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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 25(2015), No. 1, 159-172

VERTICAL FOLIATION ASSOCIATED TO A CARTAN SPACE

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Abstract The cotangent bundle of a smooth manifold, as a particular submersion, carries a natural foliation called *vertical* defined by the kernel of the differential of the projection of the cotangent bundle on its base manifold. The vertical foliation is a Lagrangian one with respect to the natural symplectic structure of the cotangent bundle. It has new properties if the cotangent bundle has additional geometrical structures, for instance those induced by a non-degenerate homogeneous Hamiltonian.

A Cartan space is a manifold whose cotangent bundle is endowed with a smooth non-degenerate Hamiltonian K^2 which is positively homogeneous of degree 2 in momenta. Then the vertical foliation becomes a semi Riemannian foliation whose transversal distribution is completely determined by K and is orthogonal on the vertical distribution with respect to a semi Riemannian metric of Sasaki type. In this framework various linear connections will be associated to and some properties of the vertical foliation will be pointed out.

Keywords and phrases: cotangent bundle, non-degenerate homogeneous Hamiltonian, foliations.
(2010) Mathematics Subject Classification: 53C12, 53C15.

Introduction

The cotangent bundle T^*M over a smooth manifold M carries the so-called vertical foliation whose leaves are the cotangent spaces T^*_xM and the corresponding distribution is the so-called vertical distribution defined by the kernel of the differential of the projection map $\tau^* : T^*M \to M$. The vertical foliation has special properties if the cotangent bundle is endowed with supplementary structures. In this paper we shall assume that the cotangent bundle is endowed with a positive non-degenerate (regular) Hamiltonian K^2 which is positively homogeneous of degree 2 in momenta. The geometry of regular Hamiltonians is useful in Mechanics as well as in some theories in Physics. It is dual by the Legendre transformation to the geometry of regular Lagrangians (Lagrange geometry) for which we refer to [6]. The square of a fundamental Finsler function is a regular Lagrangian. Thus the Lagrange geometry generalizes the Finsler geometry which was developed by P.Finsler, E. Cartan, L. Berwald and many others, see [4] and the graduate text [1]. The geometry of regular Hamiltonians (Hamilton geometry) due mainly to R. Miron is systematically presented in the monograph [5]. A manifold endowed a positive regular Hamiltonian K^2 that is positively $2-$ homogeneous in momenta was called by R. Miron a Cartan space, [5]. These Cartan spaces are dual by the Legendre transformation to Finsler spaces.

If (M, K) is a Cartan space, the vertical distribution (vertical subbundle)is naturally endowed with a (semi-)Riemannian metric and the vertical foliation becomes a (semi-) Riemannian foliation. Moreover, the vertical distribution admits a canonical complementary distribution called horizontal. This is defined by a set of local functions which generalizes the local coefficients of a linear connection, a reason to call it sometimes a nonlinear connection. Furthermore, these two new objects (metric and nonlinear connection) allow to construct a metrical structure G on T^*M similar to the Sasaki metric from the geometry of the tangent bundle. The vertical foliation and its transversal distribution provide a pair of complementary distributions. Besides the standard linear connections on T^*M adapted to this pair and named after Berwald, Cartan, Chern-Rund and Hashiguchi the Levi Civita connection of G defines the other two called the Vrănceanu connection and the Schouten-van Kampen connection. All these connections are written in the frame adapted to the nonlinear connection defined by K. Local resulting coefficients are used to compare the linear connections above. They are also used to give necessary and sufficient conditions for vertical foliation be totally geodesic and its transversal distribution be geodesically invariant or totally geodesic.

The paper is organized as follows. Section 1 contains generalities from the geometry of cotangent bundle. In Section 2 the necessary facts from the geometry of Cartan spaces are presented. Section 3 is devoted to the vertical foliation.

1. Preliminaries

Let M be an n-dimensional C^∞ manifold and $\tau^* : T^*M \to M$ its cotangent bundle. If (x^i) are local coordinates on M, then (x^i, p_i) will be taken as local coordinates on T^*M with the momenta (p_i) provided by $p = p_i dx^i$ where $p \in T^*_x M$, $x = (x^i)$ and (dx^i) is the natural basis of $T^*_x M$. The indices $i, j, k...$ will run from 1 to n and the Einstein convention on summation will be used. A change of coordinates $(x^i, p_i) \to (\widetilde{x}^i, \widetilde{p}_i)$ on T^*M has the form

$$
\begin{aligned}
&\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \quad \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\
&\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}(\widetilde{x}) p_j,
\end{aligned} \tag{1.1}
$$

where $\left(\frac{\partial x^j}{\partial \widetilde{x}^i}\right)$ is the inverse of the Jacobian matrix $\left(\frac{\partial \widetilde{x}^j}{\partial x^k}\right)$.

Let $\left(\partial_i := \frac{\partial}{\partial x^i},\ \partial^i := \frac{\partial}{\partial p_i}\right)$ be the natural basis in $T_{(x,p)}T^*M$. The change of coordinates (1.1) implies

$$
\begin{aligned}
\partial_i &= (\partial_i \widetilde{x}^j)\widetilde{\partial}_j + (\partial_i \widetilde{p}_j)\widetilde{\partial}^j, \\
\widetilde{\partial}^i &= (\partial_j \widetilde{x}^i)\partial^j.
\end{aligned} \tag{1.2}
$$

The natural cobasis (dx^i, dp_i) from $T^*_{(x,p)}T^*M$ transforms as follows

$$
d\widetilde{x}^i = (\partial_j \widetilde{x}^i) dx^j,\ d\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\, dp_j + \frac{\partial^2 x^j}{\partial \widetilde{x}^i \partial \widetilde{x}^k}\, p_j\, dx^k. \tag{1.3}
$$

The kernel $V_{(x,p)}$ of the differential $d\tau^* : T_{(x,p)}T^*M \to T_x M$ is called the *vertical* subspace of $T_{(x,p)}T^*M$ and the mapping $(x, p) \to V_{(x,p)}$ is a regular distribution on T^*M called the *vertical distribution.* This is locally spanned by (∂^i) . Hence it is integrable and defines a foliation with the leaves $T^*_x M$, $x \in M$. The vector field $L = p_i \partial^i$ is called the Liouville vector field and $\omega = p_i dx^i$ is called the Liouville 1-form on T^*M. Then $\theta = d\omega$ is the canonical symplectic structure on T^*M. For an easier handling of the geometrical objects on T^*M it is usual to consider a complementary distribution to the vertical distribution, $(x,p) \to N_{(x,p)}$, called the *horizontal distribution* and to report all geometrical objects on T^*M to the decomposition

$$T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}. \tag{1.4}$$

The pieces produced by the decomposition (1.4) are called d–geometrical objects (d is for distinguished) since their local components behave like geometrical objects on M, although they depend on momenta $p = (p_i)$.

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\delta_i := \partial_i + N_{ij}(x,p)\partial^j, \tag{1.5}$$

and for a change of coordinates (1.1), the condition

$$\delta_i = (\partial_i \widetilde{x}^j)\widetilde{\delta}_j \text{ for } \widetilde{\delta}_j := \widetilde{\partial}_j + \widetilde{N}_{jk}(\widetilde{x},\widetilde{p})\widetilde{\partial}^k, \tag{1.6}$$

is equivalent with

$$\widetilde{N}_{ij}(\widetilde{x},\widetilde{p}) = \frac{\partial x^s}{\partial \widetilde{x}^i}\,\frac{\partial x^r}{\partial \widetilde{x}^j}\,N_{sr}(x,p) + \frac{\partial^2 x^r}{\partial \widetilde{x}^i \partial \widetilde{x}^j}\,p_r. \tag{1.7}$$

The horizontal distribution is called also a *nonlinear connection* on T^*M and the functions (N_{ij}) are called the local coefficients of this nonlinear connection. It is important to note that any regular Hamiltonian on T^*M determines a symmetric nonlinear connection, that is its local coefficients verify $N_{ij} = N_{ji}$. The basis (δ_i, ∂^i) is adapted to the decomposition (1.4). The dual of it is $(dx^i, \delta p_i)$, for $\delta p_i = dp_i - N_{ji}dx^j$ and then $\delta \widetilde{p}_i = \dfrac{\partial x^j}{\partial \widetilde{x}^i}\,\delta p_j$.

2. Cartan spaces

A *Cartan structure* on M is a function $K : T^*M \to [0,\infty)$ with the following properties:

(1) K is C^∞ on $T^*_\circ M := T^*M \setminus 0$ for $0 = \{(x,0),\ x \in M\}$,
(2) K is positive on $T^*_\circ M$,
(3) $K(x,\lambda p) = \lambda K(x,p)$ for all $\lambda > 0$,
(4) The quadratic form $(g^{ij}\zeta_i\zeta_j, \zeta_i \in \mathbf{R^n})$, where $g^{ij}(x,p) = \dfrac{1}{2}\,\partial^i\partial^j K^2(x,p)$, has the same signature at all points of $T^*_\circ M$.

Some authors replace the condition "the same signature" with the stronger condition "positive defined". In this case we have $K(x,p) > 0$, whenever $p \neq 0$.

Definition 2.1. The pair (M,K) is called a *Cartan space.*

Example. Let $(\gamma_{ij}(x))$ be the matrix of the local coefficients of a Riemannian metric on M and $(\gamma^{ij}(x))$ its inverse. Then $K(x,p) = \sqrt{\gamma^{ij}(x)p_ip_j}$ gives a Cartan structure. Thus any Riemannian manifold can be regarded as a Cartan space. More examples can be found in Ch. 6 of [5].

We put $p^i = \frac{1}{2}\,\partial^i K^2$ and $C^{ijk} = -\frac{1}{4}\,\partial^i\partial^j\partial^k K^2$. The properties of K imply

$$p^i = g^{ij}p_j,\ p_i = g_{ij}p^j,\ K^2 = g^{ij}p_ip_j = p_ip^j, \\ C^{ijk}p_k = C^{ikj}p_k = C^{kij}p_k = 0. \tag{2.1}$$

One considers the *formal Christoffel symbols*

$$\gamma^i_{jk}(x,p) := \frac{1}{2}\, g^{is}(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{jk}) \tag{2.2}$$

and the contractions $\gamma^\circ_{jk}(x,p) := \gamma^i_{jk}(x,p)p_i$, $\gamma^\circ_{j\circ} := \gamma^i_{jk}p_ip^k$. Then the functions

$$N_{ij}(x,p) = \gamma^\circ_{ij}(x,p) - \frac{1}{2}\,\gamma^\circ_{h\circ}(x,p)\partial^h g_{ij}(x,p), \tag{2.3}$$

verify (1.7). In other words, these functions define a nonlinear connection on $T^*_\circ M$. This nonlinear connection was discovered by R. Miron, [5]. Thus a decomposition (1.4) holds. From now on we shall use only the nonlinear connection given by (2.3).

The Lie brackets of the local vector fields (δ_i, ∂^j) are given by

$$[\delta_j,\delta_k] = R_{ijk}\partial^i, [\delta_j,\partial^k] = -\partial^k N_{ji}\partial^i, [\partial^i,\partial^j] = 0, \tag{2.4}$$

where

$$R_{ijk} = \delta_j N_{ki} - \delta_k N_{ji}. \tag{2.5}$$

By a direct computation it comes out that under a change of coordinates (2.1), the functions $R_{ijk}(x,p)$ behave like the components of a tensor field of type (3,0) on M (thus they define a d-tensor field) and the functions $\partial^k N_{ji}(x,p)$ as the components of a linear connection on M, that is, we have

$$\widetilde{\partial^i N_{jk}} = \frac{\partial\widetilde{x}^i}{\partial x^r}\frac{\partial x^s}{\partial\widetilde{x}^j}\frac{\partial x^h}{\partial\widetilde{x}^k}\partial^r N_{sh} + \frac{\partial\widetilde{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial\widetilde{x}^j\partial\widetilde{x}^k}. \tag{2.6}$$

A vector field on T^*M is called horizontal (vertical) if its values are in the horizontal (vertical) distribution. By (2.4) the Lie bracket of any two vertical vector fields is again a vertical vector field (thus the vertical distribution is always integrable) and the Lie bracket of any two horizontal vector fields is again an horizontal one if and only if

$R_{ijk} = 0$, that is the horizontal distribution is integrable if and only if the d-tensor field defined by R_{ijk} vanishes.

A linear connection D on T^*M is said to be an *N–linear connection* if

1° D preserves by parallelism the distributions N and V, equivalently it is adapted to to decomposition (1.4),

2° $D\theta = 0$, for $\theta = \delta p_i \wedge dx^i$.

The first condition shows that in the adapted basis (δ_i, ∂^i) the linear connection D should be given as follows :

$$
\begin{aligned}
D_{\delta_j}\delta_i &= H^k_{ij}\delta_k, \quad D_{\delta_j}\partial^i = \tilde{H}^i_{kj}\partial^k,\\
D_{\partial^j}\delta_i &= V^{kj}_i\delta_k, \quad D_{\partial^j}\partial^i = \tilde{V}^{ij}_k\partial^k,
\end{aligned}
\tag{2.7.}
$$

The second condition gives $\tilde{H}^i_{kj} = -H^i_{kj}, \tilde{V}^{ij}_k = -V^{ij}_k$.

Thus any $N-$linear connection is determined by the local coefficients $\Gamma D = (H^k_{ij}, V^{kj}_i)$. As an adapted linear connection it can be thought as defined by the quadruple $\Gamma D = (H^k_{ij}, V^{kj}_i, -H^k_{ij}, -V^{kj}_i)$, where the first two coefficients define a linear connection in the horizontal bundle and the next two define a linear connection in the vertical bundle.

The coefficients V^{kj}_i define a d–tensor field of type $(2,1)$ and $H^k_{ij}(x,p)$ behave like the coefficients of a linear connection on M. We recall that any linear connection on M of local coefficients Γ^i_{jk} defines a linear connection in the cotangent bundle of local coefficients $-\Gamma^i_{jk}$. Thus $\tilde{H}^i_{kj}$ behave like the local coefficients of a linear connection in the cotangent bundle.

The functions H^k_{ij} and V^{kj}_i define operators of h–covariant and v-covariant derivatives in the algebra of d-tensor fields, denoted by $_{|k}$ and $|^k$, respectively. For g^{ij} these are given by

$$
\begin{aligned}
g^{ij}{}_{|k} &= \delta_k g^{ij} + g^{sj}H^i_{sk} + g^{is}H^j_{sk},\\
g^{ij}\big|^k &= \partial^k g^{ij} + g^{sj}V^{ik}_s + g^{is}V^{jk}_s.
\end{aligned}
\tag{2.8}
$$

An N-linear connection $D\Gamma(N) = (H^i_{jk}, V^{ik}_j)$ is called *metrical* if

$$
g^{ij}{}_{|k} = 0, \quad g^{ij}\big|^k = 0. \tag{2.9}
$$

The first equation (2.8) is equivalent to $g_{ij|k} := \delta_k g_{ij} - g_{sj}H^s_{ik} - g_{is}H^s_{jk} = 0$ and the second equation (2.8) is equivalent to $g_{ij}\big|^k := \partial^k g_{ij} - g_{sj}V^{sk}_i - g_{is}V^{sk}_j = 0$.

One verifies that the N-linear connection $C\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ with

$$\begin{aligned} H^i_{jk} &= \frac{1}{2}\, g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\ C^{jk}_i &= -\frac{1}{2}\, g_{is}(\partial^j g^{sk} + \partial^k g^{sj} - \partial^s g^{jk}) = g_{is} C^{sjk}, \end{aligned} \tag{2.10}$$

is metrical and its h-torsion $T^i_{jk} := H^i_{jk} - H^i_{kj} = 0$, v-torsion $S^{jk}_i := C^{jk}_i - C^{kj}_i = 0$ and the deflection tensor $\Delta_{ij} := N_{ij} - p_k H^k_{ij} = 0$. Moreover, it is unique with these properties. This is called the canonical metrical connection of the Cartan space (M, K). It has also the following properties:

$$\begin{aligned} &K_{|j} = 0, \quad K\Big|^j = \frac{p^j}{K}, \quad K^2{}_{|j} = 0, \quad K^2\Big|^j = 2p^j, \\ &p_{i|j} = 0, \quad p_i\Big|^j = \delta^j_i, \quad p^i{}_{|j} = 0, \quad p^i\Big|^j = g^{ij}. \end{aligned} \tag{2.11}$$

Besides $C\Gamma(N) = (H^i_{jk}, C^{jk}_i, -H^i_{jk}, -C^{jk}_i)$ one may consider on $T^*_\circ M$ three other important N-linear connections which are partially or not at all metrical:

(1) Chern–Rund connection $CR\Gamma(N) = (H^i_{jk}, 0, -H^i_{jk}, 0)$. This is only $h-$metrical :$g_{ij|k} = 0$ equivalently $g^{ij}{}_{|k} = 0$.
(2) Hashiguchi connection $H\Gamma(N) = (\partial^i N_{jk}, C^{kj}_i, -\partial^i N_{jk}, -C^{kj}_i)$. This is only $v-$ metrical : $g^{ij}\Big|^k = 0$ equivalently $g_{ij}\Big|^k = 0$.
(3) Berwald connection $B\Gamma(N) = (\partial^i N_{jk}, 0, -\partial^i N_{jk}, 0)$. This is no $h-$ metrical nor $v-$ metrical.

3. Vertical foliation of a Cartan space

Let $(M, K(x,p))$ be a Cartan space. Then on the manifold $T^*_\circ M$ we have two complementary distributions (horizontal $N_{x,p} := HT^*_\circ M$ and vertical $VT^*_\circ M$) the horizontal projector h, the vertical projector v and an almost product structure $P = h - v$. Also, we have a metrical structure $vg = g^{ij}(x,p)\delta p_i \delta p_j$ in the vertical subbundle as well as a metrical structure $hg = g_{ij}(x,p)dx^i dx^j$ in the horizontal subbundle. In other words, the pair $(VT^*_\circ M, vg)$ is a semi-Riemannian foliation and the pair $(HT^*_\circ M, hg)$ is a semi-Riemannian distribution transversal to the vertical foliation.

Using the metrical structures hg and vg one defines a metrical structure on $T^*_\circ M$ by taking their sum

$$G = g_{ij}(x,p)dx^i dx^j + g^{ij}(x,p)\delta p_i \delta p_j. \tag{3.1}$$

This G is similar to the Sasaki metric from the geometry of the tangent bundle. The horizontal and the vertical subbundles are orthogonal with respect to it.

By a theorem of Bejancu-Farran (see Theorem 5.1 in [2]) there exists a unique linear connection D in $VT^*_\circ M$ which is $v-$metrical i.e.

$$(D_{vX}vg)(vY,vZ) := vX(vg(vY,vZ)) - vg(D_{vX}vY,vZ) - vg(vY,D_{vX}vZ) = 0, \tag{3.2}$$

and $h-$torsion free i.e.

$$D_X vY - D_{vY}vX - v[X,vY] = 0,\ \forall X,Y \in \chi(T^*_\circ M). \tag{3.3}$$

Similarly, there exists a unique linear connection $D^\perp$ in $HT^*_\circ M$ which is $v-$torsion free and $h-$ metrical (in the previous formulae v should be replaced by h).

The two linear connections D and $D^\perp$ called the *intrinsic connections* define by the formula

$$\widetilde{\nabla}_X Y = D_X vY + D^\perp_X hY,\ X,Y \in \chi(T^*_\circ M), \tag{3.4}$$

a linear connection $\widetilde{\nabla}$ on $T^*_\circ M$ which preserves by parallelism the horizontal and vertical distributions.

Let ∇ be the Levi-Civita of G. Its torsion vanishes, that is

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0\ \ \forall X,Y \in \chi(T^*_\circ M). \tag{3.5}$$

The operator ∇ is determined by the equation

$$\begin{aligned} 2G(\nabla_X Y, Z) &= XG(Y,Z) + YG(Z,X) - ZG(X,Y) \\ &+G([X,Y],Z) - G([Y,Z],X) + G([Z,X],Y), \forall X,Y,Z \in \chi(T^*_\circ M). \end{aligned} \tag{3.6}$$

According to Ch. 1 in [2] one associates to ∇ the next two interesting adapted linear connections:the Schouten-van Kampen connection

$$\nabla^\circ_X Y = h\nabla_X hY + v\nabla_X vY,\ \ \forall X,Y \in \chi(T^*_\circ M) \tag{3.7}$$

and the Vrănceanu connection

$$\nabla^*_X Y = h\nabla_{hX}hY + v\nabla_{vX}vY + h[vX,hY] + v[hX,vY],\ \forall X,Y \in \chi(T^*_\circ M). \tag{3.8}$$

If the conditions determining ∇ are written in the adapted frame (δ_i, ∂^i) one obtains

Theorem 3.1. *The Levi-Civita connection ∇ of the Sasaki metric G is locally expressed as follows:*

$$\begin{aligned}
\nabla_{\delta_j}\delta_i &= H^k_{ij}\delta_k + \frac{1}{2}(R_{kji} - g_{hk}\partial^h g_{ij})\partial^k,\\
\nabla_{\partial^j}\delta_i &= B^{jk}_i\delta_k + \frac{1}{2}g_{kh}g^{jh}_{||i}\partial^k,\\
\nabla_{\delta_j}\partial^i &= B^{ik}_j\delta_k + (\frac{1}{2}g_{kh}g^{ih}_{||j} - \partial^i N_{jk})\partial^k,\\
\nabla_{\partial^j}\partial^i &= -\frac{1}{2}g^{kh}g^{ij}_{||h}\delta_k - C^{ij}_k\partial^k,
\end{aligned} \tag{3.9}$$

where $B^{ik}_j = \frac{1}{2}g^{kh}(\partial^i g_{hj} + R_{shj}g^{si})$ and $||k$ denotes the Berwald covariant derivative i.e. the covariant derivative constructed with $\partial^i N_{jk}$.

Then the Schouten-van Kampen connection defined by ∇ is locally given by

$$\begin{aligned}
&\nabla^\circ_{\delta_j}\delta_i = H^k_{ij}\delta_k, && \nabla^\circ_{\delta_j}\partial^i = (\frac{1}{2}g_{kh}g^{ih}_{||j} - \partial^i N_{jk})\partial^k,\\
&\nabla^\circ_{\partial^j}\delta_i = \frac{1}{2}g^{kh}(\partial^j g_{hi} + R_{shi}g^{sj})\delta_k, && \nabla^\circ_{\partial^j}\partial^i = -C^{ij}_h\partial^h,
\end{aligned} \tag{3.10}$$

and the Vrănceanu connection defined by ∇ is locally given by

$$\begin{aligned}
\nabla^*_{\delta_j}\delta_i &= H^k_{ij}\delta_k, & \nabla^*_{\delta_j}\partial^i &= -\partial^i N_{jk}\partial^k,\\
\nabla^*_{\partial^j}\delta_i &= 0, & \nabla^*_{\partial^j}\partial^i &= -g_{hk}C^{ijk}\partial^h.
\end{aligned} \tag{3.11}$$

One easily verifies that the linear connection D in the (semi)- Riemannian vertical bundle given locally by

$$D_{\delta_j}\partial^i = -\partial^i N_{jk}\partial^k, D_{\partial^j}\partial^i = -g_{hk}C^{ijk}\partial^h, \tag{3.12}$$

is $v-$metrical and $h-$ torsion free. By the uniqueness stated in the Theorem 5.1 from [2] this is just the intrinsic connection associated to the (semi)- Riemannian vertical foliation. Similarly, one proves that the intrinsic connection $D^\perp$ associated to its transversal distribution is locally given by

$$D^\perp_{\delta_j}\delta_i = H^k_{ij}\delta_k, D^\perp_{\partial^j}\delta_i = 0. \tag{3.13}$$

The local formulas (3.10)-(3.13) prove the following

Theorem 3.2. *Let D and $D^\perp$ the intrinsic connections associated to the vertical foliation and its transversal (horizontal) distribution. Then :*

(1) *D coincides to the restriction of the Hashiguchi connection as well as to the restriction of the Vrânceanu connection to the vertical foliation,*
(2) *$D^{\perp}$ coincides to the restriction of the Rund connection as well as to the restriction of the Vrânceanu connection to the transversal (horizontal) distribution.*

We recall some general notion following [3].

Let $\mathcal{D}$ be a distribution on a manifold M endowed with a linear connection ∇. The distribution $\mathcal{D} = \bigcup \mathcal{D}_x, x \in M$ is said to be *geodesically invariant* if for every geodesic $c : [a, b] \to M$ such that $c'(a) \in \mathcal{D}_{c(a)}$ it follows that $c'(t) \in \mathcal{D}_{c(t)}$ for each $t \in (a, b]$. Here c' denotes the tangent vector field to c. If the distribution $\mathcal{D}$ is integrable and geodesically invariant, one says that $\mathcal{D}$ is totally geodesic.

In [3] A.D. Lewis proves that the distribution $\mathcal{D}$ is geodesically invariant if and only if for all $\mathcal{D}-$ valued vector fields X, Y it results that the vector field $\nabla_X Y + \nabla_Y X$ is also $\mathcal{D}-$ valued.

Theorem 3.3. *Let (M, K) be a Cartan space and g^{ij} its metrical structure. Then :*

(1) *The vertical foliation on $T^*_\circ M$ is totally geodesic if and only if the Berwald connection of the Cartan space (M, K) is $h-$metrical, that is $g^{ij}_{||k} = 0$ or if the Berwald connection coincides to the Chern Rund connection.*
(2) *Its transversal (horizontal) distribution is geodesically invariant if and only if the Cartan space (M, K) reduces to the Riemannian space $(M, g_{ij}(x))$ and it is totally geodesic if and only if this Riemannian space is flat.*

Proof.

1. On $T^*_\circ M$ we have the Sasaki metric G with its Levi-Civita connection ∇. By the result just quoted of A. D. Lewis, the vertical foliation on $T^*_\circ M$ is totally geodesic if and only if the vector fields $\nabla_{\partial^j}\partial^i + \nabla_{\partial^i}\partial^j$ belong to the vertical distribution. By the Theorem 3.1 (the forth equation (3.9)) this fact is equivalent to $g^{ij}_{||k} = 0$. The second assertion follows from the remark that the equation $H^i_{jk} = \partial^i N_{jk}$ implies $g^{ij}_{||k} = 0$.

2. By the same Theorem 3.1, the horizontal distribution is geodesically invariant if and only if the vector fields $\nabla_{\delta_j}\delta_i + \nabla_{\delta_i}\delta_j = 2H^k_{ij}\delta_k - (g_{hk}\partial^h g_{ij})\partial^k$ are horizontal vector fields. This happens if and only if $\partial^h g_{ij} = 0$, that is the tensor field g_{ij} does not depend on momenta. Thus the Cartan space (M,K) reduces to the Riemannian space $(M, g_{ij}(x))$. In this case, by (2.3) we have $N_{ij} = \gamma^k_{ij}(x)p_k$ and $R_{ijk} = R^h_{ijk}(x)p_h$, where $R^h_{ijk}(x)$ is the curvature tensor of the Riemannian space $(M, g_{ij}(x))$. The horizontal distribution is totally geodesic if, moreover, it is integrable and this holds if and only if $R_{ijk} = 0$ or equivalently $R^h_{ijk}(x) = 0$, that is the Riemannian space $(M, g_{ij}(x))$ is flat. □

Remark. A Cartan space (M,K) is said to be a Landsberg space if its Berwald connection is $h-$metrical, that is if $g^{ij}_{||k} = 0$ or $g_{ij||k} = 0$. It follows that a Cartan space (M,K) is Landsberg if and only if $H^i_{jk} = \partial^i N_{jk}$. Thus by the Theorem 3.3 the vertical foliation is totally geodesic if and only if the Cartan space (M,K) is Landsberg.

Theorem 3.4. *Let $(M.K)$ be a Landsberg space. Assume that $g^{ij}(x,p)$ is positive defined and that the horizontal distribution is integrable. Then the sectional curvature of the Riemannian manifold $(T^*_\circ M, G)$ is non-positive.*

Proof. Indeed, if there exists a point in $T^*_\circ M$ for which the sectional curvature is positive, by the Theorem 4.6 from Ch. 3 in [2] the horizontal distribution should be non-integrable, which is a contradiction. □

Corollary 3.5. *Let (M,g) be a flat Riemannian manifold. The the sectional curvature of the Riemannian manifold $(T^*_\circ M, G)$ is non-positive.*

This Corollary can be also derived from the fundamental equations of the Riemannian submersion $(T^*_\circ M, G) \mapsto (M, g)$.

The first two equations from (3.9) are equivalent to

$$\nabla_X hY = \nabla^h_X hY + B(X, hY), \tag{3.14}$$

where $\nabla^h_X hY = h\nabla_X hY$ is a linear connection in the horizontal bundle and $B : \chi(T^*M) \times \Gamma(HT^*_\circ M) \to \Gamma(VT^*_\circ M)$ is an $\mathcal{F}(T^*_\circ M)$- bilinear mapping given by $B(X, hY) = v\nabla_X hY$.

The next two equations from (3.9) are equivalent to

$$\nabla_X vY = B'(X, vY) + \nabla^v_{X^v} vY, \tag{3.15}$$

where $\nabla^v_X vY = v\nabla_X vY$ is a linear connection in the vertical bundle, and $B' : \chi(T^*_\circ M) \times \Gamma(VT^*_\circ M) \to \Gamma(HT^*_\circ M)$ is an $\mathcal{F}(M)-$bilinear mapping given by $B'(X, vY) = h\nabla_X vY$.

The linear connections ∇^h and ∇^v are called *induced* by ∇ on horizontal and vertical bundles, respectively and the mappings

$$b(hX, hY) := B(hX, hY) = v\nabla_h XhY, b'(hX, hY) := B'(vX, vY) = h\nabla_v XvY$$

are called second fundamental forms of $HT^*_\circ M$ and $VT^*_\circ M$,respectively.

The linear connection ∇^h is locally determined by (H^k_{ij}, B^{ik}_j) and we have

$$b(\delta_i, \delta_j) = \frac{1}{2}(R_{kji} - g_{hk}\partial^h g_{ij})\partial^k. \tag{3.16}$$

The linear connection ∇^v is locally determined by $(\frac{1}{2}g_{kh}g^{ih}_{||j} - \partial^i N_{jk}, -C^{ij}_k)$ and we have

$$b'(\partial^i, \partial^j) = -\frac{1}{2}g^{kh}g^{ij}_{||h}\delta_k. \tag{3.17}$$

The pairs (∇^h, ∇^v) determines an adapted linear connection according to (3.4). This is nothing but the Schouten-van Kampen connection determined by ∇.

Next, we define the $\mathcal{F}(M)-$ linear operators

$$A^v_{hX} : \Gamma(VT^*_\circ M) \mapsto \Gamma(VT^*_\circ M), A^v_{hX}(vY) = -B(vY, hX),$$
$$A^h_{vX} : \Gamma(HT^*_\circ M) \mapsto \Gamma(HT^*_\circ M), A^h_{vX}(hY) = -B'(hY, vX).$$

These are called the shape operators of the vertical and horizontal distributions with respect to the normal sections hX and vX, respectively. The Theorem 3.1 yields

$$A^v_{\delta_i}(\partial^j) = -v\nabla_{\partial^j}\delta_i = -\frac{1}{2}g_{kh}g^{jh}_{||i}\partial^k,$$
$$A^h_{\partial^i}(\delta_j) = -h\nabla_{\delta_j}\partial^i = -B^{ik}_j\delta_k.$$

Theorem 3.6. *Let (M, K) be a Cartan space. The condition that the vertical foliation is totally geodesic is equivalent to each of the following assertions:*

(1) *The second fundamental form of the vertical foliation identically vanishes.*
(2) *The shape operators $A^v_{\delta_i}$ vanish.*
(3) *The induced connection ∇^v coincides to the intrinsic connection D on the vertical foliation.*

Proof. 1. By the Theorem 3.3 the vertical foliation is totally geodesic if and only if $g^{ij}_{||h} = 0$ and by (3.17) we have $b' \equiv 0$ and conversely.

The equivalence of **1** and **2** is obvious.

2. By (3.12) the said connections coincide if and only if $g^{ij}_{||h} = 0$, that is if and only if the vertical foliation is totally geodesic. □

Theorem 3.7. *Let (M, K) be a Cartan space. Then the second fundamental form of its horizontal distribution identically vanishes ($b = 0$) if and only if the induced connection ∇^h coincides to the intrinsic connection $D^\perp$ on the horizontal distribution. Moreover, if the horizontal distribution is integrable then $b = 0$ if and only if the horizontal foliation is totally geodesic.*

Proof. A short examination shows that $B^{ij}_k = 0$ is equivalent to $b = 0$. If the horizontal distribution is integrable i.e. $R_{kij} = 0$, from $b = 0$ it follows that (M, K) reduces to a Riemannian space and using again $R_{kij} = 0$ it comes out that the said Riemannian space is flat. By the Theorem 3.3 we conclude that the horizontal distribution is totally geodesic. The converse is clear. □

We recall that $T^*_\circ M$ is endowed with a natural symplectic structure. In the presence of the symmetric nonlinear connection defined by K this is given by

$$\theta = \delta p_i \wedge dx^i. \tag{3.18}$$

In the adapted frame (δ_i, ∂^i) we have

$$\theta(\delta_i, \delta_j) = 0,\ \theta(\delta_i, \partial^j) = -\delta^j_i,\ \theta(\partial^i, \partial^j) = 0. \tag{3.19}$$

Thus the vertical and horizontal distributions are Lagrangian distributions with respect to θ. As the vertical distribution is integrable it defines a Lagrangian foliation with the leaves $T^*_x M$, $x \in M$.

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