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MEROMORPHIC FUNCTION WITH SOME POWER SHARING A SMALL FUNCTION WITH THE DIFFERENTIAL POLYNOMIAL GENERATED BY THE FUNCTION

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Abstract. Taking Yu's [19] result in background we investigate the uniqueness of some power of meromorphic functions sharing a small function with the differential polynomial generated by the function. Our results will extend and improve a number of existing results in a different direction.

1. Introduction Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

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Throughout the paper the standard notations from Nevanlinna's theory of value distribution of meromorphic functions is used, as in [9]. We recall that $T(r,f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r,a;f)$ ($\overline{N}(r,a;f)$) denotes the counting function (reduced counting function) of a-points of meromorphic functions f.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

Also we use I to denote any set of infinite linear measure of $0 < r < \infty$.

We also recall that if $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r \longrightarrow \infty} \frac{m(r,a;f)}{T(r,f)}$$

is called Nevanlinna deficiency of the value a and by ramification index iwe mean

$$\Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

We start the discussion on the result of R. Brück [4] who first considered the uniqueness problem of an entire function sharing one value with its derivative. Below we are recalling R. Brück's result.

Theorem A. [4] *Let f be a non-constant entire function. If f and f' share the value* 1 *CM and if $N(r,0;f') = S(r,f)$ then $\frac{f'-1}{f-1}$ is a nonzero constant.*

In fact, Brück obtained the above result to justify his famous conjecture, corresponding to the uniqueness for one CM shared value of entire function with its first derivative [4]:

Conjecture: *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a non zero constant.*

Later many researchers like Zhang [20], Yang [17], Gundersen-Yang [8] et al. ponder over different aspect of the conjecture. Next we recall the following definition known as weighted sharing of values which has a remarkable influence on the subsequent results of Brück conjecture.

Definition 1.1. [10, 11] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and*

$k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [9], we explain some definitions and notations which are used in the paper.

Definition 1.2. [14]*Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(i) *$N(r,a;f \mid\geq p)$ $(\overline{N}(r,a;f \mid\geq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*

(ii) *$N(r,a;f \mid\leq p)$ $(\overline{N}(r,a;f \mid\leq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 1.3. [18] *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f \mid\geq 2)+\ldots\overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.*

Definition 1.4. [21] *For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we put*

$$\delta_p(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_p(r,a;f)}{T(r,f)}$$

Clearly $0 \leq \delta(a;f) \leq \delta_p(a;f) \leq \delta_{p-1}(a;f) \ldots \leq \delta_2(a;f) \leq \delta_1(a;f) = \Theta(a;f)$

Definition 1.5. [1] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points* *of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$.*

Definition 1.6. [10, 11] *Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f)$ $+ \overline{N}_L(r, a; g)$.

In 2003 Yu [19] tackle the conjecture in a different way than that was done previously. Actually Yu [19] stressed to the fact of getting specific relationship between a function and its derivative imposing some restrictions on the deficient values of the functions. He considered the uniqueness problem of an entire or meromorphic functions with its derivative sharing a small function a and obtained the following two theorems.

Theorem B. [19] *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0; f) > \frac{3}{4}$ then $f \equiv f^{(k)}$.*

Theorem C. [19] *Let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If*

i) *f and a have no common poles.*
ii) *$f - a$ and $f^{(k)} - a$ share the value 0 CM.*
iii) $4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k$

then $f \equiv f^{(k)}$ where k is a positive integer.

In the same paper Yu [19] posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value in *Theorem B* ?
(ii) Can the condition $\delta(0; f) > \frac{3}{4}$ of *Theorem B* be further relaxed ?
(iii) Can the condition (iii) in *Theorem C* be further relaxed ?
(iv) Can in general the condition (i) of *Theorem C* be dropped ?

Lahiri-Sarkar [14] subtly used weighted sharing of values to improve the results of Yu [19]. In 2005, Zhang [21] further extended the result of Lahiri-Sarkar to a small function and proved the following.

Theorem D. [21] *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l(\geq 2)$ and*

$$(1.1) \qquad (3+k)\,\Theta(\infty, f) + 2\,\delta_{2+k}(0; f) > 4 + k$$

or l=1 and

$$(1.2) \qquad (4+k)\,\Theta(\infty, f) + 3\,\delta_{2+k}(0; f) > 6 + k$$

or $l = 0$ and

$$(1.3) \qquad (6+2k)\,\Theta(\infty, f) + 5\,\delta_{2+k}(0; f) > 10 + 2k$$

then $f \equiv f^{(k)}$.

Recently in connection with the Yu's [19] result Zhang and Lü [22] considered the uniqueness of the n-th power of a meromorphic function sharing a small function with its k-th derivative and proved the following theorem.

Theorem E. [19] *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = \infty$ and*

$$(1.4) \qquad (3+k)\,\Theta(\infty; f) + 2\,\Theta(0; f) + \delta_{2+k}(0; f) > 6 + k - n$$

or $l = 0$ and

$$(1.5) \qquad (6+2k)\,\Theta(\infty; f) + 4\,\Theta(0; f) + 2\delta_{2+k}(0; f) > 12 + 2k - n$$

then $f^n \equiv f^{(k)}$

At the end of [22] the following question was raised by Zhang and Lü [22].
What will happen if f^n and $[f^{(k)}]^m$ share a small function ?

In 2010, Chen and Zhang [5] answered the above question. But unfortunately there were some errors in their results. Banerjee-Majumder [3] first pointed out the errors, rectified them and obtained the correct form of the same as follows.

Theorem F. [3] *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = 2$ and*

$$(1.6) \qquad (3+k)\,\Theta(\infty; f) + 2\,\Theta(0; f) + \delta_{2+k}(0; f) > 6 + k - n$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + k\right)\ \Theta(\infty; f) + \frac{5}{2}\ \Theta(0; f) + \delta_{2+k}(0; f) > 7 + k - n \tag{1.7}$$

or $l = 0$ *and*

$$(6+2k)\ \Theta(\infty; f) + 4\ \Theta(0; f) + \delta_{1+k}(0; f) + \delta_{2+k}(0; f) > 12 + 2k - n \tag{1.8}$$

then $f^n \equiv f^{(k)}$

Theorem G. [3] *Let* $k(\geq 1)$*,* $n(\geq 1)$*,* $m(\geq 2)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f*. Suppose* $f^n - a$ *and* $[f^{(k)}]^m - a$ *share* $(0, l)$*. If* $l = 2$ *and*

$$(3 + 2k)\ \Theta(\infty; f) + 2\ \Theta(0; f) + 2\delta_{1+k}(0; f) > 7 + 2k - n \tag{1.9}$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + 2k\right)\ \Theta(\infty; f) + \frac{5}{2}\ \Theta(0; f) + 2\delta_{1+k}(0; f) > 8 + 2k - n \tag{1.10}$$

or $l = 0$ *and*

$$(6 + 3k)\ \Theta(\infty; f) + 4\ \Theta(0; f) + 3\delta_{1+k}(0; f) > 13 + 3k - n \tag{1.11}$$

then $f^n \equiv [f^{(k)}]^m$.

Next we give the following definition.

Definition 1.7. *Let* $n_{0j}, n_{1j}, \ldots, n_{kj}$ *be non negative integers. The expression* $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ *is called a differential monomial generated by* f *of degree* $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ *and weight* $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i + 1)n_{ij}$.

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ *is called a differential polynomial generated by* f *of degree* $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ *and weight* $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$*, where* $T(r, b_j) = S(r, f)$ *for* $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ *and* k*(the highest order of the derivative of* f *in* $P[f]$ *are called respectively the lower degree and order of* $P[f]$.

$P[f]$ *is said to be homogeneous if* $\overline{d}(P) = \underline{d}(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

We note that $(f^{(k)})^m$ is a special differential monomial generated by f. So it will be interesting to investigate whether *Theorems D-G* can be extended up to differential polynomial generated by f. This is one of the motivations of writing the paper. We also inspect that the right hand side of all the inequalities (1.4)- (1.11) in *Theorems E-G* involves both k and n. As a result the lower bound of the inequalities depend both on k and n. So for meromorphic functions with relatively large number of poles the inequalities will be stronger. This observation is sufficient enough to explore the situation for fixed lower bounds in the above inequalities. This is another motivation of the paper.

We define for any two positive integers n and $m \leq 3$,

$$\mu_m = \min\{n, m\} \text{ and } \mu_m^* = (m+1) - \mu_m.$$

Following theorem is the main result of the paper which improve all the previous results.

Theorem 1.1. *Let f be a non-constant meromorphic function, and $n(\geq 1)$, $l(\geq 0)$ be integers. Let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a small meromorphic function. Suppose further that $P[f]$ be a differential polynomial generated by f such that $P[f]$ contains at least one derivative and $f^n - a$ and $P[f] - a$ share $(0, l)$. If $l = \infty$ and*

$$3\Theta(\infty; f) + \underline{d}(P)\delta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) > \mu_2 + 3 \tag{1.12}$$

or $2 \leq l < \infty$ and

$$3\Theta(\infty; f) + \underline{d}(P)\delta(0; f) + \mu_3\delta_{\mu_3^*}(0, f) > \mu_3 + 3 \tag{1.13}$$

or $l = 1$ and

$$4\Theta(\infty; f) + \underline{d}(P)\delta(0; f) + \Theta(0; f) + \mu_2\delta_{\mu_2^*}(0, f) > \mu_2 + 5 \tag{1.14}$$

or $l = 0$ and

$$\begin{aligned} &(2Q+6)\Theta(\infty; f) + 3\underline{d}(P)\delta(0; f) + \Theta(0; f) + \mu_2\delta_{\mu_2^*}(0; f) \\ &> 2Q + 2\overline{d}(P) + \mu_2 + 7, \end{aligned} \tag{1.15}$$

then $f^n = P[f]$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [21] *Let f be a non-constant meromorphic function and k be a positive integer, then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.2. [13] *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k)$$
$$+S(r,f).$$

Lemma 2.3. [15] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

Lemma 2.4. [6] *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r,\frac{1}{f}\right) + S(r,f).$$

Lemma 2.5. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then*

$$N(r,0;P[f]) \leq T(r,P[f]) - \underline{d}(P)T(r,f) + \underline{d}(P)N(r,0;f) + S(r,f).$$

Proof. For a fixed value of r, let $E_1 = \{\theta \in [0, 2\pi] : \left|f(re^{i\theta})\right| \leq 1\}$ and E_2 be its complement. Since by definition

$$\sum_{i=0}^{k} n_{ij} \geq \underline{d}(P),$$

for every $j = 1, 2, \ldots, t$, it follows that on E_1

$$\left|\frac{P[f]}{f^{\underline{d}(P)}}\right| \leq \sum_{j=1}^{t} |c_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} |f|^{\sum\limits_{i=0}^{k} nij - \underline{d}(P)} \leq \sum_{j=1}^{t} |c_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}}.$$

Also we note that

$$\frac{1}{f^{\underline{d}(P)}} = \frac{P[f]}{f^{\underline{d}(P)}} \, \frac{1}{P[f]}.$$

Since on E_2, $\frac{1}{|f(z)|} < 1$, we have

$$\begin{aligned}
& \underline{d}(P) m\left(r, \frac{1}{f}\right) \\
= \; & \frac{1}{2\pi} \int\limits_{E_1} \log^{+} \frac{1}{|f(re^{i\theta})|^{\underline{d}(P)}} d\theta + \frac{1}{2\pi} \int\limits_{E_2} \log^{+} \frac{1}{|f(re^{i\theta})|^{\underline{d}(P)}} d\theta \\
\leq \; & \frac{1}{2\pi} \sum_{j=1}^{l} \left[\int\limits_{E_1} \log^{+} |c_j(z)| \, d\theta + \sum_{i=1}^{k} \int\limits_{E_1} \log^{+} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} d\theta\right] \\
& + \frac{1}{2\pi} \int\limits_{E_1} \log^{+} \left|\frac{1}{P[f(re^{i\theta})]}\right| d\theta \\
\leq \; & \frac{1}{2\pi} \int\limits_{0}^{2\pi} \log^{+} \left|\frac{1}{P[f(re^{i\theta})]}\right| d\theta + S(r, f) = m\left(r, \frac{1}{P[f]}\right) + S(r, f).
\end{aligned}$$

So using *Lemmas 2.4*, *2.5* and the first fundamental theorem we get

$$\begin{aligned}
& N(r, 0; P[f]) \\
\leq \; & T(r, P[f]) - m\left(r, \frac{1}{P[f]}\right) + S(r, f) \\
\leq \; & T(r, P[f]) - \underline{d}(P) m\left(r, \frac{1}{f}\right) + S(r, f) \\
\leq \; & T(r, P[f]) - \underline{d}(P) T(r, f) + \underline{d}(P) N(r, 0; f) + S(r, f).
\end{aligned}$$

∎

Lemma 2.6. [7] *Let $P[f]$ be a differential polynomial generated by f. Then*

$$m(r,P[f]) \leq \overline{d}(P)m(r,f)+S(r,f).$$

Lemma 2.7. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r,P[f])$ can be replaced by $S(r,f)$.*

Proof. From *Lemma 2.6* it is clear that $T(r,P[f]) = O(T(r,f))$ and so the lemma follows. ∎

Lemma 2.8. *Let $P[f]$ be a differential polynomial generated by f. Then*

$$T(r,P[f]) \leq \overline{d}(P)T(r,f)+Q\overline{N}(r,\infty;f)+S(r,f).$$

Proof. Let z_0 be a pole of f of order r, such that $b_j(z_0) \neq 0,\infty; 1 \leq j \leq t$. Then it would be a pole of $P[f]$ of order at most $\max_{1\leq j\leq t}\{r.n_{0j} + (r+1)n_{1j} + \ldots + (r+k)n_{kj}\} \leq r.\overline{d}(P)+Q$. So from *Lemma 2.6* we get $T(r,P[f]) \leq \overline{d}(P)T(r,f)+Q\overline{N}(r,\infty;f)+S(r,f)$. ∎

3. Proof of the theorem

Proof of Theorem 1.1. Let $F = \frac{f^n}{a}$ and $G = \frac{P[f]}{a}$. Then $F-1 = \frac{f^n-a}{a}$ $G-1 = \frac{P[f]-a}{a}$. Since $f^n - a$ and $P[f]-a$ share $(0,l)$ it follows that F, G share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1 Let $H \not\equiv 0$.

From (2.1) we get

$$\begin{aligned}(3.1)\quad & N(r,\infty;H) \\ \leq\ & \overline{N}(r,\infty;F)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2) \\ & +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+\overline{N}(r,0;a)+\overline{N}(r,\infty;a) \\ & +\sum_{j=1}^{t}N(r,0;b_j)+\sum_{j=1}^{t}N(r,\infty;b_j),\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined. Let z_0 be a simple zero of $F-1$. Then by a simple calculation we see that z_0 is a zero of H and hence

(3.2)

$$N_E^{1)}(r,1;F) = N(r,1;F\mid=1) \leq N(r,0;H) \leq N(r,\infty;H)+S(r,F)$$

By the second fundamental theorem, *Lemma 2.7*, (3.1) and noting that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)+S(r,f)$, we get

$$\begin{aligned}(3.3)\quad & T(r,G)\\ \leq\ & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1;G)-N_0(r,0;G^{'})+S(r,G)\\ \leq\ & 2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,0;F\mid\geq 2)\\ & +\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})+S(r,f).\end{aligned}$$

While $l=\infty$, $\overline{N}_*(r,1;F,G)=0$. So

$$\begin{aligned}(3.4)\quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ \leq\ & \overline{N}(r,0;F^{'}).\end{aligned}$$

So

$$T(r,G)\leq 2\,\overline{N}(r,\infty;F)+N_2(r,0;G)+\overline{N}(r,0;F^{'})+S(r,f).$$

Using *Lemmas 2.5* and *2.1* we get

$$\begin{aligned}& T\left(r,P[f]\right)\\ \leq\ & 2\overline{N}(r,\infty;f)+N_2(r,0;P[f])+\overline{N}\left(r,0;(f^n/a)^{'}\right)+S(r,f)\\ \leq\ & 3\overline{N}(r,\infty;f)+T(r,P[f])-\underline{d}(P)T(r,f)+\underline{d}(P)N(r,0;f)\\ & +N_2(r,0;f^n)+S(r,f)\end{aligned}$$

i.e.,

$$\underline{d}(P)T(r,f)\leq 3\overline{N}(r,\infty;f)+\underline{d}(P)N(r,0;f)+\mu_2 N_{\mu_2^*}(r,0;f)+S(r,f),$$

i.e.,

$$3\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\mu_2\delta_{\mu_2^*}(0,f)\leq 3+\mu_2,$$

which contradicts (1.12)

While $l\geq 2$, (3.4) becomes

$$\begin{aligned}(3.5)\quad & \overline{N}(r,0;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ \leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,1;F\mid\geq l+1)+\overline{N}(r,1;F\mid\geq 2)\\ & +\overline{N}_0(r,0;F^{'})\leq N_2(r,0;F^{'}).\end{aligned}$$

Hence

$$T(r,G)\leq 2\,\overline{N}(r,\infty;F)+N_2(r,0;G)+N_2(r,0;F^{'})+S(r,f)$$

Again using *Lemmas 2.5* and *2.1* as above we get that

$$\begin{aligned}T(r,P[f])\ \leq\ & 3\overline{N}(r,\infty;f)+T(r,P[f])-\underline{d}(P)T(r,f)+\underline{d}(P)N(r,0;f)\\ & +N_3(r,0;f^n)+S(r,f).\end{aligned}$$

i.e.,

$\underline{d}(P)T(r,f) \leq 3\overline{N}(r,\infty;f) + \underline{d}(P)N(r,0;f) + \mu_3 N_{\mu_3^*}(r,0;f) + S(r,f)$,

i.e.,

$$3\Theta(\infty;f) + \underline{d}(P)\delta(0;f) + \mu_3\delta_{\mu_3^*}(0,f) \leq 3 + \mu_3,$$

which contradicts (1.13).

While $l = 1$ (3.4) changes to

$$\begin{aligned} & \overline{N}(r,0;F \mid\geq 2) + 2\,\overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F^{'}) \\ \leq\ & \overline{N}(r,0;F^{'}) + \overline{N}(r,0;F^{'} \mid F \neq 0). \end{aligned}$$

Similarly as above using *Lemmas 2.5*, *2.1* and *2.2* we have

$$\begin{aligned} & T\left(r,P[f]\right) \leq 2\,\overline{N}(r,\infty;f) + N_2\left(r,0;P[f]\right) + \overline{N}\left(r,0;(f^n/a)^{'}\right) \\ & +\overline{N}\left(r,0;(f^n/a)^{'} \mid (f^n/a) \neq 0\right) + S(r,f) \\ \leq\ & 4\overline{N}(r,\infty;f) + T(r,P[f]) - \underline{d}(P)T(r,f) + \underline{d}(P)N(r,0;f) \\ & +N_2\left(r,0;(f^n/a)\right) + \overline{N}\left(r,0;f^n/a\right) + S(r,f) \\ \leq\ & 4\overline{N}(r,\infty;f) + T(r,P[f]) - \underline{d}(P)T(r,f) + \underline{d}(P)N(r,0;f) \\ & +\overline{N}(r,0;f) + \mu_2 N_{\mu_2^*}(r,0;f) + S(r,f) \end{aligned}$$

i.e.,

$$4\Theta(\infty;f) + \underline{d}(P)\delta(0;f) + \Theta(0;f) + \mu_2\delta_{\mu_2^*}(0,f) \leq 5 + \mu_2,$$

which contradicts (1.14).

Subcase 1.2 $l = 0$.

In this case F and G share $(1,0)$ except the zeros and poles of $a(z)$. Also we have

$$\begin{aligned} & N(r,\infty;H) \\ \leq\ & \overline{N}(r,\infty;F) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_L(r,1;F) \\ & +\overline{N}_L(r,1;G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f). \end{aligned} \tag{3.6}$$

Let z_0 be a zero of $F-1$ with multiplicity p and a zero of $G-1$ with multiplicity q. It is easy to see that

$$N_E^{1)}(r,1;F) = N_E^{1)}(r,1;G) + S(r,f)$$

$$\overline{N}_E^{(2}(r,1;F) = \overline{N}_E^{(2}(r,1;G) + S(r,f)$$

and

$$N_E^{1)}(r,1;F) \leq N(r,\infty;H) + S(r,f) \tag{3.7}$$

By the second fundamental theorem we get using (3.6) and (3.7) that

$$\begin{aligned}
& T(r,G) \\
\leq\ & \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_E^{(2}(r,1;F) \\
& +\overline{N}_L(r,1;G)-\overline{N}_0(r,0;F^{'})+S(r,f) \\
\leq\ & 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}_L(r,1;F) \\
& +\overline{N}(r,0;G\mid\geq 2)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_L(r,1;F) \\
& +\overline{N}_L(r,1;G)+S(r,f) \\
\leq\ & 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}(r,1;G\mid\geq 2) \\
& +\overline{N}_0(r,0;G^{'})+\overline{N}(r,0;F\mid\geq 2)+2\overline{N}(r,1;F\mid\geq 2)+S(r,f) \\
\leq\ & 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G^{'})+\overline{N}(r,0;G^{'}\mid G\neq 0) \\
& +\overline{N}(r,0;F^{'})+\overline{N}(r,0;F^{'}\mid F\neq 0)+S(r,f)
\end{aligned}$$

From *Lemma 2.1* for $p=1$, $k=1$ and *Lemma 2.2* we get

$$\begin{aligned}
T(r,G)\ \leq\ & 2\overline{N}(r,0;G)+N_2(r,0;G)+\overline{N}(r,0;F^{'}) \\
& +\overline{N}(r,0;F^{'}\mid F\neq 0)+4\overline{N}(r,\infty;F)+S(r,f),
\end{aligned}$$

that is,

$$\begin{aligned}
& T\left(r,P[f]\right) \\
\leq\ & 4\overline{N}(r,\infty;f)+2\overline{N}\left(r,0;P[f]\right)+N_2\left(r,0;P[f]\right)+\overline{N}\left(r,0;(f^n/a)^{'}\right) \\
& +\overline{N}\left(r,0;(f^n/a)^{'}\mid (f^n/a)\neq 0\right)+S(r,f).
\end{aligned}$$

So as above using *Lemmas 2.1*, *2.2*, *2.5* and *2.8* we get

$$\begin{aligned}
& \{3\underline{d}(P)-2\overline{d}(P)\}T(r,f)\leq (2Q+6)\overline{N}(r,\infty;f)+3\underline{d}(P)N(r,0;f) \\
& +\overline{N}(r,0;f)+\mu_2 N_{\mu_2^*}(r,0;f)+S(r,f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& (2Q+6)\Theta(\infty;f)+3\underline{d}(P)\delta(0;f)+\Theta(0;f)+\mu_2\delta_{\mu_2^*}(0;f) \\
\leq\ & 2Q+2\overline{d}(P)+\mu_2+7.
\end{aligned}$$

This contradicts (1.15).

Case 2. Let $H\equiv 0$.

On integration we get from (2.1)

$$\frac{1}{F-1}\equiv\frac{C}{G-1}+D, \tag{3.8}$$

where C, D are constants and $C\neq 0$. We first show that $D=0$. Suppose that there exist a pole z_0 of f with multiplicity p which is not

a pole or a zero of $a(z)$. Then z_0 is the pole of F with multiplicity np and the pole of G with multiplicity r (say). We assume that $np \neq r$, since otherwise we know from (3.8) that $D = 0$ and we are done.
Subcase 2.1. Suppose $D \neq 0$.
Since $np \neq r$, we get a contradiction from (3.8). So,

$$N(r,\infty;f) \leq N(r,0;a) + N(r,\infty;a) = S(r,f),$$

and hence $\Theta(\infty;f) = 1$. Also it is clear that $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r,f)$.
From (1.12)-(1.15) we know respectively

$$\underline{d}(P)\delta(0;f) + \mu_2\delta_{\mu_2^*}(0,f) > \mu_2, \tag{3.9}$$

$$\underline{d}(P)\delta(0;f) + \mu_2\delta_{\mu_3^*}(0,f) > \mu_3, \tag{3.10}$$

$$\Theta(0;f) + \underline{d}(P)\delta(0;f) + \mu_2\delta_{\mu_2^*}(0,f) > \mu_2 + 1 \tag{3.11}$$

and

$$\Theta(0;f) + 3\underline{d}(P)\delta(0;f) + \mu_2\delta_{\mu_2^*}(0,f) > 2\overline{d}(P) + \mu_2 + 1 \tag{3.12}$$

Since $D \neq 0$, from (3.8) we get

$$-\frac{D\left(F - 1 - \frac{1}{D}\right)}{F-1} \equiv C\frac{1}{G-1}.$$

So

$$\overline{N}\left(r, 1 + \frac{1}{D}; F\right) = \overline{N}(r,\infty;G) = S(r,f).$$

Subcase 2.1.1. $D \neq -1$.
Using the second fundamental theorem for F we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1+\frac{1}{D};F) \\ &\leq \overline{N}(r,0;F) + S(r,f). \end{aligned}$$

that is

$$nT(r,f) \leq \overline{N}(r,0;f) + S(r,f).$$

If $n > 1$ we have a contradiction from above. For $n = 1$ we have $\Theta(0;f) = \delta_2(0;f) = \ldots = \delta(0;f) = 0$, which contradicts (3.9)-(3.12).
Subcase 2.1.2. $D = -1$.
Then

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}. \tag{3.13}$$

Clearly we know from above $N(r,0;F) = \overline{N}(r,\infty;G) = S(r,f)$ and hence $N(r,0;f) = S(r,f)$. If $C \neq -1$ we know from (3.13) that

$\overline{N}(r, 1+C; G) = \overline{N}(r, \infty; F) = S(r, f)$. So from *Lemmas 2.1*, *2.5* and the second fundamental theorem we get

$$\begin{aligned}
& \underline{d}(P)T(r, f) \\
\leq\ & \overline{N}(r, \infty; G) + \underline{d}(P)N(r, 0; f) + \overline{N}(r, 1+C; G) + S(r, f) \\
\leq\ & S(r, f),
\end{aligned}$$

which is absurd.

So $C = -1$ and we get from (3.13) that $FG \equiv 1$, which ultimately yields $f^n P[f] \equiv a^2$.

From above we have $N(r, 0; f) = S(r, f)$ and $N(r, \infty; f) = S(r, f)$. In view of the first fundamental theorem *Lemma 2.4* we get from above

$$\begin{aligned}
& (n + \overline{d}(P))T(r, f) \\
=\ & T\left(r, \frac{a^2}{f^{(n+\overline{d}(P))}}\right) + S(r, f) \\
\leq\ & T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + S(r, f) \\
=\ & m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) + S(r, f) \\
\leq\ & (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + N\left(r, \infty; P[f]\right) + \overline{d}(P)N(r, 0; f) \\
& + S(r, f) \\
=\ & (\overline{d}(P) - \underline{d}(P))(T(r, f) - N(r, 0; f)) + S(r, f),
\end{aligned}$$

i.e., $(n + \overline{d}(P))T(r, f) \leq S(r, f)$ which is impossible.

Subcase 2.2. $D = 0$ and so from (3.8) we get

$$G - 1 \equiv C\ (F - 1).$$

If $C \neq 1$, then

$$F \equiv \frac{G - 1 + C}{C}$$

and

$$\overline{N}(r, 0; F) = \overline{N}(r, 1 - C; G)$$

By the second fundamental theorem and using *Lemmas 2.1*, *2.5* and *2.7* we have

$$\begin{aligned}
& T(r,G) \\
\leq\ & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1-C;G)+S(r,G) \\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,0;F)+T(r,P[f])-\underline{d}(P)T(r,f) \\
& +\underline{d}(P)N(r,0;f)+S(r,f).
\end{aligned}$$

i.e.,

$$\underline{d}(P)T(r,f)\leq \overline{N}(r,\infty;f)+\underline{d}(P)N(r,0;f)+\overline{N}(r,0;f)+S(r,f),$$

which implies

$$\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\Theta(0;f)\leq 2.$$

In view of *Definition 1.4* and noting that $\mu_i\geq 1$ for $i=1,2,3$ we get from above

$$\begin{aligned}
2(1+\Theta(\infty;f)) &\geq 3\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\Theta(0;f) \\
&\geq 3\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\mu_2\delta_{\mu_2^*}(0,f) \\
&> \mu_2+3,
\end{aligned}$$

which contradicts (1.12). In a similar manner we can show that

$$2(1+\Theta(\infty;f))\geq\ 3\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\mu_3\delta_{\mu_3^*}(0,f)>\mu_3+3,$$

$$\begin{aligned}
& 2+3\Theta(\infty;f)+\mu_2\delta_{\mu_2^*}(0,f)\geq 4\Theta(\infty;f)+\underline{d}(P)\delta(0;f)+\Theta(0;f) \\
& +\mu_2\delta_{\mu_2^*}(0,f)>\mu_2+5
\end{aligned}$$

and

$$\begin{aligned}
& 2+(2Q+5)\Theta(\infty;f)+2\underline{d}(P)\delta(0;f)+\mu_2\delta_{\mu_2^*}(0,f) \\
\geq\ & (2Q+6)\Theta(\infty;f)+3\underline{d}(P)\delta(0;f)+\Theta(0;f)+\mu_2\delta_{\mu_2^*}(0,f) \\
>\ & 2Q+2\overline{d}(P)+\mu_2+7
\end{aligned}$$

which contradicts respectively (1.13)-(1.15). Hence $C=1$ and so $F\equiv G$, that is $f^n\equiv P[f]$. This completes the proof of the theorem. ∎

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