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$\omega\beta-$SEPARATION AXIOMS

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Abstract The separation axioms are important and interesting concepts in topological spaces. In this paper we introduce and investigate some weak separation axioms by using the notion of $\omega\beta-$open set.

1. Introduction

The notion of $\omega\beta-$open set was introduced by Aljarrah and Noorani [3]. In [2, 3] $\omega\beta-\mathrm{T}_1$ and $\omega\beta-\mathrm{T}_2$ were defined, respectively, by replacing the word "open" in the definition of T_1, T_2 by "$\omega\beta-$open". In this paper, we offer some new low separation axioms by utilizing $\omega\beta-$open sets. We also obtain their fundamental properties. Throughout the present paper, a space (X,τ) means a topological space on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a space (X,τ). The closure of A and interior of A in (X,τ) are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space (X,τ) is said to be $\beta-$open [6] if $A \subseteq Cl(Int(Cl(A)))$.

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Recall that a subset A of a space (X, τ) is said to be $\omega\beta-$open [3] (resp. $\omega-$open [5]) if for every $x \in A$ there exists a $\beta-$open (resp. open) set U containing x such that $U - A$ is countable. The complement of an $\omega\beta-$open set is said to be $\omega\beta-$closed [3]. The intersection of all $\omega\beta-$closed sets of X containing A is called the $\omega\beta-$closure of A and is denoted by $\omega\beta Cl(A)$. The union of all $\omega\beta-$open sets of X contained in A is called the $\omega\beta-$interior of A and is denoted by $\omega\beta Int(A)$. A subset A is called an $\omega\beta-$neighborhood of a point $x \in X$ [1] if there exists an $\omega\beta-$open set U containing x such that $U \subseteq A$.

In this paper, first we give characterizations of $\omega\beta - \mathrm{T}_i$ spaces for $i = 0, 1, 2$. Second, we give characterizations of $\omega\beta - R_i$ spaces for $i = 1, 2$. Finally, we introduce the notion of $D_{\omega\beta}-$sets as the difference sets of $\omega\beta-$open sets and investigate some preservation theorems. We recall some definitions used in the sequel.

Definition 1.1. [4] A subset $A \subseteq X$ is called a generalized $\omega\beta-$closed set (briefly $g\omega\beta-$closed) if $\omega\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). Note that every $\omega\beta-$closed set is $g\omega\beta-$closed.

Remember that a space (X, τ) is called an $\omega\beta - \mathrm{T}_{1/2}$ [4] space if every generalized $\omega\beta-$closed set is $\omega\beta-$closed.

Theorem 1.2. [4] *A space (X, τ) is an $\omega\beta - \mathrm{T}_{1/2}$ space if and only if every singleton is either closed or $\omega\beta-$open.*

Lemma 1.3. [3] *Let (X, τ) be a topological space. The following properties hold:*

(i) The union of any family of $\omega\beta-$open sets is $\omega\beta-$open.

(ii) The intersection of an $\omega\beta-$open set and an $\omega-$open set is $\omega\beta-$open.

Definition 1.4. [1] A function $f : (X, \tau) \to (Y, \sigma)$ is called $\omega\beta-$irresolute if $f^{-1}(V)$ is $\omega\beta-$open in (X, τ) for every $\omega\beta-$open set V in (Y, σ).

Lemma 1.5. [1] *A function $f : (X, \tau) \to (Y, \sigma)$ is $\omega\beta-$closed if and only if $\omega\beta Cl(f(A)) \subseteq f(\omega\beta Cl(A))$ for each subset A of X.*

Definition 1.6. [4] Let A be a subset of a space X. A point $x \in X$ is said to be a $\omega\beta-$limit point of A if for each $\omega\beta-$open set U containing x, we have $U \cap (A - \{x\}) \neq \phi$. The set of all $\omega\beta-$limit points of A is called the $\omega\beta-$derived set of A and is denoted by $D_{\omega\beta}(A)$.

Lemma 1.7. [4] *If $D(A) = D_{\omega\beta}(A)$, then we have $Cl(A) = \omega\beta Cl(A)$, where $D(A)$ is the derived set of A.*

2. $\omega\beta - \mathrm{T}_i$ SPACES

In this section we give some properties and characterizations of $\omega\beta-\mathrm{T}_0$, $\omega\beta - \mathrm{T}_1$ and $\omega\beta - \mathrm{T}_2$ spaces.

Definition 2.1. A space (X, τ) is said to be:

(i) $\omega\beta - \mathrm{T}_0$ if for each pair of distinct point x and y in X, there exist an $\omega\beta-$open set U such that $x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$.

(ii) $\omega\beta - T_1$ [2] if for each pair of distinct point x and y in X, there exist two $\omega\beta-$open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

(iii) $\omega\beta - T_2$ [3] if for each pair of distinct point x and y in X, there exist two $\omega\beta-$open sets U and V containing x and y, respectively, such that $U \cap V = \phi$.

Theorem 2.2. *A space (X, τ) is $\omega\beta - \mathrm{T}_0$ if and only if $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$ for each x, $y \in X$ and $x \neq y$.*

Proof. Necessity. Let (X, τ) be an $\omega\beta - \mathrm{T}_0$ space and x, $y \in X$ such that $x \neq y$. Then there exists an $\omega\beta-$open set U containing x, say but not y, and therefore $X - U$ is an $\omega\beta-$closed set which contains y but not x. Since $\omega\beta Cl(\{y\})$ is the smallest $\omega\beta-$closed set containing y, $\omega\beta Cl(\{y\}) \subseteq X - U$, and so $x \notin \omega\beta Cl(\{y\})$. Thus, $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$.
Sufficiency. Suppose that $x, y \in X$ and $x \neq y$. Then by assumption, $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Let $z \in X$ such that $z \in \omega\beta Cl(\{x\})$ and $z \notin \omega\beta Cl(\{y\})$, say. We claim that $x \notin \omega\beta Cl(\{y\})$. For, if $x \in \omega\beta Cl(\{y\})$, then $\omega\beta Cl(\{x\}) \subseteq \omega\beta Cl(\{y\})$, which contradicts to $z \notin \omega\beta Cl(\{y\})$. Thus, $x \in (\omega\beta Cl(\{y\}))^c$, but $(\omega\beta Cl(\{y\}))^c$ is an $\omega\beta-$open set that does not contain y. Hence, (X, τ) is $\omega\beta - \mathrm{T}_0$. ∎

Theorem 2.3. *For a topological space (X, τ), the following properties are equivalent:*

(i) (X, τ) is an $\omega\beta - \mathrm{T}_1$ space.

(ii) For each $x \in X$, $\{x\}$ is an $\omega\beta-$closed set.

(iii) Each subset of X is the intersection of all $\omega\beta-$open sets containing it.

Proof. (i) $\rightarrow$ (ii) Suppose that (X, τ) is $\omega\beta - \mathrm{T}_1$ and $x \in X$. Let $y \in \{x\}^c$. Then $x \neq y$ and so there exists an $\omega\beta-$open set U_y such

that $y \in U_y$ but $x \notin U_y$. Therefore, $y \in U_y \subseteq \{x\}^c$, i.e. $\{x\}^c = \cup\{U_y : y \in \{x\}^c\}$ which is $\omega\beta-$open.
(ii) $\to$ (iii) Suppose that for each $x \in X$, $\{x\}$ is an $\omega\beta-$closed set. Given $A \subseteq X$ and define the set A^* as follows: $A^* = \cap\{U : A \subseteq U \text{ and} U \text{ is } \omega\beta \text{ −open}\}$. We will prove that $A = A^*$. In general, $A \subset A^*$. Suppose that $x \notin A$. Then $A \subseteq X-\{x\}$ and $X-\{x\}$ is $\omega\beta-$open. Therefore, $x \notin A^*$ and hence $A^* \subseteq A$. So $A = A^*$.
(iii) $\to$ (i) Let $A^*_x = \{U : x \in U \text{ and} U \text{ is } \omega\beta - \text{open}\}$. By hypothesis, $\{x\} = \underset{U \in A^*_x}{\cap} U$. Therefore if $y \neq x$ then $y \notin \underset{U \in A^*_x}{\cap} U$ and there exists an $\omega\beta-$open set U such that $x \in U$ and $y \notin U$. Analogously, if $x \notin \underset{U_1 \in A^*_y}{\cap} U_1$ and there exists an $\omega\beta-$open set U_1 such that $y \in U_1$ and $x \notin U_1$. This shows that (X, τ) is an $\omega\beta - \mathrm{T}_1$ space. ∎

Proposition 2.4. *Let (X, τ) be a topological space.*
(i) If (X, τ) is $\omega\beta - \mathrm{T}_i$, then it is $\omega\beta - \mathrm{T}_{i-1}$, $i = 1, 2$.
(ii) If (X, τ) is $\omega\beta - \mathrm{T}_i$, then it is $\omega\beta - \mathrm{T}_{i-1/2}$, $i = 1, 1/2$, provided that $D(A) = D_{\omega\beta}(A)$.

Proof. (i) The proof is obvious.
(ii) If $i = 1$, then we claim that every $\omega\beta - \mathrm{T}_1$ space is $\omega\beta - \mathrm{T}_{1/2}$. Let (X, τ) be $\omega\beta-\mathrm{T}_1$. Then by Theorem 2.3, for each $x \in X$, the singleton $\{x\}$ is $\omega\beta-$closed. Therefore, by Lemma 1.7 and Theorem 1.2, (X, τ) is $\omega\beta - \mathrm{T}_{1/2}$. And if $i = 1/2$, we claim that every $\omega\beta - \mathrm{T}_{1/2}$ space is $\omega\beta - \mathrm{T}_0$. Let $x, y \in X$ with $x \neq y$. By Theorem 1.2, the singleton $\{x\}$ is an $\omega\beta-$open or closed. If $\{x\}$ is $\omega\beta-$open, $x \in \{x\}$ and $y \notin \{x\}$. If $\{x\}$ is closed, then $X - \{x\}$ is open and hence $\omega\beta-$open, $y \in X - \{x\}$ and $x \notin X - \{x\}$. Therefore, (X, τ) is $\omega\beta - \mathrm{T}_0$. ∎

The converse of (i) in Proposition 2.4 need not be true in general as the following example shows.

Example 2.5. *Let $X = \mathbb{R}$ and τ be a topology on X such that the all $\omega\beta-$open set has the form $(-\infty, a)$ where $a \in \mathbb{R}$. For any two distant point x, y where $x < y$, there exists an $\omega\beta-$open set $(-\infty, y)$ such that $x \in (-\infty, y)$ but not y, which means that the space is $\omega\beta - \mathrm{T}_0$. However, there is no way of getting an $\omega\beta-$open set containing y to exclude x. So that the space is not $\omega\beta - \mathrm{T}_1$.*

Example 2.6. *Let $X = \mathbb{R}$ and τ be a topology on X such that every $\omega\beta-$open set can we write as $\{U \subseteq \mathbb{R} : \mathbb{R} - U finite\}$. It is easy to check that the space is $\omega\beta - \mathrm{T}_1$. However, it is not $\omega\beta - \mathrm{T}_2$ because the intersection of any two $\omega\beta-$open set are non empty.*

Definition 2.7. A topological space (X,τ) is said to be $\omega\beta-$symmetric if for each $x,y\in X$, $x\in\omega\beta Cl(\{y\})$ implies $y\in\omega\beta Cl(\{x\})$.

Proposition 2.8. *Let (X,τ) be a topological space, then the following properties hold:*

(i) (X,τ) is $\omega\beta-$symmetric if and only if the singleton $\{x\}$ is $g\omega\beta-$closed for each $x\in X$.

(ii) (X,τ) is $\omega\beta-$symmetric and $\omega\beta-\mathrm{T}_0$ if and only if (X,τ) is $\omega\beta-\mathrm{T}_1$.

Proof. (i) Suppose that U is an $\omega\beta-$open set such that $\{x\}\subseteq U$ but $\omega\beta Cl(\{x\})\not\subseteq U$. This means that $\omega\beta Cl(\{x\})\cap(X-U)\neq\phi$. Let $y\in\omega\beta Cl(\{x\})\cap(X-U)$. Now we have $x\in\omega\beta Cl(\{y\})\subseteq(X-U)$ and $x\notin U$. But this is a contradiction. Conversely, Assume that $x\in\omega\beta Cl(\{y\})$ but $y\notin\omega\beta Cl(\{x\})$. This means that $(\omega\beta Cl(\{x\}))^c$ contains y. This implies that $\omega\beta Cl(\{y\})\subseteq(\omega\beta Cl(\{x\}))^c$. Now $(\omega\beta Cl(\{x\}))^c$ contains x which is a contradiction.
(ii) Let $x,y\in X$ such that $x\neq y$. Since (X,τ) is $\omega\beta-\mathrm{T}_0$, we can assume that $x\in U$ and $y\notin U$ for some $\omega\beta-$open set U. Then $x\notin\omega\beta Cl(\{y\})$ and hence $y\notin\omega\beta Cl(\{x\})$. Therefore, there exist an $\omega\beta-$open set V such that $y\in V$ and $x\notin V$. This shows that (X,τ) is $\omega\beta-\mathrm{T}_1$. Conversely, by Theorem 2.3, every singleton set is $\omega\beta-$closed and therefore $g\omega\beta-$closed . By (i) the space (X,τ) is $\omega\beta-$symmetric. Also, by Proposition 2.4, the space (X,τ) is $\omega\beta-\mathrm{T}_0$. ∎

Theorem 2.9. *Let (X,τ) be an $\omega\beta-$symmetric space. Then the following are equivalent:*

(i) (X,τ) is $\omega\beta-\mathrm{T}_0$.

(ii) (X,τ) is $\omega\beta-\mathrm{T}_{1/2}$.

(iii) (X,τ) is $\omega\beta-\mathrm{T}_1$.

Proof. The proof follows from Propositions 2.4 and 2.8. ∎

3. $\omega\beta-R_0$ AND $\omega\beta-R_1$ SPACES

In this section we introduce separation axioms $\omega\beta-R_0$ and $\omega\beta-R_1$ in a topological space and study some of their properties.

Definition 3.1. A topological space (X,τ) is said to be

(i) $\omega\beta-R_0$ if every $\omega\beta-$open set contains the $\omega\beta-$closure of each of its singletons.

(ii) $\omega\beta-R_1$ if for each points $x, y \in X$ with $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$, there exist disjoint $\omega\beta-$open sets U and V such that $\omega\beta Cl(\{x\}) \subseteq U$ and $\omega\beta Cl(\{y\}) \subseteq V$.

Theorem 3.2. *If (X, τ) is $\omega\beta - R_1$, then it is $\omega\beta - R_0$.*

Proof. Let U be an $\omega\beta-$open set and $x \in U$. If $y \notin U$, then $x \notin \omega\beta Cl(\{y\})$ and thus $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. So there exists an $\omega\beta-$open set V_y such that $\omega\beta Cl(\{y\}) \subseteq V_y$ and $x \notin V_y$. Thus, $y \notin \omega\beta Cl(\{x\})$. Therefore, $\omega\beta Cl(\{x\}) \subseteq U$, hence (X, τ) is $\omega\beta - R_0$. ∎

Example 2.5 shows that the converse of Theorem 3.2 need not be true in general, also we will see that in Corollary 3.22.

Theorem 3.3. *A topological space (X, τ) is $\omega\beta - R_0$ if and only if (X, τ) is $\omega\beta-$symmetric.*

Proof. Necessity. Assume that (X, τ) is $\omega\beta - R_0$. Let $x \in \omega\beta Cl(\{y\})$ and let U be any $\omega\beta-$open set such that $y \in U$. Then by hypothesis, $x \in U$. Therefore, every $\omega\beta-$open set which contains y contains x. Hence, $y \in \omega\beta Cl(\{x\})$.
Sufficiency. Let U be an $\omega\beta-$open set and $x \in U$. If $y \notin U$, then $x \notin \omega\beta Cl(\{y\})$ and thus by assumption, $y \notin \omega\beta Cl(\{x\})$. Therefore, $\omega\beta Cl(\{x\}) \subseteq U$ and hence (X, τ) is $\omega\beta - R_0$. ∎

Proposition 3.4. *A topological space (X, τ) is $\omega\beta - \mathrm{T}_1$ if and only if one of the following hold:*

(i) it is $\omega\beta - \mathrm{T}_0$ and $\omega\beta - R_0$.

(ii) it is $\omega\beta - \mathrm{T}_{1/2}$ and $\omega\beta - R_0$.

Proof. (i) This follows from Proposition 2.8 and Theorem 3.3.
(ii) This follows from (i) and Proposition 2.4. ∎

It is clear from Proposition 3.4 (part i) that if (X, τ) is $\omega\beta - \mathrm{T}_0$ but not $\omega\beta - \mathrm{T}_1$, then (X, τ) is not $\omega\beta - R_0$ (see Example 2.5).

Proposition 3.5. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) is an $\omega\beta - R_0$ space.

(ii)For each $\omega\beta-$closed set F with $x \notin F$, then $F \subseteq U$ and $x \notin U$ for some $\omega\beta-$open set U.

(iii) For any $\omega\beta-$closed set F with $x \notin F$, $F \cap \omega\beta Cl(\{x\}) = \phi$.

(iv) For any $x, y \in X$ with $x \neq y$, either $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{y\})$ or $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$.

Proof. (i) $\rightarrow$ (ii) Let F be an $\omega\beta-$closed set such that $x \notin F$. Then by (i) $\omega\beta Cl(\{x\}) \subseteq X - F$. Set $U = X - \omega\beta Cl(\{x\})$, then U is an $\omega\beta-$open set with $F \subseteq U$ and $x \notin U$.
(ii) $\rightarrow$ (iii) Let F be an $\omega\beta-$closed set such that $x \notin F$. Then by (ii), there exists an $\omega\beta-$open set U such that $F \subseteq U$ and $x \notin U$. Since U is $\omega\beta-$open, $U \cap \omega\beta Cl(\{x\}) = \phi$ and hence $F \cap \omega\beta Cl(\{x\}) = \phi$.
(iii) $\rightarrow$ (iv) Let $x, y \in X$ and $x \neq y$ such that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Then there exists a point $z \in \omega\beta Cl(\{x\})$ such that $z \notin \omega\beta Cl(\{y\})$ (or, $z \in \omega\beta Cl(\{y\})$) such that $z \notin \omega\beta Cl(\{x\})$. Thus there exists an $\omega\beta-$open set V containing z such that $y \notin V$ and $x \in V$. Thus $x \notin \omega\beta Cl(\{y\})$. Now by (iii), we have $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$. The proof for the other case is similar.
(iv) $\rightarrow$ (i) Let V be an $\omega\beta-$open set with $x \in V$. We need to show that $\omega\beta Cl(\{x\}) \subseteq V$. Let $y \notin V$. Then $x \neq y$ and $x \notin \omega\beta Cl(\{y\})$. Thus $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Hence by (iv), $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$ for each $y \notin V$ and hence $(\omega\beta Cl(\{x\})) \cap [\bigcup_{y \in X-V} \omega\beta Cl(\{y\})] = \phi$. Again, since V is an $\omega\beta-$open set and $y \in X - V$, we have $y \in \omega\beta Cl(\{y\}) \subseteq (X - V)$. Thus we have $X - V = \bigcup_{y \in X-V} \omega\beta Cl(\{y\})$. Therefore, we obtain $(X - V) \cap \omega\beta Cl(\{x\}) = \phi$ and hence $\omega\beta Cl(\{x\}) \subseteq V$. ∎

Definition 3.6. Let A be a subset of a topological space and $x \in X$.
(i) $\omega\beta-$kernel of A, denoted by $\ker_{\omega\beta}(A)$, is defined to be the set $\ker_{\omega\beta}(A) = \cap\{U : U \text{ is } \omega\beta - \text{open and A} \subseteq U\}$.
(ii) $\prec x \succ_{\omega\beta} = \omega\beta Cl(\{x\}) \cap \ker_{\omega\beta}(\{x\})$.

Note that if there is no $\omega\beta-$open set containing A, then $\ker_{\omega\beta}(A) = X$.

Lemma 3.7. *Let (X, τ) be a topological space and $A, B \subseteq X$. Then*
(i) $A \subseteq B$ implies $\ker_{\omega\beta}(A) \subseteq \ker_{\omega\beta}(B)$.
(ii) $\ker_{\omega\beta}(\ker_{\omega\beta}(A)) = \ker_{\omega\beta}(A)$.
(iii) $\ker_{\omega\beta}(A) = \{x \in X : \omega\beta Cl(\{x\}) \cap A \neq \phi\}$.
(iv) For each $x \in X$, $y \in$ $\ker_{\omega\beta}(\{x\})$ *if and only if $x \in \omega\beta Cl(\{y\})$.*
(v) For each $x \in X$, $\ker_{\omega\beta}(\prec x \succ_{\omega\beta}) = \ker_{\omega\beta}(\{x\})$.
(vi) For each $x \in X$, $\omega\beta Cl(\prec x \succ_{\omega\beta}) = \omega\beta Cl(\{x\})$.
(vii) For each $\omega\beta-$open set $U \subseteq X$, if $x \in U$ then $\prec x \succ_{\omega\beta} \subseteq U$.
(viii) For each $\omega\beta-$closed set $F \subseteq X$, if $x \in F$ then $\prec x \succ_{\omega\beta} \subseteq F$.

Proof. The proofs of (i) and (ii) are obvious.
(iii) Let $x \in \ker_{\omega\beta}(A)$ and $\omega\beta Cl(\{x\}) \cap A = \phi$. Hence $x \notin X - \omega\beta Cl(\{x\})$ which is an $\omega\beta-$open set containing A. Hence $x \notin \ker_{\omega\beta}(A)$

which is a contradiction. Consequently, $\omega\beta Cl(\{x\}) \cap A \neq \phi$. Next suppose that $x \notin \ker_{\omega\beta}(A)$, there exists an $\omega\beta-$open set U such that $A \subseteq U$ and $x \notin U$. Then $(\omega\beta Cl(\{x\})) \cap U = \phi$, thus $\omega\beta Cl(\{x\}) \cap A = \phi$. We obtain the claim.
(iv) Suppose that $y \notin \ker_{\omega\beta}(\{x\})$. Then there exists an $\omega\beta-$open set V containing x such that $y \notin V$. Therefore, we have $x \notin \omega\beta Cl(\{y\})$. The proof of the converse case can be done similarly.
(v) The proof follows easily from Definition 3.6 and Lemma 3.7 (i) and (ii).
(vi) The proof is quite similar to that of (v).
(vii) Since $x \in U$ and U is an $\omega\beta-$open set, we have $\ker_{\omega\beta}(\{x\}) \subseteq U$. Hence $\prec x \succ_{\omega\beta} \subseteq U$.
(viii) Since $x \in F$ and F is an $\omega\beta-$closed set, we have that $\prec x \succ_{\omega\beta} = \omega\beta Cl(\{x\}) \cap ker_{\omega\beta}(\{x\}) \subseteq \omega\beta Cl(\{x\}) \subseteq F$. ∎

Lemma 3.8. *For any points x and y in a topological space (X, τ), the following statements are equivalent:*
(i) $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$.
(ii) $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$.

Proof. (i) $\rightarrow$ (ii) Suppose that $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$, then there exists a point $z \in X$ such that $z \in \ker_{\omega\beta}(\{x\})$ and $z \notin \ker_{\omega\beta}(\{y\})$, it follows from $z \in \ker_{\omega\beta}(\{x\})$ that $x \in \omega\beta Cl(\{z\})$ by Lemma 3.7. By $z \notin \ker_{\omega\beta}(\{y\})$, we have $\{y\} \cap \omega\beta Cl(\{z\}) = \phi$. Since $x \in \omega\beta Cl(\{z\})$, $\omega\beta Cl(\{x\}) \subseteq \omega\beta Cl(\{z\})$ and $\{y\} \cap \omega\beta Cl(\{x\}) = \phi$. Therefore, $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$.
(ii) $\rightarrow$ (i) Suppose that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. There exists a point $z \in X$ such that $z \in \omega\beta Cl(\{x\})$ and $z \notin \omega\beta Cl(\{y\})$. Then, there exists an $\omega\beta-$open set U containing z such that $x \in U$ and $y \notin U$, *i.e* $y \notin \ker_{\omega\beta}(\{x\})$. Hence $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$. ∎

Theorem 3.9. *A topological space (X, τ) is $\omega\beta - R_0$ if and only if for any x, $y \in X$, one of the following hold:*
(i) $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$ *implies* $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$.
(ii) $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$ *implies* $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\}) = \phi$.

Proof. (i) Necessity. Suppose that (X, τ) is $\omega\beta - R_0$ and $x, y \in X$ such that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Then, there exist $z \in \omega\beta Cl(\{x\})$ such that $z \notin \omega\beta Cl(\{y\})$ (or $z \in \omega\beta Cl(\{y\})$ such that $z \notin \omega\beta Cl(\{x\})$). There exists an $\omega\beta-$open set V in (X, τ) such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \omega\beta Cl(\{y\})$. Thus

$x \in X - \omega\beta Cl(\{y\})$ which is $\omega\beta-$open in (X,τ), so $\omega\beta Cl(\{x\}) \subseteq X - \omega\beta Cl(\{y\})$ and $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$. The proof for otherwise is similar.

Sufficiency. Let V be $\omega\beta-$open in (X,τ) and let $x \in V$. We will show that $\omega\beta Cl(\{x\}) \subseteq V$. Suppose that $y \notin V$, *i.e*, $y \in X - V$. Then $x \neq y$ and $x \notin \omega\beta Cl(\{y\})$. Thus $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. By assumption, $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$. Hence $y \notin \omega\beta Cl(\{x\})$ and hence $\omega\beta Cl(\{x\}) \subseteq V$.

(ii) Necessity. Suppose that (X,τ) is an $\omega\beta - R_0$ space. Thus by Lemma 3.8, for any points $x, y \in X$ if $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$ then $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Now we prove that $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\}) = \phi$. Assume that $z \in \ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\})$. By $z \in \ker_{\omega\beta}(\{x\})$ and Lemma 3.7, it follows that $x \in \omega\beta Cl(\{z\})$. Since $x \in \omega\beta Cl(\{x\})$, by (i) $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{z\})$. Similarly, we have $\omega\beta Cl(\{y\}) = \omega\beta Cl(\{z\}) = \omega\beta Cl(\{x\})$. This is a contradiction. Therefore, we have $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\}) = \phi$.

Sufficiency. Let (X,τ) be a topological space such that for any points $x, y \in X$, $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$ implies $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\}) = \phi$. If $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$, then by Lemma 3.8, $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$. Hence $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{y\}) = \phi$ which implies $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$. Because $z \in \omega\beta Cl(\{x\})$ implies that $x \in \ker_{\omega\beta}(\{z\})$. Therefore, $\ker_{\omega\beta}(\{x\}) \cap \ker_{\omega\beta}(\{z\}) \neq \phi$. By hypothesis, we have $\ker_{\omega\beta}(\{x\}) = \ker_{\omega\beta}(\{z\})$. Then $z \in \omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\})$ implies that $\ker_{\omega\beta}(\{x\}) = \ker_{\omega\beta}(\{z\}) = \ker_{\omega\beta}(\{y\})$. This is a contradiction. Hence, $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$. By (i) the space (X,τ) is a $\omega\beta - R_0$ space. ∎

Theorem 3.10. *For a topological space (X,τ), the following properties are equivalent:*

(i) (X,τ) is an $\omega\beta - R_0$ space.

(ii) For any $A \neq \phi$ and an $\omega\beta-$open set U in (X,τ) such that $A \cap U \neq \phi$, there exists an $\omega\beta-$closed set F in (X,τ) such that $A \cap F \neq \phi$ and $F \subseteq U$.

(iii) For any $\omega\beta-$open set U in (X,τ), $U = \cup\{F : F \text{ is } \omega\beta-\text{closed}, F \subseteq U\}$.

(iv) For any $\omega\beta-$closed set F in (X,τ), $F = \cap\{U : U \text{ is } \omega\beta-\text{open}, F \subseteq U\}$.

(v) For any $x \in X$, $\omega\beta Cl(\{x\}) \subseteq \ker_{\omega\beta}(\{x\})$.

Proof. (i) $\to$ (ii) Let A be a non-empty set of (X,τ) and U be any $\omega\beta-$open set in (X,τ) such that $A\cap U\neq\phi$. There exists $x\in A\cap U$. Since $x\in U$ and U is $\omega\beta-$open in (X,τ), $\omega\beta Cl(\{x\})\subseteq U$. Set $F=\omega\beta Cl(\{x\})$, then F is $\omega\beta-$closed in (X,τ), $F\subseteq U$ and $A\cap F\neq\phi$.
(ii) $\to$ (iii) Let U be any $\omega\beta-$open set in (X,τ), then $\cup\{F: F\,\text{is}\,\omega\beta-\text{closed}\,\text{in}\,(X,\tau),\ F\subseteq U\}\subseteq U$. Let $x\in U$, there exists an $\omega\beta-$closed set F in (X,τ) such that $x\in F$ and $F\subseteq U$. Therefore, we have $x\in F\subseteq\cup\{F: F\,\text{is}\,\omega\beta-\text{closed}\,\text{in}\,(X,\tau),\ F\subseteq U\}$ and hence $U=\cup\{F: F\,\text{is}\,\omega\beta-\text{closed}\,\text{in}\,(X,\tau),\ F\subseteq U\}$.
(iii) $\to$ (iv) This is obvious.
(iv) $\to$ (v) Let $x\in X$ and $y\notin \ker_{\omega\beta}(\{x\})$. There exists an $\omega\beta-$open set V in (X,τ) such that $x\in V$ and $y\notin V$; hence $\omega\beta Cl(\{y\})\cap V=\phi$. By (iv) $(\cap\{U: U\,\text{is}\,\omega\beta-\text{open}\,\text{in}\,(X,\tau),\ \omega\beta Cl(\{y\})\subseteq U\})\cap V=\phi$ and hence there exists an $\omega\beta-$open sets U_0 such that $\omega\beta Cl(\{y\})\subseteq U_0$ and $x\notin U_0$. Therefore, $\omega\beta Cl(\{x\})\cap U_0=\phi$ and $y\notin\omega\beta Cl(\{x\})$. Consequently, we obtain $\omega\beta Cl(\{x\})\subseteq\ker_{\omega\beta}(\{x\})$.
(v) $\to$ (i) Let U be any $\omega\beta-$open in (X,τ) and $x\in U$. Suppose $y\in\ker_{\omega\beta}(\{x\})$, then $x\in\omega\beta Cl(\{y\})$ and $y\in U$. This implies that $\omega\beta Cl(\{x\})\subseteq\ker_{\omega\beta}(\{x\})\subseteq U$. Therefore, (X,τ) is an $\omega\beta-R_0$ space. ∎

Theorem 3.11. *For a topological space (X,τ), the following properties are equivalent:*
(i)(X,τ) is an $\omega\beta-R_0$ space.
(ii) If F is $\omega\beta-$closed and $x\in F$, then $\ker_{\omega\beta}(\{x\})\subseteq F$.
(iii) If $x\in X$, then $\ker_{\omega\beta}(\{x\})\subseteq\omega\beta Cl(\{x\})$.

Proof. (i) $\to$ (ii) Let F be $\omega\beta-$closed and $x\in F$. Then $\ker_{\omega\beta}(\{x\})\subseteq\ker_{\omega\beta}(F)$. By (i), it follows from Theorem 3.10 that $\ker_{\omega\beta}(F)=F$. Thus, $\ker_{\omega\beta}(\{x\})\subseteq F$.
(ii) $\to$ (iii) Since $x\in\omega\beta Cl(\{x\})$ and $\omega\beta Cl(\{x\})$ is $\omega\beta-$closed, by (ii), $\ker_{\omega\beta}(\{x\})\subseteq\omega\beta Cl(\{x\})$.
(iii) $\to$ (i) Let $x\in\omega\beta Cl(\{y\})$. Then by Lemma 3.7, $y\in\ker_{\omega\beta}(\{x\})$. By (iii), $y\in\omega\beta Cl(\{x\})$. Therefore, $x\in\omega\beta Cl(\{y\})$ implies that $y\in\omega\beta Cl(\{x\})$. Hence by Theorem 3.3 (X,τ) is $\omega\beta-R_0$. ∎

Corollary 3.12. *Let (X,τ) be a topological space, then the following properties hold for all $x\in X$:*
(i) (X,τ) is $\omega\beta-R_0$ if and only if $\omega\beta Cl(\{x\})=\ker_{\omega\beta}(\{x\})$.
(ii) If (X,τ) is $\omega\beta-R_0$ and $\prec x\succ_{\omega\beta}=\{x\}$, then $\omega\beta Cl(\{x\})=\{x\}$.

Proof. (i) The proof follows from Theorems 3.10 and 3.11.
(ii) This is a consequence of (i). ∎

Proposition 3.13. *For a topological space (X,τ), the following properties are equivalent:*

(i) (X,τ) is an $\omega\beta - R_0$ space.

(ii) For each $x \in X$, $\prec x \succ_{\omega\beta} = \omega\beta Cl(\{x\})$.

(iii) For each $x \in X$, $\prec x \succ_{\omega\beta}$ is $\omega\beta-$closed.

Proof. (i) $\rightarrow$ (ii) By Corollary 3.12, $\omega\beta Cl(\{x\}) = \ker_{\omega\beta}(\{x\})$ for each $x \in X$. Hence $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{x\}) \cap \ker_{\omega\beta}(\{x\}) = \prec x \succ_{\omega\beta}$.
(ii) $\rightarrow$ (i) Since $\omega\beta Cl(\{x\}) = \prec x \succ_{\omega\beta}$ for each $x \in X$, we have $\omega\beta Cl(\{x\}) \subseteq \ker_{\omega\beta}(\{x\})$. By Theorem 3.10, (X,τ) is $\omega\beta - R_0$.
(ii) $\leftrightarrow$ (iii) This is a consequence of Lemma 3.7. ∎

Definition 3.14. A net $\{x\alpha\}_{\alpha\in\Delta}$ in a topological space (X,τ) is called $\omega\beta-$convergent to a point $x \in X$, if for any $\omega\beta-$open subset U of (X,τ) containing x, there exists $\alpha_\circ \in \Delta$ such that $x\alpha \in U$ for each $\alpha_\circ \leq \alpha$.

Lemma 3.15. *Let (X,τ) be a topological space and let x and y be any two points of X such that every net in X $\omega\beta-$converging to y $\omega\beta-$converges to x. Then $x \in \omega\beta Cl(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a net in X that $\omega\beta-$converges to y. Thus by assumption $\{x_n\}_{n\in\mathbb{N}}$ $\omega\beta-$converges to x. Hence, $x \in \omega\beta Cl(\{y\})$. ∎

Theorem 3.16. *For a topological space (X,τ), the following statements are equivalent:*

(i) (X,τ) is an $\omega\beta - R_0$ space.

(ii) If $x, y \in X$, then $y \in \omega\beta Cl(\{x\})$ if and only if every net in (X,τ) $\omega\beta-$converging to y $\omega\beta-$converges to x.

Proof. (i $\rightarrow$ ii) Let $x, y \in X$ be such that $y \in \omega\beta Cl(\{x\})$. Let $\{x\alpha\}_{\alpha\in\Delta}$ be a net in (X,τ) such that $\{x\alpha\}_{\alpha\in\Delta}$ $\omega\beta-$converges to y. Since $y \in \omega\beta Cl(\{x\})$, by Theorem 3.3, $x \in \omega\beta Cl(\{y\})$. Since $\{x\alpha\}_{\alpha\in\Delta}$ $\omega\beta-$converges to y and $x \in \omega\beta Cl(\{y\})$, $\{x\alpha\}_{\alpha\in\Delta}$ $\omega\beta-$converges to x. Conversely, let $x, y \in X$ such that every net in X $\omega\beta-$converging to y $\omega\beta-$converges to x. Then $x \in \omega\beta Cl(\{y\})$ by Lemma 3.15. By Theorem 3.3 $y \in \omega\beta Cl(\{x\})$.
(ii $\rightarrow$ i) Assume that $x, y \in X$ such that $y \in \omega\beta Cl(\{x\})$. Suppose that $x_n = y$ for each $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a net in (X,τ) that $\omega\beta-$converges to y. Since $y \in \omega\beta Cl(\{x\})$ and $\{x_n\}_{n\in\mathbb{N}}$ $\omega\beta-$converges to y, it follows from (ii) that $\{x_n\}_{n\in\mathbb{N}}$ $\omega\beta-$converges to x. Thus, $x \in \omega\beta Cl(\{y\})$. Hence by Theorem 3.3, (X,τ) is $\omega\beta - R_0$. ∎

Theorem 3.17. *For a topological space (X,τ), the following statements are equivalent:*

(i) (X,τ) is an $\omega\beta - R_1$ space.

(ii) For each $x, y \in X$ one of the following holds:

a) For any $\omega\beta-$open set U, $x \in U$ if and only if $y \in U$.

b) There exist disjoint $\omega\beta-$open sets U and V such that $x \in U$ and $y \in V$.

(iii) If $x, y \in X$ such that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$, then there exist $\omega\beta-$closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and *$X = F_1 \cup F_2$.*

Proof. (i) $\to$ (ii) Let $x, y \in X$. Then $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{y\})$, or $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. If $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{y\})$ and U is an $\omega\beta-$open set, then $x \in U$ implies $y \in \omega\beta Cl(\{x\}) \subseteq U$ (as (X,τ) is $\omega\beta - R_0$) and $y \in U$ implies $x \in \omega\beta Cl(\{y\}) \subseteq U$. Thus consider the case $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Then there exist disjoint $\omega\beta-$open sets U and V such that $x \in \omega\beta Cl(\{x\}) \subseteq U$ and $y \in \omega\beta Cl(\{y\}) \subseteq V$.
(ii) $\to$ (iii) Let $x, y \in X$ be such that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Then $x \notin \omega\beta Cl(\{y\})$ or $y \notin \omega\beta Cl(\{x\})$. Then by (ii), there exist disjoint $\omega\beta-$open sets U and V such that $x \in U$ and $y \in V$. Let $F_1 = X - V$ and $F_2 = X - U$. Then F_1 and F_2 are $\omega\beta-$closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.
(iii) $\to$ (i) We shall first show that (X,τ) is an $\omega\beta - R_0$ space. Let U be an $\omega\beta-$open set such that $x \in U$. We claim that $\omega\beta Cl(\{x\}) \subseteq U$. For suppose $y \in \omega\beta Cl(\{x\}) \cap (X - U)$. Then $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$ (for if $\omega\beta Cl(\{x\}) = \omega\beta Cl(\{y\})$, then $y \in U$) and hence by (iii), there exist $\omega\beta-$closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Then $y \in F_2 - F_1 \subset X - F_1$ which is $\omega\beta-$open and $x \notin X - F_1$, which contradicts the fact that $y \in \omega\beta Cl(\{x\})$. Hence (X,τ) is an $\omega\beta - R_0$ space. Now let p, q be points of X such that $\omega\beta Cl(\{p\}) \neq \omega\beta Cl(\{q\})$. Then by the given condition there exists $\omega\beta-$closed sets H_1 and H_2 such that $p \in H_1$, $q \notin H_1$, $q \in H_2$, $p \notin H_2$ and $X = H_1 \cup H_2$. Thus $p \in X - H_2$ and $q \in X - H_1$, where $X - H_2$ and $X - H_1$ are disjoint $\omega\beta-$open sets. Since (X,τ) is $\omega\beta - R_0$, $\omega\beta Cl(\{p\}) \subseteq X - H_2$ and hence $\omega\beta Cl(\{q\}) \subseteq X - H_1$. Hence (X,τ) is an $\omega\beta - R_1$ space. ∎

Lemma 3.18. *A topological space (X,τ) is $\omega\beta - R_1$ if and only if for each $x, y \in X$ with $\ker_{\omega\beta}(\{x\}) \neq \ker_{\omega\beta}(\{y\})$ there exist disjoint $\omega\beta-$open sets U and V such that $\omega\beta Cl(\{x\}) \subseteq U$ and $\omega\beta Cl(\{y\}) \subseteq V$.*

Proof. This follows from Lemma 3.8. ∎

Theorem 3.19. *For a topological space (X,τ), the following are equivalent:*
(i) (X,τ) is $\omega\beta-\mathrm{T}_2$.
(ii) (X,τ) is $\omega\beta-\mathrm{T}_1$ and $\omega\beta-R_1$.
(iii) (X,τ) is $\omega\beta-\mathrm{T}_{1/2}$ and $\omega\beta-R_1$.
(iv) (X,τ) is $\omega\beta-\mathrm{T}_0$ and $\omega\beta-R_1$.

Proof. (i) $\to$ (ii) This follows from Proposition 2.4 and Theorem 2.3.
(ii) $\to$ (iii) This follows from Proposition 2.4.
(iii) $\to$ (iv) This follows from Proposition 2.4.
(iv) $\to$ (i) It follows from Theorem 3.2 and Proposition 3.4 that (X,τ) is $\omega\beta-\mathrm{T}_1$. Since (X,τ) is $\omega\beta-R_1$, by Theorem 2.3, (X,τ) is $\omega\beta-\mathrm{T}_2$. ∎

Corollary 3.20. *For an $\omega\beta-R_1$ topological space (X,τ), the following are equivalent.*
(i) (X,τ) is $\omega\beta-\mathrm{T}_2$.
(ii) (X,τ) is $\omega\beta-\mathrm{T}_1$.
(iii) (X,τ) is $\omega\beta-\mathrm{T}_{1/2}$
(iv) (X,τ) is $\omega\beta-\mathrm{T}_0$.

It is clear from Proposition 3.4 and Theorem 3.19 that any topological space that is $\omega\beta-\mathrm{T}_1$ but not $\omega\beta-\mathrm{T}_2$ is $\omega\beta-R_0$ but not $\omega\beta-R_1$ (see Example 2.6).

A point x of a topological space (X,τ) is $\omega\beta-\theta-$accumulation point of a subset $A\subseteq X$, if for each $\omega\beta-$open set U containing x, $\omega\beta Cl(U)\cap A\neq\phi$. The set of all $\omega\beta-\theta-$accumulation points of A is called the $\omega\beta-\theta-$closure of A and denoted by $\omega\beta Cl\theta(A)$. The set A is said to be $\omega\beta-\theta-$closed if $\omega\beta Cl\theta(A)=A$. The complement of an $\omega\beta-\theta-$closed set is said to be $\omega\beta-\theta-$open.

Theorem 3.21. *For a topological space (X,τ), the following are equivalent:*
(i) (X,τ) is $\omega\beta-R_1$.
(ii) For each $x\in X$, $\prec x\succ_{\omega\beta}=\omega\beta Cl\theta(\{x\})$.

Proof. (i) $\to$ (ii) First we claim that $\omega\beta Cl\theta(\{x\})\subseteq\prec x\succ_{\omega\beta}$ for each $x\in X$. For any $x\in X$, let $y\notin\prec x\succ_{\omega\beta}$. Then $\prec x\succ_{\omega\beta}\neq\prec y\succ_{\omega\beta}$ and since (X,τ) is $\omega\beta-R_0$, by Proposition 3.13 $\omega\beta Cl(\{x\})\neq\omega\beta Cl(\{y\})$.

It follows from assumption that there exist disjoint $\omega\beta-$open sets U and V such that $\omega\beta Cl(\{x\}) \subseteq U$ and $\omega\beta Cl(\{y\}) \subseteq V$. Since $U \cap \omega\beta Cl(V) = \phi$, $\{x\} \cap \omega\beta Cl(V) = \phi$ and hence $y \notin \omega\beta Cl\theta(\{x\})$. This shows that $\omega\beta Cl\theta(\{x\}) \subseteq \prec x \succ_{\omega\beta}$. Last we can obtain that $\prec x \succ_{\omega\beta} \subseteq \omega\beta Cl\theta(\{x\})$ by definitions.
(ii) $\rightarrow$ (i) For each $x \in X$, $\prec x \succ_{\omega\beta} = \omega\beta Cl\theta(\{x\}) \supset \omega\beta Cl(\{x\}) \supset \prec x \succ_{\omega\beta}$ and hence $\prec x \succ_{\omega\beta} = \omega\beta Cl(\{x\})$. By Proposition 3.13, (X, τ) is $\omega\beta - R_0$. Suppose that $\omega\beta Cl(\{x\}) \neq \omega\beta Cl(\{y\})$. Then, since (X, τ) is $\omega\beta - R_0$, by Theorem 3.9 $\omega\beta Cl(\{x\}) \cap \omega\beta Cl(\{y\}) = \phi$ and hence $\prec x \succ_{\omega\beta} \cap \prec y \succ_{\omega\beta} = \phi$. By hypothesis, $\omega\beta Cl\theta(\{x\}) \cap \omega\beta Cl\theta(\{y\}) = \phi$. Since $y \notin \omega\beta Cl\theta(\{x\})$, there exists an $\omega\beta-$open set U_y such that $y \in U_y$ and $\omega\beta Cl(U_y) \cap (\{x\}) = \phi$. Let $U_x = X - \omega\beta Cl(U_y)$, then $x \in U_x$ and U_x is $\omega\beta-$open. Since (X, τ) is $\omega\beta - R_0$, $\omega\beta Cl(\{y\}) \subset U_y$, $\omega\beta Cl(\{x\}) \subset U_x$ and $U_x \cap U_y = \phi$. This shows that (X, τ) is $\omega\beta - R_1$. ∎

Combining proposition 3.13 and Theorem 3.21, it is shown that the converse of Theorem 3.2 is true.

Corollary 3.22. *Suppose that $\omega\beta Cl\theta(\{x\}) \subseteq \omega\beta Cl(\{x\})$ for any point x of (X, τ). Then, if (X, τ) is $\omega\beta - R_0$, then it is $\omega\beta - R_1$.*

Proof. We note that $\omega\beta Cl(\{x\}) \subseteq \omega\beta Cl\theta(\{x\})$ holds for any point x of (X, τ). Let (X, τ) be an $\omega\beta - R_0$ space. Using Proposition 3.13, we have $\omega\beta Cl(\{x\}) = \prec x \succ_{\omega\beta}$ for any point x of (X, τ). It follows from assumption that $\omega\beta Cl\theta(\{x\}) = \prec x \succ_{\omega\beta}$ for any point x of (X, τ). By Theorem 3.21, (X, τ) is $\omega\beta - R_1$. ∎

4. $D_{\omega\beta}-$SETS AND ASSOCIATED SEPARATION AXIOMS

The notion of $D_{\omega\beta}-$sets is introduced in this section as the difference sets of $\omega\beta-$open sets. Also we investigate some preservation theorems.

Definition 4.1. A subset A of a space (X, τ) is called a $D_{\omega\beta}-$set if there are two $\omega\beta-$open sets U and V such that $U \neq X$ and $A = U - V$.

We see from the above definition that every proper $\omega\beta-$open set U is a $D_{\omega\beta}-$set, by setting $A = U$ and $V = \phi$. But the converse is not true as the next example shows.

Example 4.2. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and $A = \mathbb{Q}$, then A is a $D_{\omega\beta}-$set (take $U = \mathbb{Q} \cup (0, 1)$ and $V = (0, 1)$) but it is not an $\omega\beta-$open set.*

Definition 4.3. A topological space (X, τ) is said to be

(i) $\omega\beta - D_0$ if for any pair of distinct points x and y of X there exists a $D_{\omega\beta}-$set of X containing x but not y or a $D_{\omega\beta}-$set of X containing y but not x.

(ii) $\omega\beta - D_1$ if for any pair of distinct points x and y of X there exist a $D_{\omega\beta}-$set of X containing x but not y and a $D_{\omega\beta}-$set of X containing y but not x.

(iii) $\omega\beta - D_2$ if for any pair of distinct points x and y of X there exist disjoint $D_{\omega\beta}-$sets U and V of X containing x and y, respectively.

(iv) Weakly $\omega\beta - D_1$ if $\bigcap\limits_{x\in X} \omega\beta Cl(\{x\}) = \phi$.

Proposition 4.4. *For a topological space (X, τ), the following statements hold:*

(i) If (X, τ) is $\omega\beta - \mathrm{T}_i$, then it is $\omega\beta - D_i$, $i = 0, 1, 2$.

(ii) If (X, τ) is $\omega\beta - D_i$, then it is $\omega\beta - D_{i-1}$, $i = 1, 2$.

(iii) (X, τ) is $\omega\beta - D_0$ if and only if it is $\omega\beta - \mathrm{T}_0$.

(iv) (X, τ) is $\omega\beta - D_1$ if and only if it is $\omega\beta - D_2$.

(v) If (X, τ) is $\omega\beta - D_1$ then it is $\omega\beta - \mathrm{T}_0$.

Proof. The proofs of (i) and (ii) follow from definitions. And also the sufficiency for (iii) and (iv) follows from (i) and (ii).

(iii) Necessity. Let (X, τ) be $\omega\beta - D_0$. Then for all distinct points $x, y \in X$, at least one of x, y, say x, belongs to a $D_{\omega\beta}-$set U but $y \notin U$. Suppose that $U = U_1 - U_2$ where $U_1 \neq X$ and U_1, U_2 is $\omega\beta-$open in (X, τ). Then $x \in U_1$ and for $y \notin U$ we have two cases: the first one if $y \notin U_1$; so U_1 contains x but does not contain y. In second case $y \in U_1$ and $y \in U_2$; so U_2 contains y but does contain x. Hence, (X, τ) is $\omega\beta - \mathrm{T}_0$.

(iv) Necessity. Let (X, τ) be $\omega\beta - D_1$. Then for all $x, y \in X$ such that $x \neq y$, we have $D_{\omega\beta}-$sets such that $x \in V_1, y \notin V_1, y \in V_2$ and $x \notin V_2$. Let $V_1 = U_1 - U_2$ and $V_2 = U_3 - U_4$. From $x \notin V_2$, we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

1. $x \notin U_3$. From $y \notin V_1$, we obtain the following two subcases:

a) $y \notin U_1$. From $x \in U_1 - U_2$ we have $x \in U_1 - (U_2 \cup U_3)$ and from $y \in U_3 - U_4$ we have $y \in U_3 - (U_2 \cup U_4)$. It easy to see that $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4)) = \phi$. b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2$, $y \in U_2$ and $(U_1 - U_2) \cap U_2 = \phi$.

2. $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4$, $x \in U_4$ and $(U_3 - U_4) \cap U_4 = \phi$. Hence, (X, τ) is $\omega\beta - D_2$.

(v) This follows from (i) and (iii). ∎

The following diagram summarizes the implications shown in Proposition 2.4 and 4.4.

$$\begin{array}{ccccc} \omega\beta-\mathrm{T}_2 & \rightarrow & \omega\beta-\mathrm{T}_1 & \rightarrow & \omega\beta-\mathrm{T}_0 \\ \downarrow & & \downarrow & & \updownarrow \\ \omega\beta-D_2 & \leftrightarrow & \omega\beta-D_1 & \rightarrow & \omega\beta-D_0 \end{array}$$

From Example 2.5, we see that there exists an $\omega\beta-D_1$ space which is not $\omega\beta-T_1$ and also an $\omega\beta-D_2$ space which is not $\omega\beta-\mathrm{T}_2$.

Proposition 4.5. *Let (X,τ) be an $\omega\beta-$symmetric space. Then the following statements are equivalent:*
(i) (X,τ) is $\omega\beta-\mathrm{T}_0$.
(ii) (X,τ) is $\omega\beta-D_1$.
(iii) (X,τ) is $\omega\beta-\mathrm{T}_{1/2}$.
(iv) (X,τ) is $\omega\beta-\mathrm{T}_1$.

Proof. This follows from Theorem 3.3 and Propositions 3.4 and 4.4. ∎

Corollary 4.6. *For an $\omega\beta-R_0$ space (X,τ), the following are equivalent:*
(i) (X,τ) is $\omega\beta-\mathrm{T}_0$.
(ii) (X,τ) is $\omega\beta-\mathrm{T}_1$.
(iii) (X,τ) is $\omega\beta-D_1$.

Proof. This is an immediate consequence of Theorem 3.3 and Proposition 4.5. ∎

Definition 4.7. A point x of a topological space (X,τ) which has X as the only $\omega\beta-$neighborhood is called an $\omega\beta-$neat point.

Observe that if an $\omega\beta-\mathrm{T}_0$ space (X,τ) has an $\omega\beta-$neat points, then it is unique, because if x and y are distinct $\omega\beta-$neat points in X,at least one of them, say x, has an $\omega\beta-$neighborhood U containing x but not y. Thus, $U\neq X$, a contradiction.

Theorem 4.8. *Let (X,τ) be a topological space, then the following statement hold:*
(i) (X,τ) is $\omega\beta-D_1$ if and only if it is $\omega\beta-\mathrm{T}_0$ and has no $\omega\beta-$neat points.
(ii) (X,τ) is weakly $\omega\beta-D_1$ if and only if it has no $\omega\beta-$neat points.
(iii) (X,τ) is $\omega\beta-D_1$ if and only if it is $\omega\beta-\mathrm{T}_0$ and weakly $\omega\beta-D_1$.
(iv) (X,τ) is weakly $\omega\beta-D_1$ if and only if $\ker_{\omega\beta}(\{x\})\neq X$ for every $x\in X$.

Proof. (i) Necessity. Since (X,τ) is $\omega\beta-D_1$, each point $x\in X$ is contained in a $D_{\omega\beta}-$set $W=U-V$ and thus in U. By definition $U\neq X$. Hence, x is not an $\omega\beta-$neat point (X,τ) being $\omega\beta-\mathrm{T}_0$

follows from Proposition 4.4.
Sufficiency. Since (X,τ) is $\omega\beta-\mathrm{T}_0$, for each $x,y\in X$ such that $x\neq y$, there exists an $\omega\beta-$open set U containing x, say but not y. Thus U is a $D_{\omega\beta}-$set, but (X,τ) has no $\omega\beta-$neat point, so y is not an $\omega\beta-$neat point. Thus, there exists an $\omega\beta-$open set V containing y such that $V\neq X$, and therefore, $y\in V-U$, $x\notin V-U$ and $V-U$ is a $D_{\omega\beta}-$set. Hence (X,τ) is $\omega\beta-D_1$.
(ii) Necessity. Suppose that (X,τ) is weakly $\omega\beta-D_1$ and that (X,τ) has an $\omega\beta-$neat point y. Then $y\in\omega\beta Cl(\{x\})$ for each $x\in X$. This is a contradiction.
Sufficiency. Suppose that (X,τ) has no $\omega\beta-$neat points and that (X,τ) is not weakly $\omega\beta-D_1$. Then there exists $y\in\bigcap\limits_{x\in X}\omega\beta Cl(\{x\})$ and thus any $\omega\beta-$open set containing y must be X, that is, y is an $\omega\beta-$neat point of (X,τ). This is a contradiction.
(iii) This follows from (i) and (ii).
iv) Necessity. Suppose that (X,τ) is weakly $\omega\beta-D_1$. Assume that there exist a point $y\in X$ such that $\ker_{\omega\beta}(\{y\})=X$. Thus by Lemma 3.7, $y\in\bigcap\limits_{x\in X}\omega\beta Cl(\{x\})$. This is a contradiction.
Sufficiency. Assume that $\ker_{\omega\beta}(\{x\})\neq X$ for every $x\in X$. If (X,τ) is not weakly $\omega\beta-D_1$, then by Theorem 4.8, (X,τ) has an $\omega\beta-$neat point y. Hence $\ker_{\omega\beta}(\{y\})=X$. This is a contradiction. ∎

Corollary 4.9. *For an $\omega\beta-\mathrm{T}_0$ space (X,τ), the following are equivalent:*
(i) (X,τ) is $\omega\beta-D_1$.
(ii) (X,τ) has no $\omega\beta-$neat points.

Now we study and give some preservation theorems for these separation axioms.

Theorem 4.10. *Let $f:(X,\tau)\to(Y,\sigma)$ be an $\omega\beta-$irresolute function. Then the following properties hold:*
(i) If U is a $D_{\omega\beta}-$set in (Y,σ) and f is surjective, then $f^{-1}(U)$ is a $D_{\omega\beta}-$set in (X,τ).
(ii) If (Y,σ) is $\omega\beta-D_1$ and f is bijective, then (X,τ) is $\omega\beta-D_1$.

Proof. (i) Let U be a $D_{\omega\beta}-$set in (Y,σ). Then exists an $\omega\beta-$open sets U_1 and U_2 in (Y,σ) such that $U=U_1-U_2$ and $U_1\neq Y$. By $\omega\beta-$irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\omega\beta-$open in (X,τ). Since $U_1\neq Y$ and f is surjective, $f^{-1}(U_1)\neq X$. Hence, $f^{-1}(U)=f^{-1}(U_1)-f^{-1}(U_2)$ is a $D_{\omega\beta}-$set.

ii) Suppose that (Y,σ) is an $\omega\beta - D_1$ space. Let x, $y \in X$ and $x \neq y$. Since f is injective and (Y,σ) is $\omega\beta - D_1$, there exist $D_{\omega\beta}-$sets U_x and U_y of (Y,σ) containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin U_x$ and $f(x) \notin U_y$. By (i), $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are $D_{\omega\beta}-$sets in (X,τ) containing x and y, respectively, and $y \notin f^{-1}(U_x)$, $x \notin f^{-1}(U_y)$. Hence, (X,τ) is $\omega\beta - D_1$. ∎

Theorem 4.11. *A topological space (X,τ) is $\omega\beta - D_1$ if and only if for each pair of distinct points $x, y \in X$, there exists an $\omega\beta-$irresolute surjective function $f : (X,\tau) \to (Y,\sigma)$, where (Y,σ) is an $\omega\beta - D_1$ space, such that $f(x)$ and $f(y)$ are distinct.*

Proof. Necessity. For every pair of distinct points of (X,τ), it suffices to take the identity function on (X,τ).
Sufficiency. Let $x, y \in X$ and $x \neq y$. By hypothesis, there exists an $\omega\beta-$irresolute surjective function f from (X,τ) onto an $\omega\beta - D_1$ space (Y,σ) such that $f(x) \neq f(y)$. Thus by Proposition 4.4, there exist disjoint $D_{\omega\beta}-$sets U_x and U_y in (Y,σ) such that $f(x) \in U_x$ and $f(y) \in U_y$. Since f is $\omega\beta-$irresolute and surjective, by Theorem 4.10, $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are disjoint $D_{\omega\beta}-$sets in (X,τ) containing x and y,respectively. Hence, (X,τ) is $\omega\beta - D_1$. ∎

Theorem 4.12. *If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega\beta-$closed injection and (X,τ) is weakly $\omega\beta - D_1$, then (Y,σ) is weakly $\omega\beta - D_1$.*

Proof. Since (X,τ) is weakly $\omega\beta - D_1$, $\bigcap\limits_{x\in X} \omega\beta Cl(\{x\}) = \phi$. Since f is an injection, $\phi = f(\bigcap\limits_{x\in X} \omega\beta Cl(\{x\})) = \bigcap\limits_{x\in X} f(\omega\beta Cl(\{x\}))$. Since f is an $\omega\beta-$closed function, it follows from Lemma 1.5 that $\bigcap\limits_{y\in Y} \omega\beta Cl(\{y\}) \subseteq \bigcap\limits_{x\in X} \omega\beta Cl(\{f(x)\}) \subseteq \bigcap\limits_{x\in X} f(\omega\beta Cl(\{x\}))$. Hence, $\bigcap\limits_{y\in Y} \omega\beta Cl(\{y\}) = \phi$, that is, (Y,σ) is weakly $\omega\beta - D_1$. ∎

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