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FUZZY UPPER AND LOWER M-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper we introduce the notion of fuzzy upper and lower M-continuous multifunctions as generalizations of upper and lower M-continuous multifunctions [28], as fuzzy multifunctions between sets having certain minimal structures. We also prove that m-compact sets from a space with minimal structures have fuzzy m-compact images under surjective fuzzy upper M-continuous multifunctions, provided that some natural conditions are satisfied. Lastly we show that the notions of fuzzy upper and lower M-continuous multifunctions unify several forms of generalized continuity for fuzzy multifunctions from a topological space into a fuzzy topological space.

1. Introduction and preliminaries

In the past few years, different types of fuzzy open-like sets, viz., fuzzy semiopen, fuzzy preopen, fuzzy θ-open, fuzzy δ-open, fuzzy δ-preopen, fuzzy α-open, fuzzy β-open have been introduced and studied by many researchers and using these concepts several types of fuzzy non-continuous functions are introduced and studied in [1], [4], [5], [14], [18], [22].

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It has been shown that certain of these fuzzy non-continuous functions have properties similar to those fuzzy continuous function. Using these concepts different types of fuzzy multifunctions have been introduced in [6], [9], [23], [29].

Let X be a non-empty set and $I = [0, 1]$. Then I^X is the family of all fuzzy sets [36] in X. The support [33] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [33] with the singleton support $x \in X$ and the value α $(0 < \alpha \leq 1)$ at x will be denoted by x_α. 0_X and 1_X are the constant fuzzy sets taking respectively the constant values 0 and 1 on X. The complement of a fuzzy set A in X will be denoted by $1_X \setminus A$ [36] and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$. For two fuzzy sets A and B in X, we write $A \leq B$ iff $A(x) \leq B(x)$, for each $x \in X$, while we write AqB to mean A is quasi-coincident (q-coincident, for short) with B [33] if there is some $x \in X$ such that $A(x) + B(x) > 1$; the negation of these two statements are written as $A \not\leq B$ and $A\not{q}B$ respectively. It is to be noted that AqB iff BqA. Also $A\not{q}B$ iff $A \leq 1_X \setminus B$ iff $B \leq 1_X \setminus A$. If A and B are crisp sets, $A\not{q}B$ iff A and B are disjoint. For a fuzzy point x_α and a fuzzy set A in an fts X, $x_\alpha \in A$ means $x_\alpha \leq A$, i.e., $A(x) \geq \alpha$. Moreover $x_\alpha qA$ is equivalent to $A(x) > 1 - \alpha$. Let τ be a fuzzy topology (in the sense of Chang [11]) on a nonempty set X. Then (X, τ) is called a fuzzy topological space (fts, for short). clA and $intA$ of a fuzzy set A in an fts respectively stand for the fuzzy closure and fuzzy interior of A in X [11]. A set A (or a fuzzy set A) in a topological space X (in an fts X) is said to be semiopen [15] (fuzzy semiopen [1]) if there exists an open set (respectively a fuzzy open set) U in X (in an fts X) such that $U \subseteq A \subseteq clU$ (resp. $U \leq A \leq clU$), or equivalently, if $A \subseteq cl\,(intA)$ (resp. $A \leq cl(intA)$). The set of all semiopen sets in a topological space X (resp. fuzzy semiopen sets in an fts X) will be denoted by $SO(X)$ (resp. $FSO(X)$). The complement of a semiopen set (resp. fuzzy semiopen set) in a topological space X (resp. in an fts X) is called a semiclosed (fuzzy semiclosed) set. The semiclosure (resp. fuzzy semiclosure) of a set A in a topological space X (resp. of a fuzzy set A in an fts X), to be written as $sclA$, is the set of all points (resp. the union of all fuzzy points) x in a topological space X (resp. y_α in an fts X) such that for every semiopen set (resp. fuzzy semiopen set) U in a topological space X (in an fts X) with $x \in U$ (resp. $x_\alpha qU$), it follows that $U \bigcap A \neq \phi$ (resp. UqA). A set A in a topological space X (resp. a fuzzy set A in an fts X) is semiclosed (fuzzy semiclosed) iff $A = sclA$. The union of all semiopen (resp. fuzzy semiopen) sets in a topological space X (resp. in an fts X) contained in a set (fuzzy set) A is called the semi-interior (resp. fuzzy semi-interior) of A, denoted by $sintA$. It is clear that a set (fuzzy set) A is semiopen (fuzzy semiopen) iff $A = sintA$. If a set A in a topological space X is semi-open as well as semi-closed, it is called semi-regular [12]. A (fuzzy) set A in a topological space X is called (fuzzy) regular open if $A = int(clA)$ ([1]).

A (fuzzy) set A in (an fts) a topological space X is called (fuzzy) preopen [19] ([22]) if $A \subseteq int(clA)$ ($A \leq intclA$). The complement of this is called (fuzzy) preclosed. For a fuzzy set A, $pclA = \bigwedge\{B : A \leq B, B$ is fuzzy preclosed $\}$.

The fuzzy θ-closure [24] (resp., fuzzy δ-closure [14]) of a fuzzy set A in an fts (X, τ), denoted by $\theta - clA$ (resp., $\delta - clA$) is the union of all those fuzzy points x_α such that $clUqA$ (resp., UqA) whenever $x_\alpha qU \in \tau$ (resp., $x_\alpha qU$ where U is fuzzy regular open set in X). A fuzzy set A in X is called fuzzy θ-closed [24] (resp., fuzzy δ-closed [14]) if $A = \theta - clA$ (resp., $A = \delta - clA$) and the complement of fuzzy θ-closed (resp., fuzzy δ-closed) set is known as a fuzzy θ-open (resp., fuzzy δ-open) set. The union of all fuzzy δ-open sets contained in a fuzzy set A is called fuzzy δ-interior of A and is denoted by $\delta - intA$.

A (fuzzy) set A in (an fts) a topological space X ((X, τ)) is called (fuzzy) δ-preopen [34] ([5]) (resp., (fuzzy) α-open [26] ([21]), (fuzzy) β-open [13] ([3]) if $A \subseteq int(\delta clA)$ (resp., $A \leq int(\delta clA)$, $A \subseteq int(cl(intA))$ ($A \leq int(cl(intA))$), $A \subseteq cl(int(clA))$ ($A \leq cl(int(clA))$) and the complement of them are respectively called (fuzzy) δ-preclosed, (fuzzy) α-closed, (fuzzy) β-closed sets. For a fuzzy set A in an fts (X, τ), $\delta - pclA = \bigwedge\{B : A \leq B$ and B is fuzzy δ-preclosed in $X\}$ (resp., $\alpha - clA = \bigwedge\{B : A \leq B$ and B is fuzzy α-closed $\}$, $\beta clA = \bigwedge\{B : A \leq B$ and B is fuzzy β-closed $\}$. A fuzzy set A in X is fuzzy δ-preclosed (resp., fuzzy α-closed [21], fuzzy β-closed [3]) iff $A = \delta - pclA$ [5] (resp., $A = \alpha - clA$ [21], $A = \beta - clA$ [3]). We note that the collection of all fuzzy open (resp., fuzzy semiopen, fuzzy preopen, fuzzy θ-open, fuzzy δ-open, fuzzy δ-preopen, fuzzy α-open, fuzzy β-open, fuzzy regular open) sets are denoted by (resp., $FSO(X)$, $FPO(X)$, $F\theta O(X)$, $F\delta O(X)$, $F\delta - PO(X)$, $F\alpha O(X)$, $F\beta O(X)$, $FRO(X)$).

2. Some well known definitions and theorems

In this section we recall some definitions and theorems for ready references.

Definition 2.1[29]. Let (X, τ) and (Y, τ_Y) be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in Y.
Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

Definition 2.2[29, 23]. For a fuzzy multifunction $F : X \to Y$, the upper inverse F^+ and lower inverse F^- are defined as follows:
For any fuzzy set A in Y, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x)qA\}$.

The relationship between the upper and the lower inverses of a fuzzy multifunction is known to be as follows.

Theorem 2.3 [23]. *For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_X \setminus A) = X \setminus F^+(A)$, for any fuzzy set A in Y.*

Definition 2.4. A fuzzy multifunction $F : X \to Y$ is said to be fuzzy
(i) upper continuous [29] (resp. upper semi-continuous [23], upper precontinuous, upper δ-precontinuous [6], upper α-continuous, upper β-continuous, upper s-θ-continuous) if for each $x \in X$ and each fuzzy open set V of Y with $F(x) \leq V$, there exists an open (resp., semiopen, preopen, δ-preopen, α-open, β-open, semi-regular) set U of X containing x such that $F(U) \leq V$,
(ii) lower continuous [29] (resp. lower semi-continuous [23], lower precontinuous, lower δ-precontinuous [6], lower α-continuous, lower β-continuous, lower s-θ-continuous) if for each $x \in X$ and each fuzzy open set V in Y with $F(x)qV$, there exists an open (resp., semi-open, preopen, δ-preopen, α-open, β-open, semi-regular) set U in X containing x such that $F(u)qV$ for all $u \in U$.

Definition 2.5. A fuzzy multifunction $F : X \to Y$ is said to be fuzzy
(i) upper irresolute [9] (resp., upper preirrsolute, upper α-irresolute, upper β-irresolute) if for each $x \in X$ and each fuzzy semiopen (resp., fuzzy preopen, fuzzy α-open, fuzzy β-open) set V of Y with $F(x) \leq V$, there exists a semiopen (resp., preopen, α-open, β-open) set U of X containing x such that $F(U) \leq V$,

(ii) lower irresolute [9] (resp., lower preirresolute, lower α-irresolute, lower β-irresolute) if for each $x \in X$ and each fuzzy semiopen (resp., fuzzy preopen, fuzzy α-open, fuzzy β-open) set V in Y with $F(x)qV$, there exists a semiopen (resp., preopen, α-open, β-open) set U in X containing x such that $F(u)qV$ for all $u \in U$.

3. Fuzzy minimal structures

The notion of fuzzy minimal structure (in the sense of Lowen) has been introduced in [2] as follows : A family $\mathcal{M}$ of fuzzy sets in X is said to be a fuzzy minimal structure on X if $\alpha 1_X \in \mathcal{M}$ for every $\alpha \in [0,1]$. Another notion of fuzzy minimal structure, more general than the above one, has been introduced in [25] and in [7] where it is said that a family $\mathcal{F}$ of fuzzy sets in X is a fuzzy minimal structure (in the sense of Chang) on X if $0_X \in \mathcal{F}$ and $1_X \in \mathcal{F}$. It is easy to see that every fuzzy minimal structure in the sense of Lowen, as well as every minimal structure in the sense of Noiri and Popa [28], is also a fuzzy minimal structure in the sense of Chang. In the present paper the notion of fuzzy minimal structure in the sense of Chang is used. The notion of minimal structure as appears in [28] has been introduced by Popa and Noiri in [31] and [32].

Definition 3.1. A subfamily m_{I^X} of I^X is called a *fuzzy minimal structure* (m-structure, for short) on X if $0_X \in m_{I^X}$ and $1_X \in m_{I^X}$. Each member of m_{I^X} is said to be m_{I^X}-open and the complement of an m_{I^X}-open set is called m_{I^X}-closed.

Remark 3.2. Let (X,τ) be an fts. Then the families τ, $FSO(X)$, $F\delta - PO(X)$, $FRO(X)$, $FPO(X)$, $F\alpha O(X)$, $F\beta O(X)$ are all m-structures on X.

Definition 3.3. Let X be a nonempty set and m_{I^X} an m-structure on X. For $A \in I^X$, the m_{I^X}-closure of A and m_{I^X}-interior of A are defined as follows :
$m_{I^X} - clA = \bigwedge\{F : A \le F, 1_X \setminus F \in m_{I^X}\}$
$m_{I^X} - intA = \bigvee\{D : D \le A, D \in m_{I^X}\}$.

Remark 3.4. Let (X,τ) be an fts and $A \in I^X$. If $m_{I^X} = \tau$ (resp. $FSO(X)$, $FPO(X)$, $F\delta O(X)$, $F\alpha O(X)$, $F\beta O(X)$, $F\theta O(X)$), then we have

$m_{I^X} - clA = clA$ (resp., $sclA$, $pclA$, $\delta pclA$, αclA, βclA, θclA)
$m_{I^X} - intA = intA$ (resp., $sintA$, $pintA$, $\delta intA$, $\alpha intA$, $\beta intA$, $\theta intA$).

Proposition 3.5. *Let X be a nonempty set and m_{I^X}, an m-structure on X. Then for any $A \in I^X$, a fuzzy point $x_\alpha \in m_{I^X} - clA$ if and only if for any $U \in m_{I^X}$ with $x_\alpha qU$, UqA.*

Proof. Let x_α be a fuzzy point. $x_\alpha \in m_{I^X} - clA$ is equivalent, by definition of $m_{I^X} - clA$ to : $x_\alpha \in F$ whenever $A \leq F$ and $1_X \setminus F \in m_{I^X}$. We denote $1_X \setminus F = U$ and use the equivalence between $A \leq F$ and $1_X \setminus F \leq 1_X \setminus A$. It follows that $x_\alpha \in m_{I^X} - clA$ is equivalent to $x_\alpha \in 1_X \setminus U$ whenever $U \leq 1_X \setminus A$ and $U \in m_{I^X}$, which implies that $U \not{q} A$ and $U \in m_{I^X}$ imply that $x_\alpha \not{q} U$ ... (1).
Necessity: Assume that $x_\alpha \in m_{I^X} - clA$. Let $U \in m_{I^X}$ with $x_\alpha qU$. If $U \not{q} A$, then by (1) we have $x_\alpha \not{q} U$, a contradiction. Hence UqA.
Sufficiency: Assume that $U \in m_{I^X}$ and $x_\alpha qU$ implies that UqA. We have to prove that (1) is true. Let $U \in m_{I^X}$ such that $U \not{q} A$. If $x_\alpha qU$, then our assumption shows that UqA, a contradiction. Then $x_\alpha \not{q} U$, therefore (1) holds.

Lemma 3.6. *Let X be a non empty set and m_{I^X} an m-structure on X. For $A, B \in I^X$, the following hold :*
(i) $A \leq B$ implies
(a) $m_{I^X} - intA \leq m_{I^X} - intB$,
(b) $m_{I^X} - clA \leq m_{I^X} - clB$.
(ii) (a) $m_{I^X} - cl0_X = 0_X$, $m_{I^X} - cl1_X = 1_X$,
(b) $m_{I^X} - int0_X = 0_X$, $m_{I^X} - int1_X = 1_X$.
(iii) $m_{I^X} - intA \leq A \leq m_{I^X} - clA$.
(iv) (a) $m_{I^X} - clA = A$ *if* $1_X \setminus A \in m_{I^X}$,
(b) $m_{I^X} - intA = A$, *if* $A \in m_{I^X}$.
(v) $m_{I^X} - cl(1_X \setminus A) = 1_X \setminus m_{I^X} - intA$,
(b) $m_{I^X} - int(1_X \setminus A) = 1_X \setminus m_{I^X} - clA$.
(vi) (a) $m_{I^X} - cl(m_{I^X} - clA) = m_{I^X} - clA$,
(b) $m_{I^X} - int(m_{I^X} - intA) = m_{I^X} - intA$.

Proof. (i) Assume that $A \leq B$.
(a) Then $\{D : D \leq A \text{ and } D \in m_{I^X}\} \subseteq \{D : D \leq B \text{ and } D \in m_{I^X}\}$, hence applying $\bigvee$ to both members we obtain $m_{I^X} - intA \leq m_{I^X} - intB$.
(b) Similarly, $\{F : B \leq F \text{ and } 1_X \setminus F \in m_{I^X}\} \supseteq \{F : A \leq F$

and $1_X \setminus F \in m_{I^X}\}$, hence applying $\bigwedge$ to both members we get $m_{I^X} - clB \geq m_{I^X} - clA$.

(ii) (a) $m_{I^X} - cl0_X = 0_X$ since $F_0 = 0_X$ satisfies $0_X \leq F_0$ and $1_X \setminus F_0 \in m_{I^X}$.
$m_{I^X} - cl1_X = 1_X$ since the only fuzzy set F with $1_X \leq F$ and $1_X \setminus F \in m_{I^X}$ is $F = 1_X$.
(b) $m_{I^X} - int0_X = 0_X$, since the only fuzzy set D with $D \leq 0_X$ and $D \in m_{I^X}$ is $D = 0_X$.
$m_{I^X} - int1_X = 1_X$, since $D_0 \leq 1_X$ and $D_0 \in m_{I^X}$.

(iii) Let $x \in X$. Since $(m_{I^X} - intA)(x) = \bigvee\{D(x) : D \leq A$ and $D \in m_{I^X}\}$ and $\{D(x) : D \leq A$ and $D \in m_{I^X}\} \subseteq \{C(x) : C \leq A\} \subseteq [0, A(x)]$, it follows that $(m_{I^X} - intA)(x) \leq A(x)$. On the other hand, $(m_{I^X} - clA)(x) = \bigwedge\{F(x) : A \leq F$ and $1_X \setminus F \in m_{I^X}\}$ and $\{F(x) : A \leq F$ and $1_X \setminus F \in m_{I^X}\} \subseteq [A(x), 1]$, hence $(m_{I^X} - clA)(x) \geq A(x)$.

(iv) (a) $A \leq m_{I^X} - clA$ by (iii).
Assume that $1_X \setminus A \in m_{I^X}$. Then $F_1 = A$ satisfies $A \leq F_1$ and $1_X \setminus F_1 \in m_{I^X}$, hence $\bigwedge\{F : A \leq F$ and $1_X \setminus F \in m_{I^X}\} \leq F_1$, i.e., $m_{I^X} - clA \leq A$.
(b) $m_{I^X} - intB \leq B$ by (iii).
Assume that $B \in m_{I^X}$. Then $D_1 = B$ satisfies $D_1 \leq A$ and $D_1 \in m_{I^X}$, hence $D_1 \leq \bigvee\{D : D \leq B$ and $D \in m_{I^X}\}$, i.e., $B \leq m_{I^X} - intB$.

(v) (a) $m_{I^X} - cl(1_X \setminus A) = \bigwedge\{F : 1_X \setminus A \leq F$ and $1_X \setminus F \in m_{I^X}\}$. But $1_X \setminus A \leq F$ is equivalent to $1_X \setminus F \leq A$. Therefore, $m_{I^X} - cl(1_X \setminus A) = \bigwedge\{1_X \setminus (1_X \setminus F) : 1_X \setminus F \leq A$ and $1_X \setminus F \in m_{I^X}\}$. Using the substitution $D = 1_X \setminus F$, we obtain $m_{I^X} - cl(1_X \setminus A) = \bigwedge\{1_X \setminus D : D \leq A$ and $D \in m_{I^X}\}$. But $\bigwedge\{1_X \setminus M : M \in \mathcal{F}\} = 1_X \setminus \bigvee\{M : M \in \mathcal{F}\}$ for every family $\mathcal{F} \subseteq I^X$, hence $m_{I^X} - cl(1_X \setminus A) = 1_X \setminus \bigvee\{D : D \leq A$ and $D \in m_{I^X}\} = 1_X \setminus m_{I^X} - intA$.
(b) Note that $M = 1_X \setminus N$ iff $N = 1_X \setminus M$. Denote $U = m_{I^X} - clA$. Writing $U = m_{I^X} - cl(1_X \setminus (1_X \setminus A))$ and using (v) (a) we obtain $U = 1_X \setminus m_{I^X} - int(1_X \setminus A)$, hence $m_{I^X} - int(1_X \setminus A) = 1_X \setminus U$.

(vi) (a) By (iii),

$$m_{I^X} - clA \leq m_{I^X} - cl(m_{I^X} - clA) \tag{1}$$

We have to show that $m_{I^X} - cl(m_{I^X} - clA) \leq m_{I^X} - clA$. Let $x_\alpha \notin m_{I^X} - clA$. Then there exists $U \in m_{I^X}$ with $x_\alpha q U$, $U \not{q} A$. Then by

Proposition 3.5, $U\not{q}m_{I^X} - clA$ which implies that $x_\alpha \notin m_{I^X} - cl(m_{I^X} - clA)$. Consequently,

$$m_{I^X} - cl(m_{I^X} - clA) \leq m_{I^X} - clA \tag{2}$$

Combining (1) and (2) we get the result.

(b) Let $A \in I^X$. Then $1_X \setminus A \in I^X$. By (vi) (a), $m_{I^X} - cl(m_{I^X} - cl(1_X \setminus A)) = m_{I^X} - cl(1_X \setminus A)$ which implies that $m_{I^X} - cl(1_X \setminus m_{I^X} - intA) = 1_X \setminus m_{I^X} - intA$ which implies that $1_X \setminus m_{I^X} - int(m_{I^X} - intA)$ (by (v) (a)) $= 1_X \setminus m_{I^X} - intA$. Consequently, $m_{I^X} - int(m_{I^X} - intA) = m_{I^X} - intA$.

Proposition 3.7. *Let X be a non empty set with an $m-$structure m_{I^X} and $A \in I^X$. Then for any $U \in m_{I^X}$, $U\not{q}A$ this implies that $U\not{q}m_{I^X} - clA$.*

Proof. $U \not{q}A$ which implies that $U(x) + A(x) \leq 1$, for all $x \in X$ i.e., $A \leq 1_X \setminus U$ and so $m_{I^X} - clA \leq m_{I^X} - cl(1_X \setminus U) = 1_X \setminus U$, as $U \in m_{I^X}$. Therefore, $U \not{q}m_{I^X} - clA$.

Definition 3.8. An $m-$structure m_{I^X} on a non empty set X is said to have *property ($\mathcal{B}$)* if the union of any family of members of m_{I^X} belongs to m_{I^X}.

The following lemma is a generalization of Lemma 3.3 from [31].

Lemma 3.9. *For a fuzzy minimal structure m_{I^X} on a non empty set X, the following are equivalent :*
(i) m_{I^X} has property $\mathcal{B}$.
(ii) $m_{I^X} - int(A) \in m_{I^X}$ for every $A \in I^X$.
(iii) If $m_{I^X} - intV = V$, then $V \in m_{I^X}$,
(iv) $1_X \setminus m_{I^X} - cl(B) \in m_{I^X}$ for every $B \in I^X$.
(v) if $m_{I^X} - clV = V$, then $1_X \setminus V \in m_{I^X}$.

Proof. (i) $\Rightarrow$ (ii) : Let $A \in I^X$. Since $m_{I^X} - int(A)$ is the union of fuzzy sets belonging to m_{I^X}, if m_{I^X} has property $\mathcal{B}$, then $m_{I^X} - int(A) \in m_{I^X}$.

(ii) $\Rightarrow$ (iii) : Let $V \in I^X$ be such that $m_{I^X} - int(V) = V$. By (ii) $m_{I^X} - int(V) \in m_{I^X}$, hence $V \in m_{I^X}$.

(iii) $\Rightarrow$ (i) : Let $A_i \in m_{I^X}$ for every $i \in \Delta$. Denote $V = \bigvee\{A_i : i \in \Delta\}$. We prove that $V \in m_{I^X}$. By Lemma 3.6, $m_{I^X} - intV) \leq V$ ... (1).
Since $A_i \leq V$ for each $i \in \Delta$ and $A_i \in m_{I^X}$ for every $i \in \Delta$, we have $\{A_i : i \in \Delta\} \subseteq \{U : U \leq V, U \in m_{I^X}\}$, hence $V = \bigvee\{A_i : i \in \Delta\} \leq \bigvee\{U : U \leq V, U \in m_{I^X}\} = m_{I^X} - int(V)$ ... (2).
Combining (1) and (2) we get $m_{I^X} - int(V) = V$. By (iii), $V \in m_{I^X}$.

(iv) $\Rightarrow$ (v) : Let $F \in I^X$ be such that $m_{I^X} - cl(F) = F$. By (iv) $1_X \setminus m_{I^X} - cl(F) \in m_{I^X}$, hence $1_X \setminus F \in m_{I^X}$.

(v) $\Rightarrow$ (iv) : Let $B \in I^X$. By Lemma 3.6, $m_{I^X} - cl(m_{I^X} - cl(B)) = m_{I^X} - cl(B)$. Using (v) with $F = m_{I^X} - cl(B)$, it follows that $1_X \setminus m_{I^X} - cl(B) \in m_{I^X}$.

(ii) $\Leftrightarrow$ (iv) : This follows from Lemma 3.6 (v).

Lemma 3.10. *Let X be a non empty set and m_{I^X} an $m-$ structure on X satisfying (B). For $A \in I^X$, the following properties hold :*
(i) $A \in m_{I^X}$ iff $m_{I^X} - intA = A$.
(ii) $1_X \setminus A \in m_{I^X}$ iff $m_{I^X} - clA = A$.
(iii) $m_{I^X} - intA \in m_{I^X}$ and $m_{I^X} - clA$ is m_{I^X}-closed.

Proof. The proof follows from Lemma 3.6 and Lemma 3.9.

4. Fuzzy upper and lower M-continuous multifunctions

In this section M-continuous multifunctions, introduced by Noiri and Popa [28] have been generalized in fuzzy topological space by using minimal structure on a non empty set X.
We first recall the following definitions and lemma from [28].

Definition 4.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a non empty set X is called a *minimal structure* (briefly, m-structure) on X if $\varnothing \in m_X$ and $X \in m_X$. Each member of m_X is said to be m_X-open and the complement of an m_X-open set is said to be m_X-closed.

Definition 4.2. A minimal structure m_X on a non empty set X is said to have *property* $(\mathcal{B})$ if the union of any family of subsets belonging to m_X belongs to m_X.

Definition 4.3. Let X be a non empty set and m_X an m-structure on X. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined as follows :
$m_X - clA = \bigcap\{F : A \subset F, X \setminus F \in m_X\}$,
$m_X - intA = \bigcup\{U : U \subset A, U \in m_X\}$.

Lemma 4.4. *Let X be a nonempty set and m_X an m-structure on X. For subsets A and B of X, the following hold :*
(i) $m_x - cl(X \setminus A) = X \setminus m_X - intA$ and $m_X - int(X \setminus A) = X \setminus m_X - clA$,
(ii) if $X \setminus A \in m_X$, then $m_X - clA = A$ and if $A \in m_X$, then $m_X - intA = A$,
(iii) $m_X - cl\varnothing = \varnothing$, $m_X - clX = X$, $m_X - int\varnothing = \varnothing$ and $m_X - intX = X$,
(iv) if $A \subset B$, then $m_X - clA \subset m_X - clB$ and $m_X - intA \subset m_X - intB$,
(v) $A \subset m_X - clA$ and $m_X - intA \subset A$,
(vi) $m_X - cl(m_X - clA) = m_X - clA$ and $m_X - int(m_X - intA) = m_X - intA$.

Now we give the following definition which extends Definition 3.4 [28].

Definition 4.5. Let X and Y be two non empty sets and m_X and m_{I^Y} be the minimal structure and fuzzy minimal structure on X and Y respectively. A fuzzy multifunction $F : (X, m_X) \to (Y, m_{I^Y})$ is said to be
(i) fuzzy upper M-continuous (f.u.M.c., for short) if for each $x \in X$ and each $V \in m_{I^Y}$ with $F(x) \leq V$, there exists $U \in m_X$ containing x such that $F(U) \leq V$,
(ii) fuzzy lower M-continuous (f.l.M.c., for short) if for each $x \in X$ and each $V \in m_{I^Y}$ with $F(x)qV$, there exists $U \in m_X$ containing x such that $F(u)qV$, for each $u \in U$.

The next two theorems generalize Theorem 3.1 and Theorem 3.2 [28].

Theorem 4.6. *For a fuzzy multifunction $F : (X, m_X) \to (Y, m_{I^Y})$ where (Y, m_{I^Y}) satisfies property ($\mathcal{B}$), the following statements are equivalent :*
(i) F is f.u.M.c.
(ii) $F^+(V) = m_X - int(F^+(V))$, for every $V \in m_{I^Y}$.
(iii) $F^-(K) = m_X - cl(F^-(K))$, for every fuzzy m_{I^Y}-closed set K.
(iv) $m_X - cl(F^-(B)) \subseteq F^-(m_{I^Y} - clB)$, for every $B \in I^Y$.
(v) $F^+(m_{I^Y} - intB) \subseteq m_X - int(F^+(B))$, for every $B \in I^Y$.

Proof. (i) $\Rightarrow$ (ii) Let $V \in m_{I^Y}$ and $x \in F^+(V)$. Then $F(x) \leq V$. By (i), there exists $U \in m_X$ with $x \in U$ such that $F(U) \leq V$. Then $U \subseteq F^+(V)$. Therefore, $x \in U \subseteq F^+(V) \Rightarrow x \in m_X - int(F^+(V))$. Thus $F^+(V) \subseteq m_X - int(F^+(V))$. Again by Lemma 4.4, $m_X - int(F^+(V)) \subseteq F^+(V)$. Consequently, $F^+(V) = m_X - int(F^+(V))$ where $V \in m_{I^Y}$.

(ii) $\Rightarrow$ (iii) Let K be any fuzzy m_{I^Y}-closed in Y. Then $1_Y \setminus F \in m_{I^Y}$. By (ii), $F^+(1_Y \setminus K) = m_X - int(F^+(1_Y \setminus K)) = m_X - int(X \setminus F^-(K))$, i.e., $X \setminus F^-(K) = X \setminus m_X - cl(F^-(K))$. Therefore, $F^-(K) = m_X - cl(F^-(K))$.

(iii) $\Rightarrow$ (iv) Let $B \in I^Y$. Then by Lemma 3.10, $m_{I^Y} - clB$ is fuzzy m_{I^Y}-closed. Since $B \leq m_{I^Y} - clB$, $F^-(B) \subseteq F^-(m_{I^Y} - clB)$ and so $m_X - cl(F^-(B)) \subseteq m_X - cl(F^-(m_{I^Y} - clB))$. But $m_{I^Y} - clB$ is fuzzy m_{I^Y}-closed and hence by (iii), $F^-(m_{I^Y} - clB) = m_X - cl(F^-(m_{I^Y} - clB))$. Therefore, $m_X - cl(F^-(B)) \subseteq F^-(m_{I^Y} - clB)$.

(iv) $\Rightarrow$ (v) Let $B \in I^Y$. Then $1_Y \setminus B \in I^Y$ and so by (iv), $m_X - cl(F^-(1_Y \setminus B)) \subseteq F^-(m_{I^Y} - cl(1_Y \setminus B))$ which implies that $m_X - cl(X \setminus F^+(B)) \subseteq F^-(1_Y \setminus m_{I^Y} - intB)$ which implies that $X \setminus m_X - intF^+(B) \subseteq X \setminus F^+(m_{I^Y} - intB)$ which implies that $F^+(m_{I^Y} - intB) \subseteq m_X - int(F^+(B))$.

(v) $\Rightarrow$ (i) Let $x \in X$ and $V \in m_{I^Y}$ be such that $F(x) \leq V$. Then $x \in F^+(V) = F^+(m_{I^Y} - intV) \subseteq m_X - int(F^+(V))$ by (v). Then there exists $U \in m_X$ with $x \in U$ such that $U \subseteq F^+(V)$, i.e., $F(U) \subseteq V$. This proves that F is f.u.M.c.

Theorem 4.7. *For a fuzzy multifunction $F : (X, m_X) \to (Y, m_{I^Y})$ where (Y, m_{I^Y}) satisfies property ($\mathcal{B}$), the following statements are*

equivalent :
(i) F is f.l.M.c.
(ii) $F^-(V) = m_X - int(F^-(V))$, for every $V \in m_{I^Y}$.
(iii) $F^+(K) = m - cl(F^+(K))$, for every fuzzy m_{I^Y}-closed set K.
(iv) $m_X - cl(F^+(B)) \subseteq F^+(m_{I^Y} - clB)$, for every $B \in I^Y$.
(v) $F(m_X - clA) \leq m_{I^Y} - cl(F(A))$, for every subset A of X.
(vi) $F^-(m_{I^Y} - intB) \subseteq m_X - int(F^-(B))$, for every $B \in I^Y$.

Proof. (i) $\Rightarrow$ (ii) Let $V \in m_{I^Y}$ and $x \in F^-(V)$. Then $F(x)qV$. By (i), there exists $U \in m_X$ containing x such that $F(u)qV$, for every $u \in U$ ... (1).
We only have to show that $U \subseteq F^-(V)$. Let $y \in U$ be arbitrary. We have to show that $y \in F^-(V)$, i.e., $F(y)qV$. By (i), $F(y)qV$ and so $x \in U \subseteq F^-(V)$ which implies that $x \in m_X - int(F^-(V))$. Therefore, $F^-(V) \subseteq m_X - int(F^-(V))$. Again for any subset B of X, $m_X - intB \subseteq B$, and so $m_X - int(F^-(V)) \subseteq F^-(V)$. Consequently $F^-(V) = m_X - int(F^-(V))$.

(ii) $\Rightarrow$ (iii) Let K be fuzzy m_{I^Y}-closed. Then $1_Y \setminus K \in m_{I^Y}$ and so by (ii), $F^-(1_Y \setminus K) = m_X - int(F^-(1_Y \setminus K))$ which implies that $X \setminus F^+(K) = m_X - int(X \setminus F^+(K)) = X \setminus m_X - cl(F^+(K))$. Therefore, $F^+(K) = m_X - cl(F^+(K))$.

(iii) $\Rightarrow$ (iv) Let $B \in I^Y$ and by Lemma 3.10, $m_{I^Y} - clB$ is fuzzy m_{I^Y}-closed. By (iii), $F^+(m_{I^Y} - clB) = m_X - cl(F^+(m_{I^Y} - clB))$. Since $B \leq m_{I^Y} - clB$, $F^+(B) \subseteq F^+(m_{I^Y} - clB) = m_X - cl(F^+(m_{I^Y} - clB))$. Therefore, $m_X - cl(F^+(B)) \subseteq m_X - cl(F^+((m_{I^Y} - clB)) = F^+(m_{I^Y} - clB)$ (by Lemma 4.4).

(iv) $\Rightarrow$ (v) Let $A \subseteq X$. Since $A \subseteq F^+(F(A))$, $m_X - clA \subseteq m_X - clF^+(F(A)) \subseteq F^+(m_{I^Y} - clF(A))$ by (iv). Therefore, $F(m_X - clA) \leq FF^+(m_{I^Y} - clF(A)) \leq m_{I^Y} - clF(A)$.

(v) $\Rightarrow$ (vi) Let $B \in I^Y$. Then $1_Y \setminus B \in I^Y$ and so $F^+(1_Y \setminus B) \subseteq X$ and then by (v), $F(m_X - clF^+(1_Y \setminus B)) \leq m_{I^Y} - clF(F^+(1_Y \setminus B)) \leq m_{I^Y} - cl(1_Y \setminus B) \leq 1_Y \setminus m_{I^Y} - intB$.
Again $F(m_X - clF^+(1_Y \setminus B)) = F(m_X - cl(X \setminus F^-(B)) = F(X \setminus m_X - intF^-(B))$. Therefore, $F(X \setminus m_X - intF^-(B)) \leq 1_Y \setminus m_{I^Y} - intB \Rightarrow F^+F(X \setminus m_X - int(F^-(B))) \subseteq F^+(1_Y \setminus m_{I^Y} - intB)$. Therefore, $X \setminus m_X - int(F^-(B)) \subseteq F^+F(X \setminus m_X - intF^-(B)) \subseteq X \setminus F^-(m_{I^Y} - intB)$.

Consequently, $F^{-}(m_{I^Y} - intB) \subseteq m_X - intF^{-}(B)$.

(vi) $\Rightarrow$ (i) Let $x \in X$ and $V \in m_{I^Y}$ be such that $F(x)qV$. Then $x \in F^{-}(V) = F^{-}(m_{I^Y} - intV) \subseteq m_X - int(F^{-}(V))$, by (vi). Then there exists $U \in m_X$ with $x \in U$ such that $x \in U \subseteq F^{-}(V)$. Then for any $u \in U$, $u \in F^{-}(V) \Rightarrow F(u)qV$ and this proves that F is f.l.M.c.

Remark 4.8. Let (X, τ) and (Y, τ_Y) be topological space and an fts respectively.
(i) We put $m_X = \tau$ (resp., $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $SRO(X)$) and $m_{I^Y} = \tau_Y$. Then an f.u.(l.).M.c. multifunction $F : (X, m_X) \to (Y, m_{I^Y})$ is an f.u.(l.).c. (resp., semi-continuous, precontinuous, α-continuous, β-continuous, s-θ-continuous).
(ii) We put $m_X = SO(X)$ (resp., $PO(X)$, $\alpha O(X)$, $\beta O(X)$) and $m_{I^Y} = SO(Y)$ (resp., $PO(Y)$, $\alpha O(Y)$, $\beta O(Y)$), then an f.u.(l.)M.c. multifunction $F : (X, m_X) \to (Y, m_{I^Y})$ is f.u.(l.) irresolute (resp., preirresolute or M-precontinuous, α-irresolute or strongly feebly continuous, β-irresolute).

5. Fuzzy m-Compact Sets

In this section different types of fuzzy compact-like sets have been unified.

Definition 5.1[11]. Let A be a fuzzy set in a set Y. A collection $\mathcal{U}$ of fuzzy sets in Y is called a fuzzy cover of A if $sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$. In particular, if $A = 1_Y$, we get the definition of fuzzy cover of Y.

Definition 5.2[28]. A non empty set X with a minimal structure m_X is said to be m-compact if every cover of X by m_X-open sets has a finite subcover.
A subset K of a non empty set X with a minimal structure m_X is said to be m-compact if every cover of K by m_X-open sets has a finite subcover.

Definition 5.3. A non empty set Y with a fuzzy minimal structure m_{I^Y} is said to be fuzzy m-compact if every fuzzy cover of Y by fuzzy m_{I^Y}-open sets has a finite subcover.

Definition 5.4. A fuzzy set K in Y with a fuzzy minimal structure m_{I^Y} is said to be fuzzy m-compact if every fuzzy cover of K by fuzzy m_{I^Y}-open sets has a finite subcover.

Theorem 5.5. Let X and Y be two non empty sets and m_X and m_{I^Y} be the minimal structure and fuzzy minimal structure on X and Y respectively and let m_{I^Y} satisfy property $(\mathcal{B})$. If $F : (X, m_X) \to (Y, m_{I^Y})$ is an f.u.M.c. surjective multifunction such that $F(x)$ is fuzzy m-compact for each $x \in X$ and (X, m_X) is m-compact, then (Y, m_{I^Y}) is fuzzy m-compact.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y by fuzzy m_{I^Y}-open sets. Now, for each $x \in X$, $F(x)$ is fuzzy m-compact set in Y, and so there is a finite subset Λ_x of Λ such that $F(x) \leq \bigvee\{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \bigvee\limits_{\alpha \in \Lambda_x} A_\alpha$. Then $F(x) \leq A_x$ and $A_x \in m_{I^Y}$. Since F is f.u.M.c., there exists $U_x \in m_X$ containing x such that $U_x \subseteq F^+(A_x)$. The family $\{U_x : x \in X\}$ is a cover of X by m_X-open sets. Since X is m-compact, there exists finitely many points $x_1, x_2, ..., x_n$ in X such that $X = \bigcup\limits_{i=1}^{n} U_{x_i}$. As F is surjective, $1_Y = F(X) = F(\bigcup\limits_{i=1}^{n} U_{x_i}) = \bigvee\limits_{i=1}^{n} F(U_{x_i}) \leq \bigvee\limits_{i=1}^{n} A_{x_i} = \bigvee\limits_{i=1}^{n} \bigvee\limits_{\alpha \in \Lambda_{x_i}} A_\alpha$. Hence Y is fuzzy m-compact.

Corollary 5.6. *Let X and Y be two non empty sets with minimal structure m_X and fuzzy minimal structure m_{I^Y} respectively. Let m_{I^Y} satisfy property $(\mathcal{B})$. If $F : (X, m_X) \to (Y, m_{I^Y})$ is an f.u.M.c. surjective multifunction such that $F(x)$ is fuzzy m-compact for each $x \in X$ and A is an m-compact set of (X, m_X), then $F(A)$ is fuzzy m-compact in (Y, m_{I^Y}).*

Definition 5.7. A subset (resp., fuzzy set) A of a topological space (resp., an fts) X is said to be compact (α-compact [27], semi compact [8], strongly compact [20]) (resp., fuzzy compact [11], fuzzy α-compact [21], fuzzy semi compact [10], fuzzy strongly compact) relative to X if every cover (resp., fuzzy cover) of A by open (α-open, semiopen, preopen) (resp., fuzzy open, fuzzy α-open, fuzzy semiopen, fuzzy preopen) sets has a finite subcover.

A topological space (resp., an fts) X is said to be compact (α-compact [16], semicompact [8], strongly compact [20]) (resp., fuzzy compact [11], fuzzy α-compact [21], fuzzy semi compact [18], fuzzy strongly compact) if the set X is compact (α-compact, semi compact, strongly compact) (resp., fuzzy compact, fuzzy α-compact, fuzzy semi compact, fuzzy strongly compact) relative to X.

Remark 5.8. Let (X, τ) and (Y, τ_Y) be a topological space and an fts respectively and $F : (X, \tau) \to (Y, \tau_Y)$ be a surjective fuzzy multifunction such that $F(x)$ is fuzzy compact for each $x \in X$. If (X, τ) is compact (resp., α-compact, semi compact, strongly compact) and F is fuzzy upper continuous (resp., fuzzy upper α-continuous, fuzzy upper semi-continuous, fuzzy upper pre-continuous), then (Y, τ_Y) is fuzzy compact.

6. Applications

Let (X, τ) be a topological space. A subset A of X is said to be δ-open [35] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is said to be a δ-cluster point of X if $intclV \bigcap A \neq \phi$ for every open set V in X containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δclA. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the δ-interior of A and is denoted by $\delta - intA$.

A subset (fuzzy set) A of X (in an fts Y) is said to be δ-preopen [34] (resp., δ-semi-open [30]) (fuzzy δ-semi-open) if $A \subseteq int\delta clA$ (resp., $A \subseteq cl\delta intA$) ($A \leq cl\delta intA$). The family of all δ-preopen (resp., δ-semi-open) (fuzzy δ-semi-open) sets in X (in Y) is denoted by $\delta PO(X)$ (resp., $\delta SO(X)$) ($F\delta SO(X)$). The complement of a δ-preopen (resp., δ-semi-open) (fuzzy δ-semi-open) set is called δ-preclosed [34] (resp., δ-semi closed [30]) (fuzzy δ-semi closed). The intersection of all δ-preclosed (resp., δ-semi-closed) (fuzzy δ-semi closed) set of X (in Y) containing A is called the δ-preclosure [34] (resp., δ-semi-closure [30]) (fuzzy δ-semi closure) of A and is denoted by $\delta pclA$ (resp., $\delta sclA$) ($F\delta sclA$). The union of all δ-preopen (resp., δ-semiopen) (fuzzy δ-semi-open) sets of X (in Y) contained in A is called the δ-preinterior (resp., δ-semi interior) (fuzzy δ-semi interior) of A and is denoted by $\delta pintA$ (resp., δ-$sintA$) ($F\delta$-$sintA$).

A fuzzy set is called fuzzy semi-clopen [1] if it is both fuzzy semi-open as well as fuzzy semi-closed.

Let (X, τ) be a topological space and (Y, τ_Y) be an fts. We now define some new types of fuzzy generalized continuity for fuzzy multifunctions.

Definition 6.1. A fuzzy multifunction $F : (X, \tau) \to (Y, \tau_Y)$ is said to be
(i) fuzzy upper δ-almost continuous (f.u.δ.a.c., for short) (resp., fuzzy upper δ-semi continuous) (f.u.δ.s.c., for short) if for each $x \in X$ and $V \in \tau_Y$ with $F(x) \leq V$, there exists a δ-preopen (resp., δ-semi-open) set U in X containing x such that $F(U) \leq V$,
(ii) fuzzy lower δ-almost continuous (f.l.δ.a.c., for short) (resp., fuzzy lower δ-semi continuous) (f.l.δ.s.c., for short) if for each $x \in X$ and each $V \in \tau_Y$ such that $F(x)qV$, there exists a δ-preopen (resp., δ-semi-open) set U in X containing x such that $F(u)qV$, for each $u \in U$.

Definition 6.2. A fuzzy multifunction $F : (X, \tau) \to (Y, \tau_Y)$ is said to be
(i) fuzzy upper δ-almost irresolute (f.u.δ.a.i., for short) (resp., fuzzy upper δ-semi irresolute, fuzzy upper s-θ-irresolute) if for each $x \in X$ and each fuzzy δ-preopen (resp., fuzzy δ-semi open, fuzzy semi-clopen) set V in Y with $F(x) \leq V$, there exists a δ-preopen (resp., δ-semi open, semi regular) set U of X containing x such that $F(U) \leq V$,
(ii) fuzzy lower δ-almost irresolute (f.l.δ.a.i., for short) (resp., fuzzy lower δ-semi irresolute, fuzzy lower s-θ-irresolute) if for each $x \in X$ and each fuzzy δ-preopen (resp., fuzzy δ-semi open, fuzzy semi-clopen) set V in Y such that $F(x)qV$, there exists a δ-preopen (resp., δ-semi open, semi regular) set U of X containing x such that $F(u)qV$, for each $u \in U$.

Remark 6.3 [28]. Let (X, τ) be a topological space. Then the families $\delta PO(X)$ and $\delta SO(X)$ are all m-structures on X satisfying property $(\mathcal{B})$. If $m_X = \delta PO(X)$ (resp., $\delta SO(X)$), then for a subset A of X, we have
(i) $m_X - clA = \delta - pclA$ (resp., $\delta sclA$)
(ii) $m_X - intA = \delta - pintA$ (resp., $\delta sintA$).

The following two theorems are the characterizations of f.u.(l.) δ.a.c. multifunctions.

Theorem 6.4. *For a fuzzy multifunction $F : (X, \tau) \to (Y, \tau_Y)$, the following are equivalent :*
(i) F is f.u.δ.a.c.
(ii) $F^+(V) \in \delta PO(X)$ for every $V \in \tau_Y$.
(iii) $F^-(K)$ is δ-preclosed in X for every fuzzy closed set K of Y.
(iv) $\delta - pcl(F^-(B)) \subset F^-(clB)$, for every fuzzy set B of Y.
(v) $F^+(intB) \subset \delta$-$pint(F^+(B))$, for every fuzzy set B of Y.

Proof. Taking $m_X = \delta PO(X)$ and $m_{I^Y} = \tau_Y$ in Theorem 4.6, the proof is immediate.

Theorem 6.5. *For a fuzzy multifunction $F : (X, \tau) \to (Y, \tau_Y)$, the following are equivalent :*
(i) F is f.l.δ.a.c.
(ii) $F^-(V) \in \delta PO(X)$ for every $V \in \tau_Y$.
(iii) $F^+(K)$ is δ-preclosed in X for every fuzzy closed set K of Y.
(iv) $\delta - pcl(F^+(B)) \subset F^+(clB)$, for every fuzzy set B of Y.
(v) $F(\delta - pclA) \leq cl(F(A))$, for every subset A of X.
(v) $F^-(intB) \subset \delta - pint(F^-(B))$, for every fuzzy set B of Y.

Proof. Taking $m_X = \delta PO(X)$ and $m_{I^Y} = \tau_Y$ in Theorem 4.7, the proof is immediate.

Again, we define the following fuzzy multifunctions. Using Theorem 4.6 and Theorem 4.7, we can obtain their characterizations.

Definition 6.6. A fuzzy multifunction $F : (X, \tau) \to (Y, \tau_Y)$ is said to be
(i) fuzzy upper strongly β-irresolute (resp., fuzzy upper strongly semi irresolute, fuzzy upper strongly M-precontinuous, fuzzy upper strongly α-irresolute) if for each $x \in X$ and each fuzzy β-open (resp., fuzzy semi open, fuzzy α-open, fuzzy preopen) set V of Y with $F(x) \leq V$, there exists an open set U of X containing x such that $F(U) \leq V$,
(ii) fuzzy lower strongly β-irresolute (resp., fuzzy lower strongly semi irresolute, fuzzy lower strongly M-precontinuous, fuzzy lower strongly α-irresolute) if for each $x \in X$ and each fuzzy β-open (resp., fuzzy semi open, fuzzy preopen, fuzzy α-open) set V of Y with$F(x)qV$, there exists an open set U of X containing x such that $F(u)qV$, for each $u \in U$.

Remark 6.7. Let $F : (X, \tau) \to (Y, \tau_Y)$ be a fuzzy multifunction. Let us take $m_X = \tau$ and $m_{I^Y} = F\beta O(Y)$ (resp., $FSO(Y)$, $FPO(Y)$, $F\alpha O(Y)$). Then we obtain characterizations of
(i) fuzzy upper strongly β-irresolute (resp., fuzzy upper strongly semi irresolute, fuzzy upper strongly M-precontinuous, fuzzy upper strongly α-irresolute) by Theorem 4.6.
(ii) fuzzy lower strongly β-irresolute (resp., fuzzy lower strongly semi irresolute, fuzzy lower strongly M-precontinuous, fuzzy lower strongly α-irresolute) by Theorem 4.7.

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