"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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RHEONOMIC INGARDEN SPACES

OTILIA LUNGU

Abstract. This paper presents a particular type of rheonomic Finsler space, called rheonomic Ingarden space. The canonical semispray, the local coefficients of the Lorentz connection and the expressions for the curvature and for the torsion are established.

1. Preliminary. Ingarden spaces

Let M be an n-dimensional real manifold. Denote by (TM, τ, M) the tangent bundle of M and let $F^n = (M,\ F(x,\ y))$ be a Finsler space with the fundamental function $F(x,\ y) = \alpha(x,\ y) + \beta(x,\ y)$ where $\alpha(x,\ y) = \sqrt{a_{ij}(x)\ y^i y^j}$ and $\beta(x,\ y) = b_i(x)\ y^i$; $a = a_{ij}(x)\ dx^i dx^j$ is a pseudo-Riemannian metric on M and it gives the gravitational part of the metric F ; $b_i(x)$ is an electromagnetic covector on M and $\beta(x,\ dx) = b_i(x)\ dx^i$ is the electromagnetic 1-form field on M . We consider the integral of action of the energy $F^2(x,\ y)$ along a curve $c\ :\ t \in [0,\ 1] \to c(t) \in M$:

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$$I(c) = \int_0^1 F^2\left(x, \frac{dx}{dt}\right) dt = \int_0^1 \left[\alpha\left(x, \frac{dx}{dt}\right) + \beta\left(x, \frac{dx}{dt}\right)\right]^2 dt \tag{1.1}$$

The variational problem for $I(c)$ leads to the Euler-Lagrange equations:

$$E_i\left(F^2\right) := \frac{\partial(\alpha+\beta)^2}{\partial x^i} - \frac{d}{dt}\frac{\partial(\alpha+\beta)^2}{\partial y^i} = 0,\ y^i = \frac{dx^i}{dt}. \tag{1.2}$$

Let us fix a parametrization of the curve c, by natural parameter s with respect to Riemannian metric $\alpha(x, y)$. It is given by

$$ds^2 = \alpha^2\left(x, \frac{dx}{dt}\right) dt^2. \tag{1.3}$$

It follows $F^2\left(x, \frac{dx}{ds}\right) = 1$ and $\frac{d\alpha}{ds} = 0$.

Along to an extremal curve c , canonical parametrized by (1.3) , the equations (1.2) become

$$\frac{\partial\alpha^2}{\partial x^i} - \frac{d}{d\sigma}\left(\frac{\partial\alpha^2}{\partial y^i}\right) = 2F_{ij}(x)\, y^i\ , \tag{1.4}$$

where

$$F_{ij} = \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i},\ F_j^i(x) = a^{is}F_{sj}. \tag{1.5}$$

Theorem 1.1. The Euler-Lagrange equations (1.4) are equivalent to the Lorentz equations:

$$\frac{d^2x^i}{ds^2} + \gamma_{jk}^i(x)\, \frac{dx^j}{ds}\frac{dx^k}{ds} = \overset{\circ}{F}{}_j^i(x)\, \frac{dx^j}{ds}, \tag{1.6}$$

where γ_{jk}^i are the Christoffel symbols of the Riemannian metric tensor $a_{ij}(x)$.

The Euler-Lagrange equations determines a canonical semispray S on the total space of the tangent bundle :

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i} \tag{1.7}$$

where the coefficients $G^i(x, y)$ are:

$$2G^i(x, y) = \gamma_{jk}^i(x)\, y^j y^k - \overset{\circ}{F}{}_j^i(x)\, y^j. \tag{1.8}$$

This semispray determines the nonlinear connection N with the coefficients

$$N_j^i = \gamma_{jk}^i(x)\, y^k - F_j^i(x)\,, \tag{1.9}$$

where $F_j^i(x) = \frac{1}{2}\overset{\circ}{F_j^i}(x)$.

Since the autoparallel curves of N are given by the Lorentz equations (1.6), we call it the Lorentz nonlinear connection of the metric $\alpha + \beta$.

The nonlinear connection N determines the horizontal distribution, denoted by N too, with the property

$$T_u TM = N_u \oplus V_u \ , \forall u \in TM$$

V_u being the natural vertical distribution on the tangent manifold TM

The local adapted basis to the horizontal and vertical vector spaces N_u and V_u is given by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, $i = 1, ..., n$, where

(1.10)

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k} = \frac{\partial}{\partial x^i} - \gamma_{is}^k(x)\, y^s \frac{\partial}{\partial y^k} + F_i^k \frac{\partial}{\partial y^k} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F_i^k \frac{\partial}{\partial y^k}$$

and

(1.11) $$\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \gamma_{is}^k(x)\, y^s \frac{\partial}{\partial y^k} \ .$$

The adapted cobasis is $(dx^i, \delta y^i)$, $i = 1, ..., n$ with

(1.12) $$\delta y^i = dy^i + N_j^i dx^j = dy^i + \gamma_{jk}^i(x)\, y^k dx^j - F_j^i dx^j = \overset{\circ}{\delta} y^i - F_j^i dx^j$$

where

(1.13) $$\overset{\circ}{\delta} y^i = dy^i + \gamma_{jk}^i(x)\, y^k dx^j$$

Definition 1.1. The Finsler space $F^n = (M, F = \alpha + \beta)$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted IF^n .

The fundamental tensor g_{ij} of IF^n is

(1.14) $$g_{ij} = \frac{F}{\alpha}(a_{ij} - \tilde{l}_i \tilde{l}_j) + l_i l_j$$

where $\tilde{l}_i = \frac{\partial \alpha}{\partial y^i}$, $l_i = \frac{\partial F}{\partial y^i}$, $l_i = \tilde{l}_i + b_i$.

The following results holds :

Theorem 1.2. There exists an unique N -metrical connection $I\Gamma(N) = \left(F_{jk}^i, C_{jk}^i\right)$ of the Ingarden space IF^n which verifies the following axioms:

i) $\nabla_k^H g_{ij} = 0$; $\nabla_k^V g_{ij} = 0$;

ii) $T_{jk}^i = 0$; $S_{jk}^i = 0$

The connection $I\Gamma(N)$ has the coefficients expressed by the generalized Christoffel symbols:

$$(1.13)\qquad \begin{cases} F^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right), \end{cases}$$

where $\frac{\delta}{\delta x^i}$ are given by (1.10).

2. Rheonomic Finsler Spaces

We consider $E = TM \times R$ a $2n+1$- dimensional real manifold. In a local chart $U \times (a, b)$ the points $u = (x, y, t) \in E$ have the local coordinates (x^i, y^i, t). Let $rF^n = (M, F(x, y, t))$ be a rheonomic Finsler space where $F : E \to R$ is the fundamental function and the Hessian of F given by $g_{ij}(x, y, t) = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}$, called the fundamental tensor of rF^n, is positive defined.

The Cartan nonlinear connection N has the coefficients $\left(N^i_j(x, y, t), N^0_j(x, y, t)\right)$ with

$$(2.1)\qquad \begin{array}{l} N^i_j(x, y, t) = \frac{1}{2} \frac{\partial}{\partial y^j} \left(\gamma^i_{kh}(x, y, t)\, y^k y^h \right) \\ N^0_j(x, y, t) = \frac{1}{2} \frac{\partial g_{jk}}{\partial t} y^k \end{array}$$

and γ^i_{kj} are the Christoffel symbols of the fundamental tensor $g_{ij}(x, y, t)$. N determines the horizontal distribution on E which is supplementary to the vertical distribution. The adapted basis to these distribution is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t} \right)$ with

$$(2.2)\qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y, t) \frac{\partial}{\partial y^j} - N^0_i(x, y, t) \frac{\partial}{\partial t}.$$

The dual adapted basis is $(dx^i, \delta y^i, dt)$ where

$$(2.3)\qquad \begin{array}{l} \delta y^i = dy^i + N^i_j(x, y, t)\, dx^j \\ \delta t = dt + N^0_j(x, y, t)\, dx^j. \end{array}$$

Theorem 2.1 The canonical metrical N-connection has the coefficients expressed by the generalized Christoffel symbols:

$$(2.4)\qquad \begin{cases} F^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right) \\ C^i_{j0} = \frac{1}{2} g^{is} \left(\frac{\partial g_{s0}}{\partial y^j} + \frac{\partial g_{sj}}{\partial t} - \frac{\partial g_{j0}}{\partial y^s} \right) \end{cases}$$

where $\frac{\delta}{\delta x^i}$ are given by (2.2).

3. Rheonomic Ingarden spaces

Let $\alpha(x,y,t)=\sqrt{a_{ij}(x,t)\,y^iy^j}$ be a rheonomic Riemannian structure on $M\times R$ and $\beta(x,y,t)=b_i(x,t)\,y^i$ with $b_i(x,t)$ a covector field on $M\times R$.

Now we consider the rheonomic metric

$$F(x,y,t)=\alpha(x,y,t)+\beta(x,y,t)\,. \tag{3.1}$$

The pair $rIF^n=(M,F(x,y,t))$ is a rheonomic Finsler space. The tensor field g_{ij} of rIF^n

$$g_{ij}(x,y,t)=\tfrac{1}{2}\tfrac{\partial^2F^2}{\partial y^i\partial y^j}, \tag{3.2}$$

is positively defined on $(TM\setminus\{0\})\times R$ and it is called the fundamental tensor of the space rIF^n.

We denote

$$\gamma^i_{jk}=\frac{1}{2}g^{is}\left(\frac{\delta\,g_{sj}}{\delta x^k}+\frac{\delta\,g_{sk}}{\delta x^j}-\frac{\delta\,g_{jk}}{\delta x^s}\right) \tag{3.3}$$

the Christoffel symbols for the fundamental tensor $g_{ij}(x,y,t)$.

We also denote

$$F_{ij}(x,t)=\frac{\partial b_j}{\partial x^i}-\frac{\partial b_i}{\partial x^j} \tag{3.4}$$

and we consider

$$F^i_j(x,y,t)=g^{ik}(x,y,t)\,F_{kj}(x,t). \tag{3.5}$$

Theorem 3.1. For a rheonomic Finsler space rIF^n the Euler-Lagrange equations are equivalent with the Lorentz equations:

$$\tfrac{d^2x^i}{dt^2}+\overset{\alpha}{\gamma}{}^i_{jk}(x,y,t)\,\tfrac{dx^i}{dt}\,\tfrac{dx^k}{dt}=F^i_j\left(x,\tfrac{dx}{dt},t\right)\tfrac{dx^i}{dt}, \tag{3.6}$$

with $\overset{\alpha}{\gamma}{}^i_{jk}(x,y,t)$ the Christoffel symbols of the rheonomic Riemannian structure a and

$$F^i_j\left(x,\tfrac{dx}{dt},t\right)=-g^{ih}\tfrac{\partial g_{hj}}{\partial t}. \tag{3.7}$$

The canonical spray S of rIF^n is

$$S=y^i\frac{\partial}{\partial x^i}-\left(N^i_0(x,y,t)+N^i_k(x,y,t)\,y^k\right)\frac{\partial}{\partial y^i}+\frac{\partial}{\partial t} \tag{3.8}$$

where

$$
\begin{array}{ll}
N_j^i(x,y,t) = \frac{1}{2}\frac{\partial}{\partial y^j}\left(\gamma_{rs}^i(x,y,t)\,y^r y^s\right) \\
N_j^0(x,y,t) = \frac{1}{2}\frac{\partial g_{jk}}{\partial t} y^k
\end{array} \tag{3.9}
$$

Let us consider in a rIF^n space the nonlinear connection $\overset{L}{N}$ whose local coefficients are $\left(\overset{L}{N}{}_j^i(x,y,t), \overset{L}{N}{}_j^0(x,y,t)\right)$ with

$$
\begin{cases}
\overset{L}{N}{}_j^i = N_j^i - F_j^i \\
\overset{L}{N}{}_j^0 = N_j^0 - F_j^i.
\end{cases} \tag{3.10}
$$

N is called Lorentz nonlinear connection of the space rIF^n.

The local basis adapted to the Lorentz nonlinear connection is $\left(\frac{\overset{L}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$, where

$$
\frac{\overset{L}{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + F_j^i\frac{\partial}{\partial y^j} + F_j^i\frac{\partial}{\partial t}. \tag{3.11}
$$

The dual adapted basis is $\left(dx^i, \overset{L}{\delta}y^i, \overset{L}{\delta}t\right)$ with

$$
\begin{array}{l}
\overset{L}{\delta}y^i = \delta y^i - F_j^i dx^j \\
\overset{L}{\delta}t = \delta t - F_j^i dx^j.
\end{array} \tag{3.12}
$$

The weak torsion $\overset{L}{T}{}_{jk}^i$ of rIF^n space is

$$
\overset{L}{T}{}_{jk}^i = \frac{\partial \overset{L}{N}{}_j^i}{\partial y^k} - \frac{\partial \overset{L}{N}{}_k^i}{\partial y^j} \tag{3.13}
$$

and the curvature tensor $\overset{L}{R}{}_{jk}^i$ is

$$
\overset{L}{R}{}_{jk}^i = \frac{\overset{L}{\delta}\overset{L}{N}{}_j^i}{\partial x^k} - \frac{\overset{L}{\delta}\overset{L}{N}{}_k^i}{\partial x^j}. \tag{3.14}
$$

By a direct calculus we can state the following theorem:

Theorem 3.2. The torsion $\overset{L}{T}{}_{jk}^i$ of the Lorentz nonlinear connection of the rIF^n space vanishes and the curvature tensor is given by

(3.15)

$$\overset{L}{R}{}^{i}_{jk} = \left(\frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}\right) - \left(\frac{\delta F^i_j}{\delta x^k} - \frac{\delta F^i_k}{\delta x^j}\right) + \left(F^i_j \frac{\partial N^i_j}{\partial y^k} - F^i_k \frac{\partial N^i_k}{\partial y^j}\right)$$
$$- \left(F^i_j \frac{\partial F^i_j}{\partial y^k} - F^i_k \frac{\partial F^i_k}{\partial y^j}\right) + \left(F^i_j \frac{\partial N^i_j}{\partial t} - F^i_k \frac{\partial N^i_k}{\partial t}\right) - \left(F^i_j \frac{\partial F^i_j}{\partial t} - F^i_k \frac{\partial F^i_k}{\partial t}\right).$$

References

[1] Bucătaru I., Miron R., **Finsler-Lagrange Geometry- Applications to dynamical systems**, Ed. Academiei Romane, Bucuresti, (2007).

[2] Frigioiu C.,**On the geometrization of the rheonomic Finslerian Mechanical Systems**, Analele Ştiinţifice Univ. " Al.I. Cuza", Iaşi, vol. LIII, 145-155, (2007).

[3] Frigioiu C., Hedrih K., Bîrsan I.G., **Lagrangian geometrical model of the rheonomic mechanical systems**, World Academy of Science, Engineering and Technology, vol 5, 283-289, (2011).

[4] Miron R., **General Randers Spaces**, Lagrange and Finsler Geometry, Ed. By P.L. Antonelli and R. Miron, FTPH 59, Kluwer Acad. Publ., 123-140,(1996).

[5] Miron R., **Lagrangian and Hamiltonian Geometries- Applications to Analytical Mechanics**, Ed. Academiei Romane, Ed. Fair Partners, Bucureşti, (2011).

[6] Miron R., **The Geometry of Ingarden Spaces,**Rep. on Math. Phys., 54, 131-147, (2004).

[7] Z.Shen, **Differential Geometry of Spray and Finsler Spaces,** Kluwer Academic Publishrs, Dodrecht, (2001).

Department of Mathematics, Informatics and Educational Sciences,
Faculty of Sciences, "Vasile Alecsandri" University of Bacău,
157 Calea Mărăşeşti, Bacău, 600115, ROMANIA
e-mail : otilia.lungu@ub.ro , otilas@yahoo.com