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A GENERAL FIXED POINT THEOREM FOR MAPPINGS SATISFYING A CYCLICAL CONTRACTIVE CONDITION IN G - METRIC SPACES

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Abstract. In this paper a general fixed point theorem for mappings satisfying a cyclical implicit contractive relation which extends the results from [3], [10], [15], [17], [21], [23], [24], [27] to G - metric spaces is proved.

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1. Introduction

In 2003, Kirk et al. [10] extended Banach's contraction principle to a case of cyclic contractive mappings. In [23], most of the fundamental metrical fixed point theorems in literature (Chatejeea, Reich, Hardy - Rogers, Ćirić) are extended to cyclic contractive mappings. Other new results are obtained in [21], [22], [33]. Several extensions of these results have appeared in literature.

In [5], [6], Dhage introduced a new class of generalized metric spaces, named D - metric spaces. Mustafa and Sims [11], [12] proved

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that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [13], [14], [15], [16], [17], [20], [34] and other papers.

Several classical fixed point theorems and common fixed point theorems have been recently unified by considering a general condition by an implicit relation in [25], [26] and in other papers.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single valued functions, hybrid pairs of functions and set valued functions.

Quite recently, this method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces and intuitionistic metric spaces.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [27], [28], [29], [30] and in other papers.

The study of cyclic contractions on G - metric spaces in initiated in [1], [2], [9], [18].

The study of implicit - relation - type cyclic contractive mappings is initiated in [19].

2. Preliminaries

Definition 2.1. [12] Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$(G_3) : G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

$(G_4) : G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ (symmetry in all three variables),

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G - metric on X and the pair (X, G) is called a G - metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$.

Definition 2.2 ([12]). Let (X, G) be a G - metric space. A sequence (x_n) in (X, G) is said to be:

a) G - convergent if for $\varepsilon > 0$, there is an $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}, m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

b) G - Cauchy if for $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that for all $m, n, p \in \mathbb{N}$, $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$.

c) A G - metric space (X, G) is said to be G - complete if every G - Cauchy sequence is G - convergent.

Lemma 2.3 ([12]). *Let (X, G) be a G - metric space. Then, the following properties are equivalent:*

1) (x_n) is G - convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 2.4 ([12]). *If (X, G) is a G - metric space, the following properties are equivalent:*

1) (x_n) is G - Cauchy;

2) For $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \in \mathbb{N}$, $n, m > k$, .

Lemma 2.5 ([12]). *Let (X, G) be a G - metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Note that each G - metric on X generates a topology τ_G on X [11] whose base is a family of open G - balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_G(x, \varepsilon) = \{y \in X : G(x, y, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A nonempty set $A \subset X$ is G - closed if $\overline{A} = A$.

Hence, $x \in \overline{A} \Leftrightarrow B_G(x, \varepsilon) \cap A \neq \emptyset$, $\forall \varepsilon > 0$.

Lemma 2.6 ([9]). *Let (X, G) be a G - metric space and A be a nonempty subset of X. A is closed if for any G - convergent sequence (x_n) in A with $\lim_{n\to\infty} x_n = x$, then $x \in \overline{A}$.*

Theorem 2.7 (Theorem 2.1 [15]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping which satisfy the following inequality for all $x, y \in X$:*

$$G(Tx, Ty, Ty) \leq k \max\{G(y, Ty, Ty) + G(x, Ty, Ty), \\ 2G(y, Tx, Tx)\}, \tag{2.1}$$

where $k \in \left[0, \frac{1}{3}\right)$. Then T has an unique fixed point.

Theorem 2.8 (Theorem 3.1 [17]). *Let (X, G) be a G - complete metric space and let $T : X \to X$ be a mapping such that for all $x, y \in X$:*

$$
\begin{aligned}
G(Tx, Ty, Ty) \leq \max\{&aG(x, y, y), \\
&b[G(x, Tx, Tx) + 2G(y, Ty, Ty), \\
&b[G(x, Ty, Ty) + G(y, Ty, Ty) + (y, Tx, Tx)]\},
\end{aligned} \tag{2.2}
$$

where $a \in [0, 1)$ and $b \in \left[0, \frac{1}{3}\right)$. Then T has an unique fixed point.

Let $p > 1$, $p \in \mathbb{N}$ be, T a self mapping of a metric space (X, d) and $\{A_i\}_{i=1}^p$ nonempty closed subsets of X. The mapping T is said to be cyclical if

$$
T(A_i) \subset A_{i+1}, i = \overline{1, p}, \text{ where } A_{p+1} = A_1. \tag{2.3}
$$

In [10] the following theorem is proved.

Theorem 2.9 ([10]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space and suppose that $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ satisfy the condition (2.3) and there exists a constant $\alpha \in (0, 1)$ such that*

$$
d(Tx, Ty) \leq \alpha d(x, y), \ \forall x \in A_i, y \in A_{i+1}, 1 \leq i \leq p. \tag{2.4}
$$

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

A cyclical extension of Kanan's theorem [8] is obtained in [33].

Theorem 2.10 ([33]). *Let (X, d) be a complete metric space, $A_1, ..., A_p$ be nonempty closed subsets of X and $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$. We suppose that T satisfies (2.3) and*

$$
d(Tx, T^2x) \leq kd(x, Tx), \tag{2.5}
$$

for all $x \in A_i$ and $k \in \left[0, \frac{1}{2}\right)$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

A cyclical extension of Kanan's theorem [8] is obtained in [23].

Theorem 2.11 ([23]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X, d) and suppose that $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ satisfy the condition (2.3) and there exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that*

$$
d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)], \tag{2.6}
$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = \overline{1, p}$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

A cyclical extension of Reich [31] - Rus [32] theorem is obtained in [23].

Theorem 2.12 ([23]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d) and suppose that $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ satisfy (2.3) and there exist $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + 2\beta < 1$ such that*

$$d(Tx,Ty) \le \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)], \tag{2.7}$$

for all $x \in A_i$, $y \in A_{i+1}, 1 \le i \le p$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

The following results are obtained in [24].

Theorem 2.13 ([24]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d) and suppose that $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ satisfy (2.3) and*

a) there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that

$$d(Tx,Ty) \le ad(x,y) + bd(x,Tx) + cd(y,Ty), \tag{2.8}$$

or,

b) there exist $a,b \in \mathbb{R}_+$ with $a+b<1$ such that

$$d(Tx,Ty) \le ad(x,y) + b\max\{d(x,Tx), d(y,Ty)\}, \tag{2.9}$$

for all $x \in A_i$, $y \in A_{i+1}, 1 \le i \le p$.

c) there exist $a,c \in \mathbb{R}_+$ with $a+2c<1$ such that

$$d(Tx,Ty) \le ad(x,y) + c\max\{d(x,Ty), d(y,Tx)\}, \tag{2.10}$$

for all $x \in A_i$, $y \in A_{i+1}, 1 \le i \le p$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

A cyclical extension of Reich [31] - Rus [32] theorem is obtained in [23].

Theorem 2.14 ([23]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space (X,d) and suppose that $f : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ satisfy (2.3) and there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\begin{aligned} d(fx,fy) \le \alpha \max\{&d(x,y), d(x,fx), \\ &d(y,fy), d(x,fy), d(y,fx)\}, \end{aligned} \tag{2.11}$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = \overline{1,p}$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

The following two results are obtained in [24].

Theorem 2.15 ([24])**.** *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, ..., A_p$ be nonempty closed subsets of X and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A$ satisfy (2.3) and*

$$d(Tx,Ty) \le k \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \tag{2.12}$$

where $k \in (0,1)$ for all $x \in A_i$, $y \in A_{i+1}$, $i = \overline{1,p}$.

Then T has an unique fixed point in $\cap_{i=1}^{p} A_i$.

Remark 2.16. *This result is an extension of Theorem 2.1 from* [10], [22].

Theorem 2.17 ([24])**.** *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, ..., A_p$ be nonempty closed subsets of X and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A$ satisfy (2.3) and*

$$\begin{aligned} d(Tx,Ty) \le{}& \alpha \max\{d(x,y), d(x,Tx), d(y,Ty)\}+ \\ &+(1-\alpha)[ad(x,Ty)+bd(y,Tx)], \end{aligned} \tag{2.13}$$

where $0 \le \alpha \le 1$, $0 \le a < \frac{1}{2}$, $0 \le b < \frac{1}{2}$, for all $x \in A_i$, $y \in A_{i+1}$, $i = \overline{1,p}$.

Then T has an unique fixed point which is in $\cap_{i=1}^{p} A_i$.

The purpose of this paper is to prove a general fixed point theorem for mappings satisfying a cyclical implicit contractive condition, which extends Theorems 2.7 - 2.17 to G - metric spaces.

3. Implicit relations

Definition 3.1 ([3], [19])**.** Let $\mathfrak{F}_G$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variable t_5;

(F_2) : There exists $h_1 \in [0,1)$ such that for all $u, v \ge 0$, $F(u, v, v, u, u+v, 0) \le 0$ implies $u \le h_1 v$;

(F_3) : There exists $h_2 \in [0,1)$ such that for all $t, t' \ge 0$, $F(t,t,0,0,t,t') \le 0$ implies $t \le h_2 t'$.

Example 3.2. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \ge 0$ and $a+b+c+2d+e < 1$.*

(F_1) : Obviously.

(F_2) : Let $u, v \ge 0$ and $F(u,v,v,u,u+v,0) = u - av - bv - cv - d(u+v) \le 0$ which implies $u \le h_1 v$, where $0 \le h_1 = \dfrac{a+b+d}{1-(c+d)} < 1$

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - at - dt - et' \leq 0$ *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \dfrac{e}{1-(a+d)} < 1$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{1}{2}(t_5 + t_6)\right\}$, *where* $k \in [0, 1)$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - k \max\left\{u, v, \dfrac{u+v}{2}\right\} \leq 0$. *If* $u \leq v$, *then* $u(1-k) \leq 0$, *a contradiction. Hence* $u \leq v$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = k < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - k \max\left\{t, \dfrac{t+t'}{2}\right\} \leq 0$. *If* $t > t'$, *then* $t(1-k) \leq 0$, *a contradiction. Hence* $t \leq t'$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = k < 1$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1-\alpha)(at_5 + bt_6)$, *where* $0 \leq \alpha < 1, 0 \leq a < \dfrac{1}{2}, 0 \leq b < \dfrac{1}{2}$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - \alpha \max\{u, v\} - (1-\alpha)a(u+v) \leq 0$. *If* $u > v$, *then* $u(1-\alpha)(1-2a) \leq 0$, *a contradiction. Hence,* $u \leq v$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \dfrac{\alpha + (1-\alpha)a}{1-(1-\alpha)a} < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - \alpha t - (1-\alpha)at - (1-\alpha)bt' \leq 0$ *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \dfrac{b}{1-a} < 1$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $a + b + 2c < 1$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - av - b \max\{u, v\} - c(u+v) \leq 0$. *If* $u > v$, *then* $u[1-(a+b+2c)] \leq 0$, *a contradiction. Hence,* $u \leq v$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \dfrac{\alpha + b + c}{1-c} < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - at - c \max\{t, t'\} \leq 0$. *If* $t > t'$, *then* $t[1-(a+c)] \leq 0$, *a contradiction. Hence* $t \leq t'$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \frac{c}{1-a} < 1$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $k \in \left[0, \dfrac{1}{2}\right)$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - k \max\{u, v, u+v\} \leq 0$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \dfrac{k}{1-k} < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - k \max\{t, t'\} \leq 0$. *If* $t > t'$, *then* $t(1-k) \leq 0$, *a contradiction. Hence* $t \leq t'$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = k < 1$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{t_4, t_5, 2t_6\}$, *where* $a, b, c \geq 0$ *and* $a + b + 3c < 1$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - av - bv - c(2u+v) \leq 0$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \dfrac{a+b+c}{1-2c} < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - at - c\max\{t, 2t'\} \leq 0$. *If* $t > 2t'$, *then* $t[1-(a+c)] \leq 0$, *a contradiction. Hence* $t \leq 2t'$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \dfrac{2c}{1-a} < 1$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_4 + t_5 + t_6)\}$, *where* $a \in (0, 1)$ *and* $b \in \left[0, \dfrac{1}{3}\right)$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u - \max\{av, b(2u + v)\} \leq 0$. *If* $u > v$, *then* $u[1 - \max\{a, 3b\}] \leq 0$, *a contradiction. Hence,* $u \leq v$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \max\{a, 3b\} < 1$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t - \max\{at, b(t + t')\} \leq 0$. *If* $t > t'$, *then* $t[1 - \max\{a, 2b\}] \leq 0$, *a contradiction. Hence* $t \leq t'$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \max\{a, 2b\} < 1$.

Example 3.9. $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6$, *where* $a, b, c, d \geq 0$ *and* $0 < a + b + c + d < 1$.

(F_1) : *Obviously.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, 0) = u^2 - u(av + bv + cu) \leq 0$. *If* $u > 0$, *then* $u - av - bv - cu \leq 0$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \dfrac{a+b}{1-c} < 1$. *If* $u = 0$, *then* $u \leq h_1 v$.

(F_3) : *Let* $t, t' \geq 0$ *and* $F(t, t, 0, 0, t, t') = t^2 - at^2 - ctt' \leq 0$, *which implies* $t \leq h_2 t'$, *where* $0 \leq h_2 = \dfrac{c}{1-a} < 1$.

Example 3.10. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - b\dfrac{t_5 t_6}{1 + t_3^2 + t_4^2}$, *where* $a, b \geq 0$ *and* $0 < a + b < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, u+v, 0) = u^2 - av^2 \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a} < 1$. If $u = 0$, then $u \leq h_1 v$.

(F_3) : Let $t, t' \geq 0$ and $F(t, t, 0, 0, t, t') = t^2(1-a) - btt' \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{b}{1-a} < 1$.

Example 3.11. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4 + t_5, t_1 + t_4 + t_5 + t_6\}$, where $a, b, c \geq 0$ and $0 < a + b + 4c < 1$.*

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, u+v, 0) = u - av - bv - c(3u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b+c}{1-3c} < 1$.

(F_3) : Let $t, t' \geq 0$ and $F(t, t, 0, 0, t, t') = t - at - c(t+t') \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{c}{1-(a+c)} < 1$.

Example 3.12. *$F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3 + t_4, t_5 + t_6\}$, where $0 \leq k < \dfrac{1}{2}$.*

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, u+v, 0) = u - k(u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{k}{1-k} < 1$.

(F_3) : Let $t, t' \geq 0$ and $F(t, t, 0, 0, t, t') = t - k(t+t') \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{k}{1-k} < 1$.

Example 3.13. *$F(t_1, ..., t_6) =$*

$$= t_1 - k\max\left\{t_2, t_3, t_4, \frac{2t_4 + t_6}{3}, \frac{2t_4 + t_5}{3}, \frac{t_5 + t_6}{2}\right\},$$ *where $0 \leq k < 1$.*

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ and

$F(u, v, v, u, u+v, 0) = u - k\max\left\{u, v, \dfrac{2u}{3}, \dfrac{2u+v}{3}, \dfrac{u+v}{3}\right\} \leq 0$.If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t, t' \geq 0$ and $F(t, t, 0, 0, t, t') = t - k\max\left\{t, \dfrac{t}{3}, \dfrac{t+t'}{3}\right\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence, $t \leq t'$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

4. Main results

Theorem 4.1. *Let (X,G) be a complete G - metric spaces and let $\{A_i\}_{i=1}^p$ be a family of nonempty closed subsets of X. Let $Y=\cup_{i=1}^p A_i$ and let $T:Y\to Y$ be a mapping satisfying*

$$T(A_i)\subset A_{i+1}, i=\overline{1,p}, \text{ where } A_{p+1}=A_1 \tag{4.1}$$

If the inequality

$$\begin{gathered} F(G(Tx,Ty,Ty),G(x,y,y),G(x,Tx,Tx),\\ G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx))\leq 0 \end{gathered} \tag{4.2}$$

holds for all $x\in A_i$, $y\in A_{i+1}$, $i=\overline{1,p}$ and $F\in\mathfrak{F}_G$.

Then T has an unique fixed point in $\cap_{i=1}^p A_i$.

Proof. Let x_0 be an arbitrary point in A_1. We define the sequence $x_n=Tx_{n-1}, n=1,2,\ldots$. Then, by (4.1) we have $x_0\in A_1$, $x_1=Tx_0\in A_2$, ..., $x_{p-1}=Tx_{p-2}\in A_p$, $x_p=Tx_{p-1}\in A_{p+1}=A_1$, $x_{p+1}=Tx_p\in A_{p+2}=A_2$, ...

By (4.2) we have successively

$$\begin{gathered} F(G(Tx_0,Tx_1,Tx_1),G(x_0,x_1,x_1),G(x_0,Tx_0,Tx_0),\\ G(x_1,Tx_1,Tx_1),G(x_0,Tx_1,Tx_1),G(x_1,Tx_0,Tx_0))\leq 0, \end{gathered}$$

$$\begin{gathered} F(G(x_1,x_2,x_2),G(x_0,x_1,x_1),G(x_0,x_1,x_1),\\ G(x_1,x_2,x_2),G(x_0,x_2,x_2),0)\leq 0. \end{gathered}$$

By (F_1) and (G_5) we obtain

$$\begin{gathered} F(G(x_1,x_2,x_2),G(x_0,x_1,x_1),G(x_0,x_1,x_1),\\ G(x_1,x_2,x_2),G(x_0,x_1,x_1)+G(x_1,x_2,x_2),0)\leq 0. \end{gathered}$$

By (F_2) we obtain

$$G(x_1,x_2,x_2)\leq h_1G(x_0,x_1,x_1).$$

Similarly, we obtain

$$G(x_{p-2},x_{p-1},x_{p-1})\leq h_1G(x_{p-3},x_{p-2},x_{p-2}).$$

Again, by (4.2) we obtain

$$\begin{aligned} G(x_{p-1},x_p,x_p) &\leq h_1G(x_{p-2},x_{p-1},x_{p-1})\\ &\ldots \end{aligned}$$

By induction we have

$$G(x_n,x_{n+1},x_{n+1})\leq h_1G(x_{n-1},x_n,x_n)\leq\ldots\leq h_1^nG(x_0,x_1,x_1).$$

By a routine computation, using (G_5), it follows that (x_n) is a G - Cauchy sequence in X. Since (X, G) is G - complete it follows that (x_n) is G - convergent to x^*. Then, also the subsequences

$$(x_{np+1}) \subset A_1,\ (x_{np+2}) \subset A_2,\ ..., (x_{(n+1)p-1}) \subset A_p,\ n = 0, 1, 2, ...$$

also converges to x^*. Hence $x^* \in \cap_{i=1}^{p} A_i$.

We prove that x^* is a fixed point of T.

Let $(x_n) \in A_{n+1}$ and $x^* \in \cap_{i=1}^{p} A_i$. Then by (4.2) we have successively

$$F(G(Tx_n, Tx^*, Tx^*), G(x_n, x^*, x^*), G(x_n, Tx_n, Tx_n),\\ G(x^*, Tx^*, Tx^*), G(x_n, Tx^*, Tx^*), G(x^*, Tx_n, Tx_n)) \leq 0,$$

$$F(G(x_{n+1}, Tx^*, Tx^*), G(x_n, x^*, x^*), G(x_n, x_{n+1}, x_{n+1}),\\ G(x^*, Tx^*, Tx^*), G(x_n, Tx^*, Tx^*), G(x^*, x_{n+1}, x_{n+1})) \leq 0.$$

Letting n tend to infinity we have

$$F(G(x^*, Tx^*, Tx^*), 0, 0, G(x^*, Tx^*, Tx^*), G(x^*, Tx^*, Tx^*), 0) \leq 0.$$

By (F_2) we have $G(x^*, Tx^*, Tx^*) = 0$, i.e. $x^* = Tx^*$. Hence, x^* is a fixed point and $x^* \in \cap_{i=1}^{p} A_i$. Suppose that there exists another fixed point $y^* \in \cap_{i=1}^{p} A_i$. Since, $x^*, y^* \in \cap_{i=1}^{p} A_i$, then $x^* \in A_1, y^* \in A_2$. By (4.2) we have successively

$$F(G(Tx^*, Ty^*, Ty^*), G(x^*, y^*, y^*), G(x^*, Tx^*, Tx^*),\\ G(y^*, Ty^*, Ty^*), G(x^*, Ty^*, Ty^*), G(y^*, Tx^*, Tx^*)) \leq 0,$$

$$F(G(x^*, y^*, y^*), G(x^*, y^*, y^*), 0, 0, G(x^*, y^*, y^*), G(y^*, x^*, x^*)) \leq 0.$$

By (F_3) we obtain

$$G(x^*, y^*, y^*) \leq h_2 G(y^*, x^*, x^*).$$

Since, $x^*, y^* \in \cap_{i=1}^{p} A_i$, then $y^* \in A_2$ and $x^* \in A_3$. By (4.2) we obtain

$$G(y^*, x^*, x^*) \leq h_2 G(x^*, y^*, y^*).$$

Hence, $G(x^*, y^*, y^*)(1 - h_2^2) \leq 0$, a contradiction. Therefore, $x^* = y^*$. □

Remark 4.2. *1) By Theorem 4.1 and Example 3.7 with $a = b = 0$ we obtain a cyclical extension of Theorem 2.7.*

2) By Theorem 4.1 and Example 3.8 we obtain a cyclical extension of Theorem 2.8.

3) By Theorem 4.1 and Example 3.2 with $a = \alpha$, $b = c = d = e = 0$ we obtain an extension of Theorem 2.9 in G - metric spaces.

4) By Theorem 4.1 and Example 3.2 with $a = b = c = 0$ *and* $d = e = \alpha$ *we obtain an extension of Theorem 2.11 in G - metric spaces.*

5) By Theorem 4.1 and Example 3.2 with $a = \alpha$, $b = c = \beta$, $d = e = 0$ *we obtain an extension of Theorem 2.12 in G - metric spaces.*

6) By Theorem 4.1 and Example 3.2 with $d = e = 0$ *we obtain an extension of Theorem 2.13 a) in G - metric spaces.*

7) By Theorem 4.1 and Example 3.5 with $c = 0$ *we obtain an extension of Theorem 2.13 b) in G - metric spaces.*

8) By Theorem 4.1 and Example 3.5 with $b = 0$ *we obtain an extension of Theorem 2.13 c) in G - metric spaces.*

9) By Theorem 4.1 and Example 3.6 we obtain an extension of Theorem 2.14 in G - metric spaces.

10) By Theorem 4.1 and Example 3.3 we obtain an extension of Theorem 2.15 in G - metric spaces.

11) By Theorem 4.1 and Example 3.8 we obtain an extension of Theorem 2.17 in G - metric spaces.

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