a unique common fixed point.*

If f, g, F and G are single valued mappings, then bt Theorem 5.2 we obtain

Theorem 5.4. *Let (X, d) be a metric space and f, g, F, G be self mappings of (X, d) satisfying*

$$\begin{aligned}(5.5) \qquad & \phi(\textstyle\int_0^{d(fx,gy)} h(t)dt, \int_0^{d(Fy,Gy)} h(t)dt, \int_0^{d(fx,Fx)} h(t)dt, \\ & \textstyle\int_0^{d(gy,Gy)} h(t)dt, \int_0^{d(fx,Gy)} h(t)dt, \int_0^{d(Fx,gy)} h(t)dt) \leq 0,\end{aligned}$$

for all $x, y \in X$, with $fx \neq gy$ and $\phi \in \phi_d$. If the pairs (f, F) and (g, G) are owc then f, g, F, G have a unique common fixed point.

Remark 5.2. *By Examples 3.1 - 3.10 and Theorem 5.3 we obtain a similar Corollary with Corollary 5.1.*

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