"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 25(2015), No. 1, 145-158

SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE PRODUCTS WITH EXPONENTIALS ARE CONVEX

S. S. DRAGOMIR AND I. GOMM

Abstract. We prove several inequalities of Hermite-Hadamard type for real functions having a convex product with an exponential $\exp(\alpha\cdot)$ for some nonzero real number α.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the papers [1] – [59] and the references therein.

Keywords and phrases: Convex functions, Hermite-Hadamard inequality, Special means
(2010)Mathematics Subject Classification: 26D15; 25D10.

Following [20], we denote by $\mathfrak{Expconv}(I)$ the class of all functions defined on the interval I of real numbers such that $\exp(f)$ is convex on I. If $\mathfrak{Conv}(I)$ is the class of convex functions defined on I then we have the following fact [20]:

Proposition 1. *We have the strict inclusion*

$$\mathfrak{Conv}(I) \subsetneq \mathfrak{Expconv}(I).$$

Now, if $g \in \mathfrak{Expconv}(I)$, then by the Hermite-Hadamard inequality for $\exp(g)$ we have for $a, b \in I$ with $a < b$ that

$$\exp g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \exp g(t)\,dt \leq \frac{1}{2}\left[\exp g(a) + \exp g(b)\right]. \tag{2}$$

Consider the *identric mean* of two positive numbers

$$I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0.$$

We observe that

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln u du$$

for $a, b > 0$, $a \neq b$.

The following result holds [20]:

Theorem 2. *Assume that $f \in \mathfrak{Expconv}(I)$ and $a, b \in I$ with $a < b$. Then we have*

$$\exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq I(\exp f(a), \exp f(b)) \tag{3}$$

and

$$\begin{aligned} &\exp f\left(\frac{a+b}{2}\right) \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right]dx\right) \\ &\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx. \end{aligned} \tag{4}$$

Remark 1. *If the function $g : I \to (0, \infty)$ is convex on I, then $f = \ln g \in \mathfrak{Expconv}(I)$ and for $a, b \in I$ with $a < b$ we have, by (3) and (4),* *the following inequalities*

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(t)\,dt\right) \le I(g(a), g(b)) \tag{5}$$

and

$$g\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{g(x)+g(a+b-x)}{2}\right]dx\right) \le \frac{1}{b-a}\int_a^b g(x)\,dx. \tag{6}$$

We denote by $\mathfrak{Expproconv}_\alpha(I)$ for $\alpha \neq 0$, the class of all functions f defined on the interval I of real numbers such that the function $g_\alpha(t) := f(t)\exp(\alpha t)$ is convex on I.

Consider the function $\ell(t) = t$, $t \in \mathbb{R}$. Then $g_\alpha(t) = t\exp(\alpha t)$. Taking the second derivative of g_α we have

$$g''_\alpha(t) = \alpha e^{\alpha t}(\alpha t + 2).$$

We observe that for either $\alpha > 0$ or $\alpha < 0$ the function g_α is concave on $\left(\infty, -\frac{2}{\alpha}\right]$ and convex on $\left[-\frac{2}{\alpha}, \infty\right)$. Therefore for any $\alpha \neq 0$ the function $\ell \in \mathfrak{Expproconv}_\alpha\left(\left[-\frac{2}{\alpha}, \infty\right)\right)$.

The interested reader can give other examples of such functions, for instance all functions $f_p(t) = t^p$, $p < 0$ and $t \in (0, \infty)$ are in the class $\mathfrak{Expproconv}_\alpha(0, \infty)$ for every $\alpha \neq 0$. Indeed, if we take the second derivative of the function $g_{\alpha,p}(t) = t^p \exp(\alpha t)$, then we have

$$g''_{\alpha,p}(t) = e^{\alpha t} t^{p-2}\left((\alpha t + p)^2 - p\right) > 0,$$

for any $\alpha \neq 0$, $t \in (0, \infty)$ and $p < 0$.

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for functions from the class $\mathfrak{Expproconv}_\alpha(I)$.

2. Some Inequalities of Hermite-Hadamard Type

We have the following result:

Theorem 3. *Assume that the function $f : I \to \mathbb{R}$ has the property that $f \in \mathfrak{Expproconv}_\alpha(I)$ for some $\alpha \neq 0$. Then f is integrable on any* *interval* $[a,b] \subset I$ *and*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\exp\left[\alpha\left(t-\frac{a+b}{2}\right)\right]dt \tag{7}$$
$$\leq \frac{1}{2}\left[f(a)\exp\left(-\alpha\frac{b-a}{2}\right)+f(b)\exp\left(\alpha\frac{b-a}{2}\right)\right].$$

Proof. Since $f \in \mathfrak{Expproconv}_{\alpha}(I)$, $\alpha \neq 0$, the function $g_{\alpha} : I \to \mathbb{R}$ defined by $g_{\alpha}(t) = f(t)\exp(\alpha t)$ is convex, which gives that $f(t) = g_{\alpha}(t)\exp(-\alpha t)$, $t \in I$. This shows that f is integrable on $[a,b]$.

Now, using the Hermite-Hadamard inequality for g_{α} we have

$$f\left(\frac{a+b}{2}\right)\exp\left(\alpha\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\exp(\alpha t)\,dt$$
$$\leq \frac{1}{2}\left[f(a)\exp(\alpha a)+f(b)\exp(\alpha b)\right].$$

Dividing this inequality by $\exp\left(\alpha\frac{a+b}{2}\right)$ we get the desired result (7).

An alternative direct proof for the first inequality is as follows.

By the fact that $g_{\alpha}(t) = f(t)\exp(\alpha t)$ is convex on I, we have

$$f\left(\frac{x+y}{2}\right)\exp\left[\alpha\left(\frac{x+y}{2}\right)\right] \leq \frac{1}{2}\left[f(x)\exp(\alpha x)+f(y)\exp(\alpha y)\right],$$

which is equivalent, by division with $\exp\left[\alpha\left(\frac{x+y}{2}\right)\right]$, to

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f(x)\exp\left(-\frac{1}{2}\alpha(y-x)\right)+f(y)\exp\left(\frac{1}{2}\alpha(y-x)\right)\right] \tag{8}$$

for any $x,y \in I$.

If we take

$$y = (1-t)a + tb \text{ and } x = (1-t)b + ta$$

then we have

$$y - x = (2t-1)(b-a) \text{ and } \frac{x+y}{2} = \frac{a+b}{2},$$

and by (8) we get

$$
\begin{aligned}
(9) \qquad & f\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{2} f\left((1-t) b+t a\right) \exp \left(-\frac{1}{2} \alpha(2 t-1)(b-a)\right) \\
& +\frac{1}{2} f\left((1-t) a+t b\right) \exp \left(\frac{1}{2} \alpha(2 t-1)(b-a)\right) \\
& =\frac{1}{2} f\left((1-t) b+t a\right) \exp \left(-\alpha\left(t-\frac{1}{2}\right)(b-a)\right) \\
& +\frac{1}{2} f\left((1-t) a+t b\right) \exp \left(\alpha\left(t-\frac{1}{2}\right)(b-a)\right)
\end{aligned}
$$

for any $t \in [0,1]$.

Integrate over $t \in [0,1]$ to get

$$
\begin{aligned}
(10) \qquad & f\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{2} \int_0^1 f\left((1-t) b+t a\right) \exp \left(-\alpha\left(t-\frac{1}{2}\right)(b-a)\right) dt \\
& +\frac{1}{2} \int_0^1 f\left((1-t) a+t b\right) \exp \left(\alpha\left(t-\frac{1}{2}\right)(b-a)\right) dt.
\end{aligned}
$$

If we change the variable by putting $s=1-t$, then we have

$$
\begin{aligned}
& \int_0^1 f\left((1-t) a+t b\right) \exp \left(\alpha\left(t-\frac{1}{2}\right)(b-a)\right) dt \\
& =\int_0^1 f\left(s a+(1-s) b\right) \exp \left(-\alpha\left(s-\frac{1}{2}\right)(b-a)\right) ds
\end{aligned}
$$

and by (10) we get

(11)

$$
f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left((1-t) b+t a\right) \exp \left(-\alpha\left(t-\frac{1}{2}\right)(b-a)\right) dt.
$$

Now, if we change the variable $u=(1-t) b+t a$, then $t=\frac{b-u}{b-a}$,

$$
t-\frac{1}{2}=\frac{b-u}{b-a}-\frac{1}{2}=\frac{2 b-2 u-b+a}{2(b-a)}=\frac{1}{b-a}\left(\frac{b+a}{2}-u\right),
$$

which implies that

$$\exp\left(-\alpha\left(t-\frac{1}{2}\right)(b-a)\right) = \exp\left[\frac{-\alpha}{b-a}\left(\frac{b+a}{2}-u\right)(b-a)\right] = \exp\left[\alpha\left(u-\frac{b+a}{2}\right)\right].$$

Then by (11) we get

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(u)\exp\left[\alpha\left(u-\frac{b+a}{2}\right)\right] du.$$

∎

The following result also holds:

Theorem 4. *Assume that the function $f : I \to \mathbb{R}$ has the property that $f \in \mathfrak{Expproconv}_{\alpha}(I)$ for some $\alpha \neq 0$. Then for any $[a,b] \subset I$ with $a < b$ we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{\alpha(b-a)} \times \left(\left[1-\frac{\exp(-\alpha(b-a))-1}{\alpha(b-a)}\right] f(a) + \left[\frac{1-\exp(\alpha(b-a))}{\alpha(b-a)}-1\right] f(b)\right). \tag{12}$$

Proof. Since $f \in \mathfrak{Expproconv}_{\alpha}(I)$, $\alpha \neq 0$, then

$$f((1-t)a+tb)\exp[\alpha((1-t)a+tb)] \le (1-t)f(a)\exp(\alpha a) + tf(b)\exp(\alpha b)$$

and if we divide by $\exp[\alpha((1-t)a+tb)]$ we get

$$f((1-t)a+tb) \le (1-t)f(a)\exp(\alpha t(a-b)) + tf(b)\exp(\alpha(1-t)(b-a)) \tag{13}$$

for any $a, b \in I$ and $t \in [0,1]$.

Integrating (13) over t on $[0,1]$ we get

$$\int_0^1 f((1-t)a+tb)\,dt \le f(a)\int_0^1 (1-t)\exp(\alpha t(a-b))\,dt + f(b)\int_0^1 t\exp(\alpha(1-t)(b-a))\,dt. \tag{14}$$

Observe, for $a \neq b$, that integrating by parts

$$\int_0^1 (1-t)\exp(\alpha t(a-b))\,dt = \frac{1}{\alpha(a-b)}\left[\frac{\exp(\alpha(a-b))-1}{\alpha(a-b)} - 1\right]$$

and

$$\int_0^1 t\exp(\alpha(1-t)(b-a))\,dt = \frac{1}{\alpha(a-b)}\left[1 - \frac{1-\exp(-\alpha(a-b))}{\alpha(a-b)}\right]$$

and by (14) we get the desired result (12). ∎

Remark 2. *Since all functions $f_p(t) = t^p$, $p < 0$ and $t \in (0,\infty)$ are in the class $\mathfrak{Expproconv}_\alpha(0,\infty)$ for every $\alpha \neq 0$, then by (7) we have for $0 < a < b$ that*

$$\left(\frac{a+b}{2}\right)^p \le \frac{1}{b-a}\int_a^b t^p \exp\left[\alpha\left(t - \frac{a+b}{2}\right)\right] dt \le \frac{1}{2}\left[a^p \exp\left(-\alpha\frac{b-a}{2}\right) + b^p \exp\left(\alpha\frac{b-a}{2}\right)\right] \tag{15}$$

and by (12) we have

$$\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \le \frac{1}{\alpha(b-a)} \times \left(\left[1 - \frac{\exp(-\alpha(b-a))-1}{\alpha(b-a)}\right]a^p + \left[\frac{1-\exp(\alpha(b-a))}{\alpha(b-a)} - 1\right]b^p\right), \tag{16}$$

here $p \in (-\infty,0)\setminus\{-1\}$.

3. Further Related Results

We also have the following result:

Theorem 5. *Assume that the function $f : I \to \mathbb{R}$ has the property that $f \in \mathfrak{Expproconv}_\alpha(I)$ for some $\alpha \neq 0$. Then for any $[a,b] \subset I$ we have*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right) dx dy \le E\left(-\frac{1}{2}\alpha b, -\frac{1}{2}\alpha a\right)\frac{1}{b-a}\int_a^b f(y)\exp\left(\frac{1}{2}\alpha y\right) dy, \tag{17}$$

where the exponential mean E is defined as

$$E(c,d) := \frac{\exp d - \exp c}{d-c}, \ \ d,c \in R, d \neq c.$$

Proof. Using the inequality (8) and integrating on the square $[a,b] \times [a,b]$ we have

$$\begin{aligned}
(18) \qquad & \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \\
& \leq \frac{1}{2} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(x) \exp\left(-\frac{1}{2}\alpha(y-x)\right) dxdy \\
& + \frac{1}{2} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(y) \exp\left(\frac{1}{2}\alpha(y-x)\right) dxdy.
\end{aligned}$$

Changing x with y we get

$$\begin{aligned}
& \frac{1}{(b-a)^2} \int_a^b \int_a^b f(x) \exp\left(-\frac{1}{2}\alpha(y-x)\right) dxdy \\
& = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(y) \exp\left(\frac{1}{2}\alpha(y-x)\right) dxdy
\end{aligned}$$

and by (18) we get

$$\begin{aligned}
(19) \qquad & \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy \\
& \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f(y) \exp\left(\frac{1}{2}\alpha(y-x)\right) dxdy.
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{1}{(b-a)^2} \int_a^b \int_a^b f(y) \exp\left(\frac{1}{2}\alpha(y-x)\right) dxdy \\
& = \frac{1}{(b-a)^2} \int_a^b \exp\left(-\frac{1}{2}\alpha x\right) dx \int_a^b f(y) \exp\left(\frac{1}{2}\alpha y\right) dy \\
& = E\left(-\frac{1}{2}\alpha b, -\frac{1}{2}\alpha a\right) \frac{1}{b-a} \int_a^b f(y) \exp\left(\frac{1}{2}\alpha y\right) dy,
\end{aligned}$$

then we get from (19) the desired result (17). ∎

Finally we have:

Theorem 6. *Assume that the function $f : I \to \mathbb{R}$ has the property that $f \in \mathfrak{Expproconv}_{\alpha}(I)$ for some $\alpha \neq 0$. Then for any $[a,b] \subset I$ with $a < b$ we have*

$$f(x)\exp(\alpha x) E(-\alpha a, -\alpha b) - \alpha \frac{1}{b-a}\int_a^b f(y)(x-y)\,dy \geq \frac{2}{b-a}\int_a^b f(y)\,dy - \frac{f(b)(b-x) + f(a)(x-a)}{b-a} \tag{20}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\exp\left(\alpha\frac{a+b}{2}\right) E(-\alpha a, -\alpha b) \\ &- \alpha\frac{1}{b-a}\int_a^b f(y)\left(\frac{a+b}{2} - y\right)dy \\ &\geq \frac{2}{b-a}\int_a^b f(y)\,dy - \frac{f(b)+f(a)}{2}. \end{aligned} \tag{21}$$

Proof. Since $f\exp(\alpha\cdot)$ is convex then $f\exp(\alpha\cdot)$ has lateral derivative on the interior $\mathring{I}$ of I and therefore f has lateral derivatives on $\mathring{I}$.

Using the gradient inequality

$$g(x) - g(y) \geq g'_+(y)(x-y)$$

for any $x, y \in \mathring{I}$, that holds for a convex function g defined on the interval I we have

$$f(x)\exp(\alpha x) - f(y)\exp(\alpha y) \geq \left[\alpha f(y)\exp(\alpha y) + f'_+(y)\exp(\alpha y)\right](x-y)$$

for any $x, y \in \mathring{I}$.

Dividing by $\exp(\alpha y)$ we get

$$f(x)\exp(\alpha x)\exp(-\alpha y) - f(y) \geq \left[\alpha f(y) + f'_+(y)\right](x-y)$$

that is equivalent to

$$f(x)\exp(\alpha x)\exp(-\alpha y) - \alpha f(y)(x-y) \geq f(y) + f'_+(y)(x-y) \tag{22}$$

for any $x, y \in \mathring{I}$.

Integrating (22) over $y \in [a,b]$ and dividing by $b-a$ we get

(23)

$$f(x)\exp(\alpha x)\frac{1}{b-a}\int_a^b \exp(-\alpha y)\,dy - \alpha\frac{1}{b-a}\int_a^b f(y)(x-y)\,dy$$
$$\geq \frac{1}{b-a}\int_a^b f(y)\,dy + \frac{1}{b-a}\int_a^b f'_+(y)(x-y)\,dy.$$

Since

$$\frac{1}{b-a}\int_a^b \exp(-\alpha y)\,dy = E(-\alpha a, -\alpha b)$$

and

$$\int_a^b f'_+(y)(x-y)\,dy = f(y)(x-y)\big|_a^b + \int_a^b f(y)\,dy$$
$$= \int_a^b f(y)\,dy - f(b)(b-x) - f(a)(x-a),$$

then by (23) we get the desired result (20). ∎

Corollary 7. *With the assumptions of Theorem 6 and if*

$$\frac{\int_a^b f(y)\,y dy}{\int_a^b f(y)\,dy} \in [a,b], \tag{24}$$

then

(25)

$$f\left(\frac{\int_a^b f(y)\,y dy}{\int_a^b f(y)\,dy}\right)\exp\left(\alpha\frac{\int_a^b f(y)\,y dy}{\int_a^b f(y)\,dy}\right)E(-\alpha a, -\alpha b)$$
$$\geq \frac{2}{b-a}\int_a^b f(y)\,dy$$
$$-\frac{1}{b-a}\left[f(b)\left(b - \frac{\int_a^b f(y)\,y dy}{\int_a^b f(y)\,dy}\right) + f(a)\left(\frac{\int_a^b f(y)\,y dy}{\int_a^b f(y)\,dy} - a\right)\right].$$

Remark 3. *We observe that a sufficient condition for (24) to hold is that $f(y) \geq 0$ for any $y \in [a,b]$.*

Acknowledgement. The authors would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

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Victoria University
College of Engineering and Science
Ballarat Road, Footscray, Melbourne 8001
Victoria, AUSTRALIA
e-mail: sever.dragomir@vu.edu.au, ian.gomm@vu.edu.au